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Efficient Analytical Approach for Nonlinear System of Delay Differential Equations

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Abstract

Delay differential equations (DDEs) are class of differential equations that have been frequently used as efficient mathematical tool to model many real-life problems. Recently, series of techniques have encountered problem in finding convergent analytical solution for the system of DDEs, in particular nonlinear type. This article introduced analytical scheme for solving nonlinear system of DDEs via Homotopy analysis method and Natural transform method. By this technique, the He's polynomial is adjusted to ease the difficulties of computing nonlinear terms for the system of DDEs. The method gives solution to Different initial value problems in a series form which converges to exact solution or approximate solution. The convergence analysis is sufficient enough to guarantee the convergence of approximate solutions obtained by the proposed technique. The obtained results reveal that the approach is accurate, efficient, avoids large computational work and round off error and can be applied to different form of nonlinear problems.

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1 Introduction

Delay differential equations (DDEs) have been widely applied to model real life phenomena from different fields of applied sciences such as biology, chemistry, physics, economy, engineering, control problems electrodynamics and medicine, to mentioned but few [1, 2, 3, 4, 5]. Contrary to ordinary differential equations (ODEs) the derivatives of the independent variables in DDEs at a certain time are express in terms of the values of the function at previous time. For this purpose, DDEs are used in various application instead of ODEs. Furthermore, systems of DDEs also play a vital role in different setting of real-life application. Numerous researchers have used different techniques for obtaining solutions of DDEs. [6, 7, 8, 9, 10]. However, it is observed that most of these techniques have some deficiencies in finding approximate analytical solution of different form of nonlinear system of DDEs. These include divergent of result, restrictive assumptions, lengthy calculation and dependency of small or large parameter. Therefore, nonlinear systems of DDEs are very difficult to solve analytically [11] and consequently numerical techniques are widely used to obtain the approximate solution of such equations. Hence, new methods for finding analytical solutions to these types of equations are highly needed.

In this paper analytical approach based on Homotopy analysis method (HAM) and Natural transform is proposed. Liao [12] was first introduced the concept of HAM to obtain solutions of linear and nonlinear problems. HAM is based on Homotopy, fundamental concept in topology and differential geometry that modify the traditional Homotopy and provides a convenient way to adjust and control the convergence region for the series solution using an auxiliary parameter. The HAM has been successfully applied to solved different types of problems [6, 13, 14, 15]. Khan and Khan [16] was first defined an integral transform called N-Transform and later Belgacem and Silambarasan [17] defined its inverse and provided the detailed of its fundamental properties and application. Since then, the Natural Transform was applied to solve different types of differential and integral equations [17, 18, 19].

This research introduces a new approach for finding approximate analytical solution of nonlinear systems of DDEs from the combination of these powerful method.The present work also studies the convergence of the proposed technique and adopt the use of He's polynomial for computation of nonlinear terms. Hence, the motive toward the conduct of this study is to establish a reliable method that provides a convergent analytical solution to different types of nonlinear system of DDEs in a series form from only few numbers of iterations and minimal error as compared with the existing methods. Some experimental examples are given in order to show the efficiency of the proposed method over the reference techniques.

The remaining parts of this paper are organized as follows: In Section 2, some basic definitions and properties of Natural transform are given. The analysis of the new approach is presented in Section 3. The convergence analysis of the proposed technique was rendered in Section 4. In section 5, the proposed method is applied to three different problems in order to show its efficiency. Finally, the conclusion follows in Section 6.

2 Definitions and Properties of Natural Transform

The basic definitions and some important properties of Natural transform that could be used further in this paper are presented in this section.

Definition 2.1. *[20]*

Suppose that the function $f(t) > 0$ and $f(t) = 0$ for $t < 0$. Then the Natural Transform is defined over the set
$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, if t \in (-1)^j \times [0, \infty), j \in \mathbf{Z}^+ \right\}$
as in the given integral:

$$\mathbf{N}^+ [f(t)] = R^+(s, u) = \int_0^\infty e^{-st} f(ut)dt; \quad s, u \in (0, \infty] \tag{1}$$

where s, u are transform variables. In this case the function $f(t)$ defined in Eq.(1) is called the inverse of $R^+(s, u)$ defined as

$$\mathbf{N}^- \left[R^+(u, s)\right] = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}} R(s, u)ds. \tag{2}$$

Below are some important properties of Natural transform method (see [20], [21])

Property 2.1. *The Generalized Natural Transform of the function $f(t)$ is given by*

$$\mathbf{N}^+[f(t)] = R(s, u) = \sum_{n=0}^{\infty} \frac{n! a_n u^n}{s^{n+1}} \tag{3}$$

Property 2.2. *Let f^n be the nth derivatives of the function $f(t)$ then its natural transform is given by*

$$\mathbf{N}^+[f^n(t)] = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=1}^{n} \frac{s^{n-k}}{u^{(n-k)+1}} f^{k-1}(0) \tag{4}$$

For $n = 1$ and $n = 2$ Eq. (4) gives the Natural transform of first and second derivatives of $f(t)$ respectively.

$$\mathbf{N}^+[f^{'}(t)] = R_1(s,u) = \frac{s}{u}R(s,u) - \frac{f(0)}{u} \tag{5}$$

$$\mathbf{N}^+[f^{''}(t)] = R_2(s,u) = \frac{s^2}{u^2}R(s,u) - \frac{sf(0)}{u^2} - \frac{f^{'}(0)}{u} \tag{6}$$

Property 2.3. *If α and β are non-zero constants, then for the functions $f(t)$ and $g(t)$ in set A we define*

$$\mathbf{N}^+[\alpha f(t) + \beta g(t)] = \alpha \mathbf{N}^+[f(t)] + \beta \mathbf{N}^+[g(t)]. \tag{7}$$

3 Analysis of the Method

Consider the following n-order nonlinear system of Retarded (RDDEs)

$$v_i^{(n)} = f_i(t, v_\lambda^{(k)}(t), v_\lambda^{(r)}(\alpha_j(t))),\ t \in [0,d],\ i = 1,2...,N, j = 1,2..,M \tag{8}$$

with initial condition

$$v_i^{(k)}(0) = v_{i,0}^{(k)},\ v_i(t) = P_i(t),\ t < 0 \tag{9}$$

where
$v_\lambda^{(k)}(t) = (v_1^{(k)}, v_2^{(k)}, \ldots, v_N^{(k)}), v_\lambda^r(\alpha_j(t)) = (v_1^{(r)}(\alpha_j(t)), v_2^{(r)}(\alpha_j(t)), \ldots, v_N^{(r)}(\alpha_j(t)))$
for $k = 0,1,2,\ldots,n-1, r \leq n$ and $\alpha_j(t)$ are delay functions.
Now Eq. (8) can be written in the form

$$L_i(v_i) + R_i(v_\lambda) + F_i(v_\lambda) = g_i(t) \tag{10}$$

with specified initial condition. Where v_λ is N-dimensional vector form of dependent variables define as $v_\lambda = [v_1(t), v_2(t), \ldots v_N(t)]$. The linear terms are split into $L_i + R_i$ where L_i is the highest order bounded linear operators, R_i are remaining of the linear operators which are also bounded, and F_i are

continuous functions satisfy the Lipschitz condition with Lipschitz constants $\mu_i \in [0, d]$ represent the non-linear terms.
Take the Natural Transform of both sides of Eq.(10) to obtain:

$$\mathbf{N}^+[L_i(v_i)] + \mathbf{N}^+[R_i(v_\lambda)] + \mathbf{N}^+[F_i(v_\lambda)] = \mathbf{N}^+[g_i(t)] \tag{11}$$

Substitute Eq.(9) into Eq.(11), and simplify using the differential properties of Natural Transform in Eq.(4) to obtain:

$$\mathbf{N}^+[(v_i)] - \frac{u^n}{s^n}\sum_{k=1}^{n}\frac{s^{n-k}}{u^{(u-k)+1}}v_i^{k-1}(0) + \frac{u^n}{s^n}\mathbf{N}^+[R_i(v_\lambda) + F_i(v_\lambda) - g_i(t)] = 0 \tag{12}$$

From Eq. (12) we can define the following nonlinear operator

$$\begin{aligned} N_i[\phi_i(t;q)] = \mathbf{N}^+[\phi_i(t;q)] - \frac{u^n}{s^n}\sum_{k=1}^{n}\frac{s^{n-k}}{u^{(u-k)+1}}\phi_i^{k-1}(0) \\ +\frac{u^n}{s^n}\mathbf{N}^+\{R_i[\phi_\lambda(t;q)] + F_i[\phi_\lambda(t;q)] - g_i(t;q)\} \end{aligned} \tag{13}$$

where $q \in [0, 1]$ is an embedding parameter, $\phi_i(t; q)$ are functions of real variables t and q.
Then, by means of HAM, we construct the Homotopy equations as

$$(1 - q)\mathbf{N}^+[\phi_i(t;q) - v_{i,0}(t)] = h_i q H_i(t) N_i[\phi_\lambda(t;q)] \tag{14}$$

where $\mathbf{N}^+$ denotes Natural transform, $h_i \neq 0$ are auxiliary parameters, $H_i(t) \neq 0$ are auxiliary functions, $v_{i,0}(t)$ are initial approximations of $v_i(t)$ and $\phi_i(t; q)$ are unknown functions. From Eq. (14) when $q = 0$ and $q = 1$ we respectively have the following equation.

$$\begin{aligned} \phi_i(t, 0) = v_{i,0}(t) \\ \phi_i(t, 1) = v_i(t) \end{aligned} \tag{15}$$

Thus, as q increases from 0 to 1, the solutions $\phi_i(t, q)$ varies from the initial approximations $v_{i,0}(t)$ to the exact solutions $v_i(t)$. In topology such kind of variation is called deformation and equation (14) is called zero- order deformation equation. Expand $\phi_i(t; q)$ in Taylor series with respect to q

$$\phi_i(t, q) = \phi_i(t, 0) + \sum_{m=1}^{\infty} v_{i,m}(t)q^m \tag{16}$$

where

$$v_{im}(t) = \frac{1}{m!}\frac{\partial^m \phi_i(t;q)}{\partial q^m}|_{q=0}$$

Suppose that the initial approximations $v_{i0}(t)$, auxiliary parameters h_i and the auxiliary function $H_i(t)$ are properly chosen so that the series in Eq. (16) converges at $q = 1$; that is,

$$\phi_i(t,1) = v_{i,0}(t) + \sum_{m=1}^{\infty} v_{i,m}(t) \tag{17}$$

Now define the vectors

$$\overrightarrow{v}_{i,n}(t) = [v_{i,0}(t), v_{i,1}(t), ...v_{i,n}(t)] \tag{18}$$

If Eq. (14) is differentiated m-times with respect to q, then dividing by $m!$ and finally Setting $q = 0$, the following mth-order deformation equation is obtained

$$\mathbf{N}^{+}[v_{i,m}(t) - \chi_m v_{i,m-1}(t)] = h_i H_i(t) R_{y_i m}[\overrightarrow{v}_{i,m-1}(t)] \tag{19}$$

where

$$R_{y_i,m}[\overrightarrow{v}_{i,m-1}(t)] = \frac{1}{(m-1)!}\frac{\partial^{m-1} N_i[\phi_i(t,q)]}{\partial q^{m-1}}|_{q=0} \tag{20}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1. \end{cases}$$

Now, taking the inverse Natural Transform on both sides of Eq. (19), we have

$$v_{i,m}(t) = \chi_m v_{i,m-1}(t) + h_i \mathbf{N}^{-}[H_i(t) R_{y_i m}[\overrightarrow{v}_{i,m-1}(t)]]. \tag{21}$$

Therefore, $v_{i,m}(t)$ for $m \geq 1$ can be easily obtained from Eq. (21), at Mth order we have

$$v_i(t) = \sum_{m=0}^{M} v_{i,m}(t) \tag{22}$$

Therefore, as $M \to \infty$ the recursive relation of Equations (8) and (9) can be obtained as follows

$$v_{m,i}(t) = (\chi_m + h)v_{i,m-1}(t) - h_i(1-\chi_m)\mathbf{N}^{-}\sum_{k=1}^{n}\frac{u^{k-1}}{s^k}v_i^{k-1}(0)$$

$$+h_i\mathbf{N}^{-}\{\frac{u^n}{s^n}\mathbf{N}^{+}[R(v_{\lambda,m-1}(t))+H_{i,m-1}(v_{\lambda 1}, v_{\lambda 2}\dots v_{\lambda N})-g_i(t)]\}, m = 1, 2, 3\dots \tag{23}$$

Here, the nonlinear operators $F_i(v_\lambda)$ are expanded as series of He's polynomials $H_{i,m-1}(v_{\lambda 1}, v_{\lambda 2} \ldots v_{\lambda n})\}$ defined as

$$H_{i,m}(v_{\lambda 1}, v_{\lambda 2} \ldots v_{\lambda N}) = \frac{1}{m!}\frac{\partial^m}{\partial q^m} F_i(\sum_{p=0}^{m} q^p v_{\lambda,p})|_{q=0} \tag{24}$$

where $v_{\lambda,i} = (v_{i,1}, v_{i,2}, \ldots v_{i,N})$ and $v_{\lambda,p} = (v_{1,p}, v_{2,p}, \ldots v_{N,p})$.

4 Convergence of the Proposed Method

To ensure the convergence of the proposed method, this section provides sufficient conditions with the aim to prove the convergence of the algorithm derived in Eq. (23). This will be established based on Banach's Fixed-Point Theorem. Thus Eq. (21) can be written in the following form:

$$v_{i,m}(t) = A_i(v_{i,m-1}(t)), \ \ , i = 1, 2, \ldots, N, \ m = 1, 2, \ldots, M \tag{25}$$

where A_i are operators of the form

$$A_i(v) = -h_i \mathbf{N}^{-} R_{y_i,m}[\overrightarrow{v}_{im-1}(t)]. \tag{26}$$

Theorem 4.1. *Assume that Y is a Banach space. Let $A_i : Y \longrightarrow Y$ be nonlinear functions such that*

$$\parallel A_i(v) - A_i(u) \parallel \leq \rho_i \parallel u - v \parallel, \ \ \forall v, u \in Y \tag{27}$$

where $\rho_i < 1$ such that $\rho_i = (\mu_i + s_i + w_i)h_i$ for some positive numbers μ_i, s_i and w_i. Then A_i have unique fixed points in Y. In addition, the sequences (25) by the means of the proposed method and choosing an arbitrary $v_{i,0} \in Y$ converge to their respective fixed points of Y.

Proof

Since L_i and R_i are bounded continuous functions, therefore there exists some positive numbers s_i and w_i such that $\|L_i(v_i)\| \leq s_i\|v_i(t)\|$ and $\|R_i(v_\lambda)\| \leq w_i\|v_\lambda\|$. Also since $F_i(v_\lambda)$ satisfies the Lipschitz condition with Lipchitz constants μ_i, then the following hold.

$$|F_i(v) - F_i(u)| \leq \mu_i|v - u|, \ \ \forall t \in [0, d].$$

Let $[C(D); \|.\|]$ be the Banach space of all continuous functions on $D = [0, d]$, and let the norm be defined by

$$\|f(t)\| = |f(t)|, \ \ t \in D \tag{28}$$

Now it suffices to show that the $\{v_{i,m}\}$ are Cauchy sequences in $[C(D); \|.\|]$. From Equations (13) and (21) we have

$$\begin{aligned} \|v_{i,m} - v_{i,p}\| &= |v_{i,m} - v_{i,p}| = | - h_i \mathbf{N}^- [R_{y_i,m}[\overrightarrow{v}_{i,m-1}(t)] - R_{y_i,p}[\overrightarrow{v}_{i,p-1}(t)]]| \\ &= | - h_i \mathbf{N}^- [2\frac{s^n}{u^n} R(s,u) - \sum_{k=1}^{\infty} \frac{s^{n-k}}{u^{(n-1)+k}} (v_{i,m-1}^{k-1}(0) + v_{i,p-1}^{k-1}(0)) \\ &+ N^+[R_i(v_{i,m-1} - v_{i,p-1}) + F_i(v_{i,m-1} - v_{i,p-1})]]| \\ &\leq h_i |\mathbf{N}^- \{\mathbf{N}^+ [L_i(v_{i,m-1} - v_{i,p-1}) + R_i(v_{i,m-1} - v_{i,p-1}) \\ &+ F_i(v_{i,m-1} - v_{i,p-1})]\} \\ &\leq h_i(\mu_i + s_i + w_i) \|v_{i,m-1} - v_{i,p-1}\| \leq \rho_i \|v_{im-1} - v_{ip-1}\| \end{aligned} \tag{29}$$

For $m = p + 1$, we have

$$\|v_{i,p+1} - v_{i,p}\| \leq \rho_i \|v_{i,p} - v_{i,p-1}\| \leq \rho_i^2 \|v_{i,p-1} - v_{i,p-2}\| \leq \cdots \leq \rho_i^q \|v_{i,1} - v_{i,0}\| \tag{30}$$

Therefore, for $m, p \in N$, such that $m > p$ the following hold

$$\begin{aligned} \|v_{i,m} - v_{i,p}\| &\leq |v_{i,m} - v_{i,m-1}\| + \|v_{i,m-1} - v_{i,m-2}\| + \cdots + |v_{i,p+1} - v_{i,p}\| \\ &\leq \rho_i^{m-1} \|v_{i,1} - v_{i,0}\| + \rho_i^{m-2} \|v_{i,1} - v_{i,0}\| + \cdots + \rho_i^p \|v_{i,1} - v_{i,0}\| \\ &= \rho_i^p \|v_{i1} - v_{i,0}\| \sum_{k=0}^{m-n-1} \rho_i^k \leq \rho_i^p \|v_{i,1} - v_{i,0}\| \sum_{k=0}^{\infty} \rho_i^k \\ &= \rho_i^p \|v_{i1} - v_{i,0}\| (\frac{1}{1-\rho_i}). \end{aligned} \tag{31}$$

Since $\rho_i < 1$, for arbitrary ε_i we can have some large $\eta_i \in N$ such that $\rho_i^{\eta_i} < \frac{\varepsilon_i(1-\rho_i)}{\|v_{i,1} - v_{i0}\|}$
Hence by choosing $p, m > N$ we obtain

$$\|v_{i,m} - v_{i,p}\| \leq \rho_i^p \|v_{i,1} - v_{i,0}\| (\frac{1}{1-\rho_i}) < \frac{\varepsilon_i(1-\rho_i)}{\|v_{i,1} - v_{i,0}\|} \|v_{i,1} - v_{i,0}\| (\frac{1}{1-\rho_i}) = \varepsilon_i \tag{32}$$

This shows that $\{v_{i,m}\}$ are Cauchy sequences in $C[D]$. Hence the sequences converge. This completes the proof

Theorem 4.2. *Suppose $\{v_{i,m}\}$ are Cauchy sequences in $[C(D); \|.\|]$, and let $\delta = (\mu_i + w_i)d$ for some $\delta \in (0,1)$. Then $\{v_{i,m}\}$ converge uniformly to the unique solution $v_i(t)$ of equations (8) and (9).*

Proof

From Theorem 4.1 and Eq. (31) we have

$$\|v_{i,m} - v_i(t)\| = |v_{i,m} - v_i(t)| = |v_i - v_{i,m}| \leq \frac{\rho_i^m}{1-\rho_i}\|v_{i,1} - v_{i,0}\| \tag{33}$$

Now, since $\rho_i < 1$, we obtain

$$\lim_{m\to\infty} \|v_{i,m} - v_i(t)\| \leq \lim_{m\to\infty} \frac{\rho_i^m}{1-\rho_i}\|v_{i,1} - v_{i,0}\| = 0 \tag{34}$$

This shows $v_{i,m}$ converge uniformly to solutions $v_i(t)$ on the interval D. To show the uniqueness of these solutions, suppose that $v_i(t)$ and $u_i(t)$ are two distinct solutions of equations (8) and (9). From Eq. (11) solution of equations (8) and (9) can be obtained as

$$v_i(t) = L^{-1}[g_i(t) - R_i(v_\lambda) - F_i(v_\lambda)] \tag{35}$$

where L^{-1} is an inverse operator defined by $\int_0^t (.)dt$. Since $v_i(t)$ and $u_i(t)$ are distinct solutions of Equations (8) and (9), so from Eq. (35) we obtain the following equation

$$\begin{aligned}\|v_i - u_i\| &= |-\int_0^t [R_i(v_\lambda - u_\lambda) + F_i(v_\lambda - u_\lambda)]dt| \\ &\leq \int_0^t [|R_i(v_i - u_i)| + |F_i(v_i) - F_i(u_i)|]dt \\ &\leq (w_i|v_i - u_i| + \mu_i|v_i - u_i|)d \leq \delta|v_i - u_i|\end{aligned} \tag{36}$$

From Eq. (36) we obtain $(1-\delta)|v_i - u_i| \leq 0$. But $\delta \in (0,1)$. Therefore $|v_i - u_i| \leq 0$ implies that $v_i = u_i$.

5 Application

In this section the proposed method shall be applied to solve some problems of nonlinear system of DDEs.

Example 5.1. *[7] Consider the first order nonlinear system of DDEs*

$$\begin{aligned} v_1'(t) &= v_2^2(t) \\ v_2'(t) &= \frac{1}{2}v_1(\frac{t}{2}) \\ v_3'(t) &= e^{\frac{5t}{2}}v_2(t) + 9e^{2t}v_3(\frac{t}{3}) \end{aligned} \tag{37}$$

with initial condition

$$v_1(0) = 1, \ \ v_2(0) = 1, \ \ v_3(0) = 0 \tag{38}$$

Eq. (37), together with the initial condition Eq. (38), has the exact solution

$$v_1(t) = e^t, \ \ v_2(t) = e^{\frac{t}{2}}, \ \ v_3(t) = te^{3t}$$

Take the Natural Transform on both sides of Eq. (37) and simplify further using Eq. (38) to get

$$\begin{aligned} \mathbf{N}^+[v_1(t)] - \frac{1}{s} - \frac{u}{s}\mathbf{N}^+[v_2^2(t)] &= 0 \\ \mathbf{N}^+[v_2(t)] - \frac{1}{s} - \frac{u}{s}\mathbf{N}^+[v_1(\frac{t}{2})] &= 0 \\ \mathbf{N}^+[v_3(t)] - \frac{u}{s}\mathbf{N}^+[e^{\frac{5t}{2}}v_2(t) + 9e^{2t}v_3(\frac{t}{3})] &= 0 \end{aligned} \tag{39}$$

From Eq.(39) define a nonlinear operator

$$\begin{aligned} N[\phi_1(t;q)] &= \mathbf{N}^+[\phi_1(t;q)] - \frac{1}{s} - \frac{u}{s}\mathbf{N}^+[\phi_2^2(t)] \\ N[\phi_2(t;q)] &= \mathbf{N}^+[\phi_2(t)] - \frac{1}{s} - \frac{1}{2}\frac{u}{s}\mathbf{N}^+[\phi_1(\frac{t}{2})] \\ N[\phi_3(t)] &= \mathbf{N}^+[\phi_3(t;q)] - \frac{u}{s}\mathbf{N}^+[e^{\frac{5t}{2}}\phi_2(t;q) + 9e^{2t}\phi_3(\frac{t}{3};q)] \end{aligned} \tag{40}$$

Now, using Eq. (23), the recursive relation of Example 5.1 can be obtained as

$$\begin{aligned} v_{1,m}(t) &= (\chi_m + h_1)v_{1,m-1}(t) - h_1(1-\chi_m)\mathbf{N}^-[\frac{1}{s}] - h_1\mathbf{N}^-\{\frac{u}{s}\mathbf{N}^+[H_{1,m-1}] \\ v_{2,m}(t) &= (\chi_m + h_2)v_{2,m-1}(t) - h_2(1-\chi_m)\mathbf{N}^-[\frac{1}{s}] - \frac{1}{2}h_2\mathbf{N}^-\{\frac{u}{s}\mathbf{N}^+[R_2(v_{1,m-1})(t)] \\ v_{3,m}(t) &= (\chi_m + h_3)v_{3,m-1}(t) - h_3\mathbf{N}^-\{\frac{u}{s}\mathbf{N}^+[R_3(v_{2,m-1}(t), v_{3,m-1}(t))], \ m \geq 1 \end{aligned} \tag{41}$$

From Eq. (38) the initial approximations can be chosen as

$$v_{1,0}(t) = 1, \ \ v_{2,0}(t) = 1, \ \ v_{3,0}(t) = t \tag{42}$$

From Eq.(41) we obtain the following

$$\begin{aligned}
v_{1,1}(t) &= -h_1 t \quad v_{2,1}(t) = -\frac{1}{2}h_2 t, \quad v_{3,1}(t) = -h_3[\frac{11}{4}t^2 + \frac{73}{24}t^3 + \frac{413}{192}t^4] \\
v_{1,2}(t) &= -(h_1^2 + h_1)t + \frac{1}{2}h_1 h_2 t^2 \quad v_{2,2}(t) = -\frac{1}{2}(h_2^2 + h_2)t + \frac{1}{8}h_1 h_2 t^2 \\
v_{3,2}(t) &= \frac{-1}{4}(h_3^2 - h_2 h_3 + h_3)t^2 - \frac{1}{24}(41h_3^2 + 73h_3)t^3 \\
&\quad - \frac{1}{576}(571h_3^2 + 225h_2 h_3 + 1239h_3)t^4 \\
v_{1,3}(t) &= -(h_1^3 + 2h_1^2 + h_1)t + \frac{1}{2!}(2h_1 h_2 + h_1 h_2^2 + h_1^2 h_2)t^2 - \frac{1}{3!}h_1^2 h_2 t^3 \\
v_{2,3}(t) &= \frac{-1}{2}(h_2^3 + 2h_2^2 + h_2)t + \frac{1}{8}(2h_1 h_2 + h_1 h_2^2 + h_1^2 h_2)t^2 - \frac{1}{48}h_1 h^2 t^3 \\
v_{3,3}(t) &= \frac{-1}{4}(h_3^3 + 2h_3^2 - h_2^2 h_3 - h_2 h_3^2 - 2h_2 h_3 + h_3)t^2 \\
&\quad - \frac{1}{24}(12h_2 h_3^2 + 82h_3^2 + 73h_3)t^3 - \frac{1}{576}(906h_1 h_2 h_3 + 1235h_2 h_3 + 1239h_3)t^4
\end{aligned} \tag{43}$$

etc. Hence, the solution is given by

$$v_1(t) = \sum_{m=0}^{\infty} v_{1,m}(t), \quad v_2(t) = \sum_{m=0}^{\infty} v_{2,m}(t), \quad v_3(t) = \sum_{m=0}^{\infty} v_{3,m}(t) \tag{44}$$

The series solutions Eq. (43) contains the auxiliary parameters $h_i (i = 1, 2, 3)$. To obtain the values of h_i for the convergence of these series, the h-curves for the 3rd order approximation solutions of equations (37) and (38) is plotted in Fig. 1. Observe from the h-curves the range of values of h_i for $v_1(t), v_2(t)$ and $v_3(t)$ are in between $-1.3 \leq h_1 \leq -0.8, -1.5 \leq h_2 \leq -0.5$ and $-1.7 \leq h_3 \leq -0.3$ respectively. Thus, the proper values are found to be at $h_i = -1$, and series solution of equations (37) and (38) is given by

$$\begin{aligned}
v_1(t) &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \dots \\
v_2(t) &= 1 + \frac{1}{2}t^2 + \frac{1}{8}t^2 + \frac{1}{48}t^3 + \frac{1}{384}t^4 + \dots \\
v_3(t) &= t + 3t^2 + \frac{9}{2}t^3 + \frac{27}{6}t^4 + \dots
\end{aligned} \tag{45}$$

As $m \to \infty$ these series converge to

$$v_1(t) = e^t \quad v_2(t) = e^{\frac{t}{2}} \quad v_3(t) = te^{3t}$$

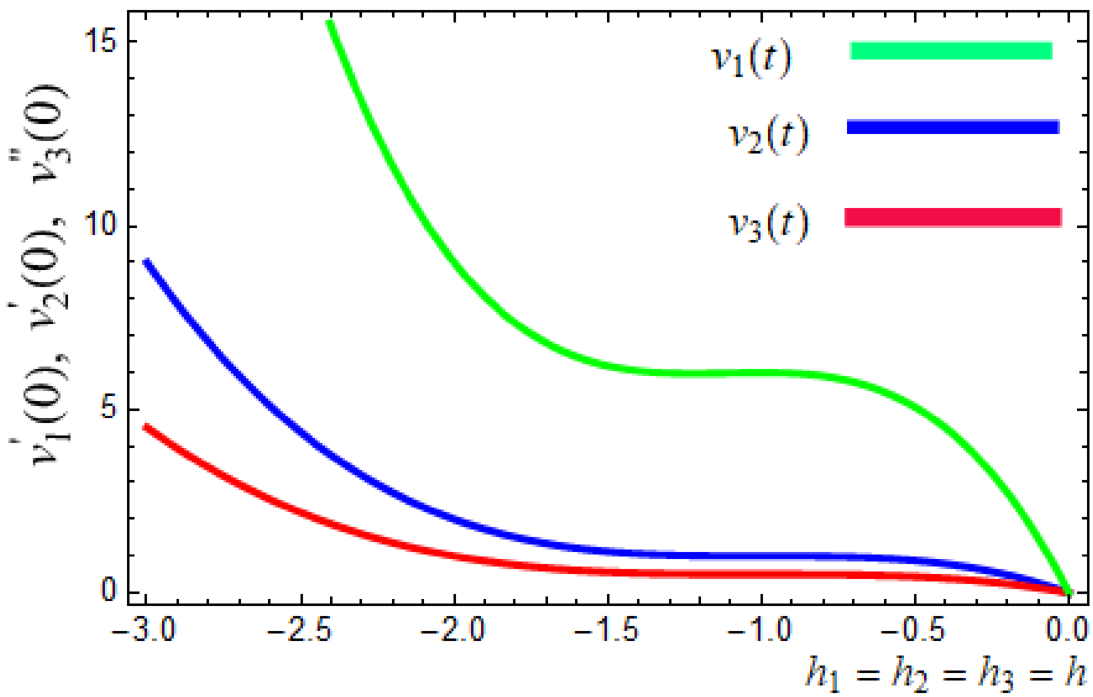

Figure 1: The $h-$curve for the third order approximation of Example 5.1

Adomian decomposition method [7] was applied to solve this example but only approximate solution was able to obtain. Likewise, Rebenda el at [8] obtained solution of the same problem from modern three iterations. While using our derived algorithm a good approximate solution of this problem was obtained from only three iterations.

Example 5.2. *[10] Consider the first order nonlinear system of DDEs*

$$\begin{aligned} v_1^{'}(t) &= 2v_2(\frac{t}{2}) + v_3(t) - t\cos(\frac{t}{2}) \\ v_2^{'}(t) &= -2v_3^2(t)(\frac{t}{2}) - t\sin t + 1 \\ v_3^{'}(t) &= -t\cos t - v_1(t) + v_2(t) \end{aligned} \tag{46}$$

with initial condition

$$v_1(0) = -1,\ \ v_2(0) = 0,\ \ v_3(0) = 0 \tag{47}$$

Take the Natural Transform on both sides of Eq. (46) and simplify further using Eq. (47) to get

$$\begin{aligned}
&\mathbf{N}^+[v_1(t)] + \frac{1}{s} - \frac{u}{s}\mathbf{N}^+[2v_2(\frac{t}{2}) + v_3(t) - t\cos(\frac{t}{2})] = 0\\
&\mathbf{N}^+[v_2(t)] - \frac{u}{s^2} + \frac{u}{s}\mathbf{N}^+[2v_1^2(\frac{t}{2}) + t\sin t] = 0\\
&\mathbf{N}^+[v_3(t)] - \frac{u}{s}\mathbf{N}^+[v_2(t) - v_1(t) - t\cos t] = 0
\end{aligned} \tag{48}$$

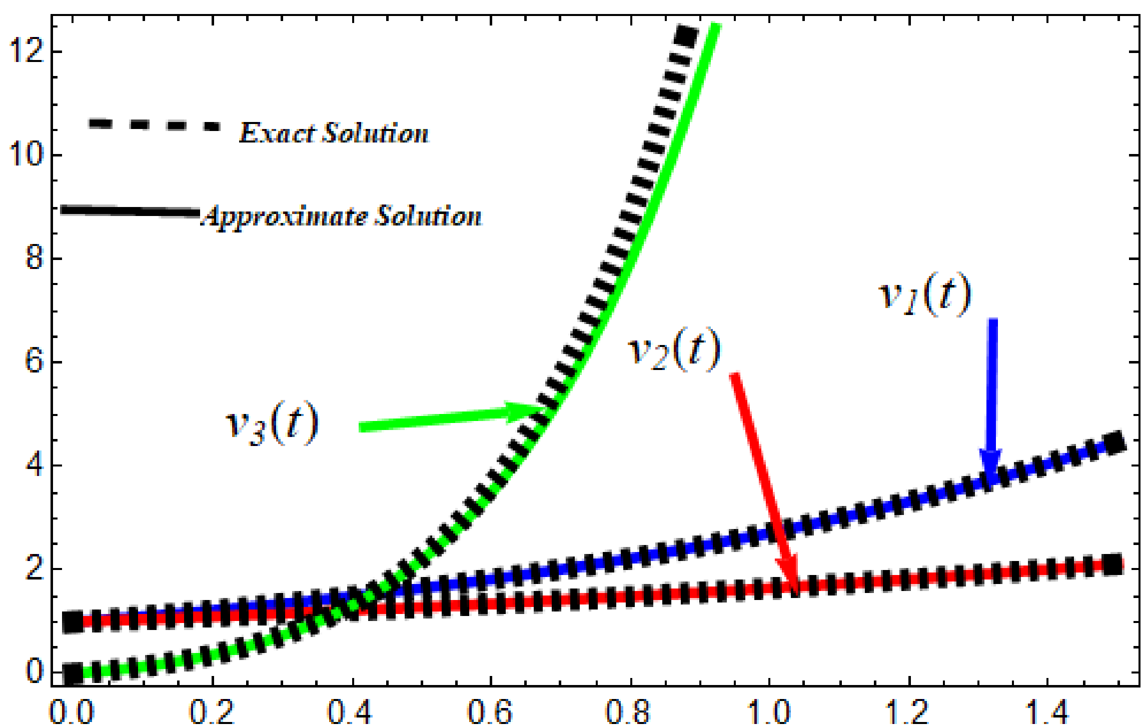

Figure 2: Exact Solution and Approximate Solution of Example 5.1

From Eq.(48) define a non-linear operator

$$\begin{aligned} N[\phi_1(t;q)] &= \mathbf{N}^+[\phi_1(t;q)] + \frac{1}{s} - \frac{u}{s}\mathbf{N}^+[2\phi_2(\frac{t}{2};q) + \phi_3(t;q) - t\cos(\frac{t}{2})] \\ N[\phi_2(t;q)] &= \mathbf{N}^+[\phi_2(t)] - \frac{u}{s^2} + \frac{u}{s}\mathbf{N}^+[2\phi_3^2(\frac{t}{2};q) + t\sin t] \\ N[\phi_3(t)] &= \mathbf{N}^+[\phi_3(t;q)] - \frac{u}{s}\mathbf{N}^+[\phi_2(t;q) - \phi_1(t;q) - t\cos t] \end{aligned} \tag{49}$$

Now, using Eq. (23), the recursive relation of Example 5.2 can be obtained as

$$\begin{aligned} v_{1,m} &= (\chi_m + h_1)v_{1,m-1} - h_1(1-\chi_m)\mathbf{N}^-[\frac{1}{s}] - h_1\mathbf{N}^-\{\frac{u}{s}\mathbf{N}^+[R_1(v_{2,m-1}, v_{3,m-1}) - g_1] \\ v_{2,m} &= (\chi_m + h_2)v_{2,m-1} - h_2(1-\chi_m)\mathbf{N}^-[\frac{u}{s^2}] + h_2\mathbf{N}^-\{\frac{u}{s}\mathbf{N}^+[H_{2,m-1}(v_{2,1}) + g_2(t)] \\ v_{3,m} &= (\chi_m + h_3)v_{3,m-1} - h_3\mathbf{N}^-\{\frac{u}{s}\mathbf{N}^+[R_3(v_{1,m-1}, v_{2,m-1}) - g_3], \;\; m \geq 1 \end{aligned} \tag{50}$$

From the given initial condition the initial approximations can be chosen as

$$v_{1,0}(t) = -1, \;\; v_{2,0}(t) = t, \;\; v_{3,0}(t) = t \tag{51}$$

From Eq. (50) we obtain the following

$$\begin{aligned} v_{1,1}(t) &= -h_1(\frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{2304}t^6), v_{2,1} = h_2(\frac{1}{2}t^3 - \frac{1}{24}t^5 + \frac{1}{720}t^7) \\ v_{3,1}(t) &= h_3(-\frac{1}{8}t^4 + \frac{1}{144}t^6) \\ v_{1,2}(t) &= -\frac{1}{2}(h_1^2 + h_1)t^2 1\frac{1}{24}h_1(h_1^2 + 2h_1 + h_1h_2)t^4 - \frac{1}{40}(h_1h_3 + h_2)t^5 \\ v_{2,2}(t) &= \frac{1}{2}(h_2^2 + h_2)t^3 - \frac{1}{24}h_1(h_2^2 + h_2)t^5 + \frac{1}{720}t^7 + \dots \\ v_{3,2}(t) &= -\frac{-1}{6}h_1h_3t^3 - \frac{1}{8}(h_3^2 + h_3 + h_2h_3)t^4 + \frac{1}{120}h_1h_3t^5 \\ &+ \frac{1}{720}(5h_3^2 + 5h_3 + 4h_2h_3)t^6 + .. \end{aligned} \tag{52}$$

Continuing this process the subsequent terms can be obtained for $m \geq 3$. By putting the optimal of values of $h_i = -1$ obtained from Fiq. 3, the series

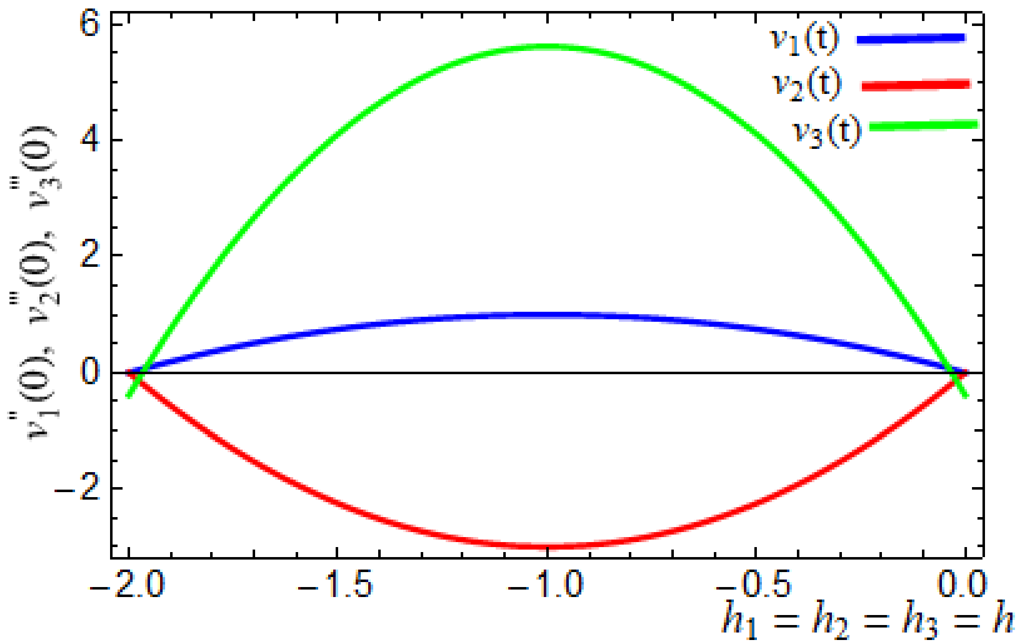

Figure 3: The $h-$curve for the second order approximation of Example 5.2

solution Eq. (52) is given by

$$\begin{aligned} v_1(t) &= \sum_{m=0}^{\infty} v_{1m}(t) = -1 + \frac{1}{2!}t^2 - \frac{1}{4!}t^4 + \frac{1}{6!}t^6 + \dots \\ v_2(t) &= \sum_{m=0}^{\infty} v_{2m}(t) = t - \frac{1}{2!}t^3 + \frac{1}{4!}t^5 - \frac{1}{6!}t^7 + \dots \\ v_3(t) &= \sum_{m=0}^{\infty} v_m(t) = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + \dots \end{aligned} \tag{53}$$

Which converges to the series expansion of the closed form solution
$v_1(t) = -\cos t, \quad v_2(t) = t\cos t, \quad v_3(t) = \sin t$

Two iterations of our derived algorithms provide a series of solution which converges to a good approximate solution of Example 5.2 while the same problem was solve using Efficient Homotopy Analysis Method to System of DDEs [10], but no good approximate convergent series was obtained.

Example 5.3. *[8] Consider the following second order nonlinear system of*

DDEs

$$\begin{aligned} v_1^{''}(t) - v_1^{'}(t) - 2v_1(\frac{t}{2})v_2(t-1) - 4 &= -t^4 + 2t^3 - t^2 - 4t \\ v^{''}(t) - v_2^{'}(t-2) - v_2(\frac{t}{3})v_1(\frac{t}{2}) - 6 &= -\frac{t^4}{18} - 2t \end{aligned} \tag{54}$$

with initial condition

$$v_1(0) = v_1^{'}(0) = 0, \quad v_2(0) = v_2^{'}(0) = 0 \tag{55}$$

Take the Natural Transform on both sides of Eq.(54) and simplify further using Eq. (55) to get

$$\begin{aligned} &\mathbf{N}^+[v_1(t)] - \frac{4u^2}{s^2} - \frac{u^2}{s^2}\mathbf{N}^+[v_1^{'} + 2v_1(\frac{t}{2})v_2(t-1) - (t^4 - 2t^3 + t^2 + 4t)] = 0 \\ &\mathbf{N}^+[v_2(t)] - \frac{6u^2}{s^2} - \frac{u^2}{s^2}\mathbf{N}^+[v_2^{'}(t-2) + v_2(\frac{t}{3})v_1(\frac{t}{2}) - (\frac{t^4}{18} - 2t)] = 0 \end{aligned} \tag{56}$$

Now define a nonlinear operator based on Eq. (56)

$$\begin{aligned} N[\phi_1(t;q)] &= \mathbf{N}^+[\phi_1(t;q)] - \frac{4u^2}{s^2} - \frac{u^2}{s^2}\mathbf{N}^+[\phi_1^{'}(t;q) \\ &\quad + 2\phi_1(\frac{t}{2}, q)\phi_2((t-1);q) - (t^4 - 2t^3 + t^2 + 4t)] \\ N[\phi_2(t;q)] &= \mathbf{N}^+[\phi_2(t;q)] - \frac{6u^2}{s^2} - \frac{u^2}{s^2}\mathbf{N}^+[\phi_2^{'}((t-2);q) \\ &\quad + \phi_1(\frac{t}{3};q)\phi_1(\frac{t}{2};q) - (\frac{t^4}{18} - 2t)] \end{aligned} \tag{57}$$

Now using Eq. (23) we obtained the recursive relation of equations (54) and (55) as follows

$$\begin{aligned} v_{1,m}(t) &= (\chi_m + h_1)v_{1,m-1}(t) - h_1(1-\chi_m)\mathbf{N}^-[\frac{4u^2}{s^2}] \\ &\quad - h_1\mathbf{N}^-\{\frac{u^2}{s^2}\mathbf{N}^+[R_1(v_{1,m-1}(t)) + H_{1,m-1}(v_{1,i}(t), v_{2,i}(t)) + g_1(t)]\} \\ v_{2,m}(t) &= (\chi_m + h_2)v_{2,m-1}(t) - h_2(1-\chi_m)\mathbf{N}^-[\frac{6u^2}{s^3}] \\ &\quad - h_2\mathbf{N}^-\{\frac{u^2}{s^2}\mathbf{N}^+[R_2(v_{2,m-1}(t)) + H_{2,m-1}(v_{1,i}(t), v_{2,i}(t)) + g_2(t)]\}, m \geq 1 \end{aligned} \tag{58}$$

From the given initial conditions the initial approximations can be chosen as

$$v_{1,0}(t) = 2t^2, \quad v_{2,0}(t) = t^2 \tag{59}$$

From Eq.(58) we obtain the following

$$\begin{aligned}
v_{1,1}(t) &= h_1 v_{1,0} - 2h_1 t^2 - h_1 \mathbf{N}^{-}\{\frac{u^2}{s^2}\mathbf{N}^{+}[R_1(v_{1,0}) + H_{1,0}(v_{1,0}, v_{2,0}) + g_1]\} \\
&= 2h_1 t^2 - 2h_1 t^2 - h_1 \mathbf{N}^{-}\{\frac{u^2}{s^2}\mathbf{N}^{+}[0]\} = 0 \\
v_{2,1}(t) &= h_2 v_{2,0} - 2h_2 t^2 - h_2 \mathbf{N}^{-}\{\frac{u^2}{s^2}\mathbf{N}^{+}[R_2(v_{2,0}) + H_{2,0}(v_{1,0}, v_{2,0}) + g_2]\} \\
&= h_2 t^2 - 3h_2 t^2 - h_2 \mathbf{N}^{-}\{\frac{u^2}{s^2}\mathbf{N}^{+}[-4]\} = h_2 t^2 - 3_2 t^2 + 2h_2 t^2 = 0
\end{aligned} \tag{60}$$

It is now obvious that for $m \geq 2$, we have $v_{i,m}(t) = 0$. Therefore, the solution of Equations (54) and (55) can be obtained as

$$v_1(t) = 2t^2 \quad v_2(t) = t^2$$

which give the exact solution of equations (54) and (55). Therefore, using only one iteration of our derived algorithm we successfully obtained an approximate solution which converged to exact solution of equations (54) and (55). Rebenda el at [8] obtained the same solution of the problem using a Taylor series approach from exactly three iterations.

6 Conclusion

In this paper, a combination of Homotopy Analysis Method (HAM) and Natural Transform Method (NTM) provides a robust analytical approach suitable for solving nonlinear systems of Delay Differential Equations (DDEs). The developed algorithm gives a solution in series form which converges to an exact or approximate solution with few computational numbers of terms. The main advantage of this work is that a new generating function Eq.(23) is constructed which adjusts the He's polynomial to ease the difficulties of computing nonlinear terms for the system of DDEs. Thus, unlike some existing methods the presented technique obtained solution to various form of nonlinear delay differential equations without linearization, perturbation and restrictive assumptions. The method also adjusts the interval of convergence for the series solution, avoids round-off of errors and reduces the computational size compared to reference methods.

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