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Hybrid generalized bi-ideals in semigroups

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Abstract

In this paper, the notion of hybrid generalized bi-ideal of a semigroup is introduced with some of their important properties investigated. We obtain some characterizations of regular and left quasiregular semigroups in terms of hybrid generalized bi-ideals.

1 Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. For any subsets A and B of S, the multiplication of A and B is defined as $AB = \{ab \ : a \in A$ and $b \in B\}$. A non-empty subset A of S is called a subsemigroup of S if $A^2 \subseteq S$. A subsemigroup X of S is called a left (resp., right) ideal of S if $SX \subseteq X(resp., XS \subseteq X)$. If X is both a left and right ideal of S, then X is called a two-sided ideal or ideal of S. It can easily be verified that for any $a \in S,\ L(a) = \{a, Sa\}$ (resp.,$R(a) = \{a, aS\}$) is a left (resp., right) ideal of S

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generated by a in S. A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$. In [6], S. Lajos defined the concept of generalized bi-ideal and characterized semigoups, which are regular, and both intraregular and left quasi regular in terms of generalized bi-ideals. Following [6], a non-empty subset I of S is said to be a generalized bi-ideal of S if $ISI \subseteq I$. A semigroup S is called regular if for each $a \in S$, there exists an element x in S such that $a = axa$. A semigroup S is called intra-regular if for each $a \in S$, there exist $x, y \in S$ such that $a = xa^2y$.

In his classic paper [10], Zadeh introduced the notion of a fuzzy subset A of a set X as a mapping from X into $[0, 1]$. Rosenfeld [9] and Kuroki [5] applied this concept in group theory and semigroup theory. Since then, several authors have been pursued the study of fuzzy algebraic structure in many directions such as groups, rings, modules, vector spaces and so on(See[4], [5] and [8]). As a new mathematical tool for dealing with uncertainties, Molodtsov [7] introduced the soft set theory. In the past few years, the fundamentals of soft set theory have been studied by various researchers. In 2017, S. Anis et al.[1] introduced the notions of hybrid sub-semigroups and hybrid left(resp., right) ideals in semigroups and obtained several properties. The notion of hybrid bi-ideals in semigroup was introduced in [3] and investigated some of their important properties, and provided various equivalent conditions for a semigroup to be regular.

In this paper, we introduce and discuss some properties of hybrid generalized bi-ideal of a semigroup, which is an extension of the concept of of a hybrid bi-ideal, and characterize regular semigroups, and both intra-regular and left quasiregular semigroups in terms of hybrid generalized bi-ideals.

2 Hybrid structures in semigroup

In this section, we present some elementary definitions of hybrid structures in semigroup that we will use later in this paper.

Definition 2.1. *[1]Let I be the unit interval and $\mathscr{P}(U)$ denote the power set of an initial universal set U. A hybrid structure in S over U is defined to be a mapping $\tilde{f}_\lambda := (\tilde{f}, \lambda) : S \to \mathscr{P}(U) \times I,\ x \mapsto (\tilde{f}(x), \lambda(x))$, where $\tilde{f} : S \to \mathscr{P}(U)$ and $\lambda : S \to I$ are mappings.*

Let us denote by $H(S)$ the set of all hybrid structures in S over U. We define an order $\ll$ in $H(S)$ as follows : For all $\tilde{f}_\lambda, \tilde{g}_\gamma \in H(S)$, $\tilde{f}_\lambda \ll \tilde{g}_\gamma$ if *and only if* $\tilde{f} \tilde{\subseteq} \tilde{g}, \lambda \succeq \gamma$, *where* $\tilde{f} \tilde{\subseteq} \tilde{g}$ *means that* $\tilde{f}(x) \subseteq \tilde{g}(x)$ *and* $\lambda \succeq \gamma$ *means that* $\lambda(x) \geq \gamma(x)$ *for all* $x \in S$. *For any* $x, y \in S$, $\tilde{f}_\lambda(x) = \tilde{g}_\gamma(y)$ *if and only if* $\tilde{f}_\lambda(x) \ll \tilde{g}_\gamma(y)$ *and* $\tilde{g}_\gamma(y) \ll \tilde{f}_\lambda(x)$, *where* $\tilde{f}_\lambda(x) \ll \tilde{g}_\gamma(y)$ *means that* $\tilde{f}(x) \subseteq \tilde{g}(y)$ *and* $\lambda(x) \geq \gamma(y)$. *Also* $\tilde{f}_\lambda = \tilde{g}_\gamma$ *if and only if* $\tilde{f}_\lambda \ll \tilde{g}_\gamma$ *and* $\tilde{g}_\gamma \ll \tilde{f}_\lambda$. *Note that* $(H(S), \ll)$ *is a poset.*

Definition 2.2. *[1] Let* A *be a non-empty subset of* S. *Then the characteristic hybrid structure in* S *over* U *is denoted by* $\chi_A(\tilde{f}_\lambda) = (\chi_A(\tilde{f}), \chi_A(\lambda))$, *where* $\chi_A(\tilde{f}) : S \to \mathscr{P}(U), x \mapsto \begin{cases} U & if \;\; x \in A \\ \phi & otherwise \end{cases}$
and $\chi_A(\lambda) : S \to I, x \mapsto \begin{cases} 0 & if x \in A \\ 1 & otherwise \end{cases}$

Definition 2.3. *[1] For any hybrid structures* $\tilde{f}_\lambda$ *and* $\tilde{g}_\gamma$ *in* S *over* U, *the hybrid product of* $\tilde{f}_\lambda$ *and* $\tilde{g}_\gamma$ *in* S *is defined to be a hybrid structure* $\tilde{f}_\lambda \odot \tilde{g}_\gamma = (\tilde{f} \tilde{\circ} \tilde{g}, \lambda \tilde{\circ} \gamma)$ *in* S *over* U, *where*

$$(\tilde{f} \tilde{\circ} \tilde{g})(x) = \begin{cases} \bigcup_{x=yz} \{\tilde{f}(y) \cap \tilde{g}(z)\} & if \; \exists \; y, z \in S \; such \; that \; x{=}yz \\ \phi & otherwise \end{cases}$$

and

$$(\lambda \tilde{\circ} \gamma)(x) = \begin{cases} \bigwedge_{x=yz} \bigvee \{\lambda(y), \gamma(z)\} & if \; \exists \; y, z \in S \; such \; that \; x = yz \\ 1 & otherwise \end{cases}$$

for all $x \in S$.

Definition 2.4. *[1] Let* $\tilde{f}_\lambda$ *and* $\tilde{g}_\gamma$ *be hybrid structures in* S *over* U. *Then the hybrid intersection of* $\tilde{f}_\lambda$ *and* $\tilde{g}_\gamma$ *is denoted by* $\tilde{f}_\lambda \Cap \tilde{g}_\gamma$ *and is defined to be a hybrid structure* $\tilde{f}_\lambda \Cap \tilde{g}_\gamma : S \to \mathscr{P}(U) \times I, x \mapsto ((\tilde{f} \tilde{\cap} \tilde{g})(x), (\lambda \vee \gamma)(x))$, *where* $\tilde{f} \tilde{\cap} \tilde{g} : S \longrightarrow \mathscr{P}(U), x \mapsto \tilde{f}(x) \cap \tilde{g}(x)$ *and* $\lambda \vee \gamma : S \to I, x \mapsto \bigvee \{\lambda(x), \gamma(x)\}$.

Definition 2.5. *[1] Let S be a semigroup. A hybrid structure $\tilde{f}_\lambda$ in S is called a hybrid subsemigroup of S over U if $\tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y)$ and $\lambda(xy) \leq \bigvee\{\lambda(x), \lambda(y)\}$ for all $x, y \in S$.*

Definition 2.6. *[1] Let S be a semigroup. A hybrid structure $\tilde{f}_\lambda$ in S over U is called a hybrid left (resp., right) ideal of S over U if*

(i) $\tilde{f}(xy) \supseteq \tilde{f}(y)$ (resp., $\tilde{f}(xy) \supseteq \tilde{f}(x)$),

(ii) $\lambda(xy) \leq \lambda(y)(res., \lambda(xy) \leq \lambda(x))$ for all $x, y \in S$.

$\tilde{f}_\lambda$ is called a hybrid ideal of S if it is both a hybrid left and a hybrid right ideal of S over U.

Definition 2.7. *[3] Let S be a semigroup. A hybrid subsemigroup $\tilde{f}_\lambda$ in S over U is called a hybrid bi-ideal of S over U if for all $x, y, z \in S$,*

(i) $\tilde{f}(xyz) \supseteq \tilde{f}(x) \cap \tilde{f}(z)$,

(ii) $\lambda(xyz) \leq \bigvee\{\lambda(x), \lambda(z)\}$.

Clearly, every hybrid left and hybrid right ideals of S over U are hybrid bi-deals of S over U, however, hybrid bi-deals of S over U need not be either hybrid left or hybrid right ideals of S over U as can be seen by the following example.

Example 2.1. *[3] Let $S = \{0, a, b, c\}$ be a semigroup with the following Cayley table:*

.	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	c

Let $\tilde{f}_\lambda$ be a hybrid structure in S over $U = [0, 1]$ defined by $f(0) = [0, 0.6]; f(a) = [0, 0.5]; f(b) = [0, 0.4]; f(c) = [0, 0.2]$ and λ be any constant mapping form S to I. Then $\tilde{f}_\lambda$ is a hybrid bi-ideal of S over U, which is neither hybrid left nor hybrid right ideal of S over U as $f(ca) \not\supseteq f(a)$ and $f(ac) \not\supseteq f(a)$. □

As an extension of the concept of a hybrid bi-ideal of a semigroup, we have the following.

Definition 2.8. *Let S be a semigroup. A hybrid structure $\tilde{f}_\lambda$ in S over U is called a hybrid generalized bi-ideal of S over U if for all $x, y, z \in S$,*
(i) $\tilde{f}(xyz) \supseteq \tilde{f}(x) \cap \tilde{f}(z)$,
(ii) $\lambda(xyz) \leq \bigvee\{\lambda(x), \lambda(z)\}$.

It is clear that every hybrid bi-ideal of S is a hybrid generalized bi-ideal of S but converse is need not be true shown by the following example.

Example 2.2. *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:*

$.$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\tilde{f}_\lambda$ be a hybrid structure in S over $U = [0, 1]$ defined by $f(a) = [0, 0.5]$; $f(b) = \{0\}$; $f(c) = [0, 0.2]$; $f(d) = \{0\}$ and λ be any constant mapping form S to I. Then it can be checked easily that $\tilde{f}_\lambda$ is a hybrid generalized bi-ideal of S over U, but not a hybrid bi-ideal of S. □

3 Hybrid generalized bi-ideals

Theorem 3.1. *Let A be a non-empty subset of S. Then A is a generalized bi-ideal of S if and only if the characteristic hybrid structure $\chi_A(\tilde{f}_\lambda)$ is a hybrid generalized bi-ideal of S.*

Proof: Assume that A is a bi-ideal of S. Let $x, y, z \in S$. If $x \in A$ and $z \in A$, then $xyz \in A$, so $\tilde{f}(xyz) = U = \tilde{f}(x) \cap \tilde{f}(z)$ and $\lambda(xyz) = 0 = \bigvee\{\lambda(x), \lambda(z)\}$. If $x \notin A$ or $z \notin A$, then $\tilde{f}(x) \cap \tilde{f}(z) = \phi \subseteq \tilde{f}(xyz)$ and

$\bigvee\{\lambda(x), \lambda(z)\} = 1 \geq \lambda(xyz)$. So $\chi_A(\tilde{f}_\lambda)$ is a hybrid generalized bi-ideal of S.

Conversely, assume that $\chi_A(\tilde{f}_\lambda)$ is a hybrid generalized bi-ideal of S. Let $x, z \in A$ and $y \in S$. Then $\chi_A(\tilde{f})(xyz) \supseteq \chi_A(\tilde{f})(x) \cap \chi_A(\tilde{f})(z) = \phi$ which implies $xyz \in A$. So A is a bi-ideal of S. □

Theorem 3.2. *Let $\tilde{f}_\lambda$ be a hybrid structure of S. Then the following conditions are equivalent:*

(i) $\tilde{f}_\lambda$ is a hybrid generalized bi-ideal of S,

(ii) $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$ for any hybrid structure $\tilde{g}_\mu$ of S.

Proof:$(i) \Rightarrow (ii)$ Assume that $\tilde{f}_\lambda$ is a generalized hybrid bi-ideal of S and $x \in S$. Let $a \in S$. Suppose that there exist $x, y, p, q \in S$ such that $a = xy$ and $x = pq$. Then

$$\begin{aligned}(\tilde{f}\tilde{\circ}\chi_S(\tilde{g})\tilde{\circ}\tilde{f})(a) &= \bigcup_{a=xy}\{(\tilde{f}\tilde{\circ}\chi_S(\tilde{g}))(x) \cap \tilde{f}(y)\}\\ &= \bigcup_{a=xy}\{\bigcup_{x=pq}\{\tilde{f}(p) \cap \chi_S(\tilde{g})(q)\} \cap \tilde{f}(y)\}\\ &= \bigcup_{a=xy}\{\bigcup_{x=pq}\{\tilde{f}(p) \cap U\} \cap \tilde{f}(y)\}\\ &= \bigcup_{a=pqy}\{\tilde{f}(p) \cap \tilde{f}(y)\}\\ &\subseteq \bigcup_{a=pqy} \tilde{f}(pqy) = \tilde{f}(a).\end{aligned}$$

Also $$\begin{aligned}(\lambda\tilde{\circ}\chi_S(\mu)\tilde{\circ}\lambda)(a) &= \bigwedge_{a=xy} \bigvee\{(\lambda\tilde{\circ}\chi_S(\mu))(x), \lambda(y)\}\\ &= \bigwedge_{a=xy} \bigvee\{\bigwedge_{x=pq} \bigvee\{(\lambda(p), \chi_S(\mu)(q)\}, \lambda(y)\}\\ &= \bigwedge_{a=xy} \bigvee\{\bigwedge_{x=pq} \bigvee\{(\lambda(p), 0\}, \lambda(y)\}\\ &= \bigwedge_{a=pqy} \bigvee\{(\lambda(p), \lambda(y)\}\\ &\geq \bigwedge_{a=pqy} \lambda(pqy) = \lambda(a).\end{aligned}$$

Otherwise $a \neq xy$ or $x \neq pq$ for all $x, y, p, q \in S$. Then $(\tilde{f}\tilde{\circ}\chi_S(\tilde{g})\tilde{\circ}\tilde{f})(a) = \phi \subseteq \lambda(a)$. $(\lambda\tilde{\circ}\chi_S(\mu)\tilde{\circ}\lambda)(a) = 1 \geq \lambda(a)$.

Therefore $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds and let $x, y, z \in S$. Then

$$\begin{aligned}\tilde{f}(xyz) &\supseteq (\tilde{f}\tilde{\circ}\chi_S(\mu)\tilde{\circ}\tilde{f})(xyz)\\ &\supseteq (\tilde{f}\tilde{\circ}\chi_S(\mu))(xy) \cap \tilde{f}(z)\\ &\supseteq (\tilde{f}(x) \cap \chi_S(\mu)(y)) \cap \tilde{f}(z)\\ &= (\tilde{f}(x) \cap U) \cap \tilde{f}(z)\\ &= \tilde{f}(x) \cap \tilde{f}(z).\end{aligned}$$

and

$$\begin{aligned}\lambda(xyz) &\leq (\lambda\tilde{\circ}\chi_S(\mu)\tilde{\circ}\lambda)(xyz)\\ &\leq \bigvee\{\lambda\tilde{\circ}\chi_S(\mu))(xy), \lambda(z)\}\\ &\leq \bigvee\{\bigvee\{\lambda(x), \chi_S(\mu)(y)\}, \lambda(z)\}\\ &= \bigvee\{\bigvee\{\lambda(x), 0\}, \lambda(z)\}\\ &= \bigvee\{\lambda(x), \lambda(z)\}.\end{aligned}$$

Therefore $\tilde{f}_\lambda$ is a hybrid generalized bi-ideal of S. □

Lemma 3.1. *Let S be a semigroup. Then the product of two hybrid generalized bi-ideals of S is a hybrid generalized bi-ideal of S.*

Proof: Let $\tilde{f}_\lambda$ and $\tilde{g}_\mu$ be hybrid generalized bi-ideals of S. Then $(\tilde{f}_\lambda \odot \tilde{g}_\mu) \odot (\tilde{f}_\lambda \odot \tilde{g}_\mu) = \tilde{f}_\lambda \odot [\tilde{g}_\mu \odot (\tilde{f}_\lambda \odot \tilde{g}_\mu)] \ll \tilde{f}_\lambda \odot (\tilde{g}_\mu \odot \chi_S(\tilde{g}_\mu) \odot \tilde{g}_\mu) \ll \tilde{f}_\lambda \odot \tilde{g}_\mu$. By Theorem 3.12 of [1], $\tilde{f}_\lambda \odot \tilde{g}_\mu$ is a hybrid subsemigroup of S. Also, for any hybrid structure $\tilde{h}_\gamma$ of S, we have $(\tilde{f}_\lambda \odot \tilde{g}_\mu) \odot \chi_S(\tilde{h}_\gamma) \odot (\tilde{f}_\lambda \odot \tilde{g}_\mu) = \tilde{f}_\lambda \odot [\tilde{g}_\mu \odot (\chi_S(\tilde{h}_\gamma) \odot \tilde{f}_\lambda) \odot \tilde{g}_\mu] \ll \tilde{f}_\lambda \odot (\tilde{g}_\mu \odot \chi_S(\tilde{h}_\gamma) \odot \tilde{g}_\mu) \ll \tilde{f}_\lambda \odot \tilde{g}_\mu$. By Theorem 3.2, $\tilde{f}_\lambda \odot \tilde{g}_\mu$ is hybrid bi-ideal of S. Therefore $\tilde{f}_\lambda \odot \tilde{g}_\mu$ is a hybrid bi-ideal of S. □

Theorem 3.3. *Let S be a semigroup. If S is regular, then every hybrid generalized bi-ideal $\tilde{f}_\lambda$ of S is hybrid bi-ideal of S.*

Proof: Let $\tilde{f}_\lambda$ be a hybrid generalized bi-ideal of S, and $a, b \in S$. Then there exists $x \in S$ such that $b = bxb$. Now $\tilde{f}(ab) = \tilde{f}(a(bxb)) = \tilde{f}(a(bx)b) \supseteq \tilde{f}(a) \cap \tilde{f}(b)$ and $\lambda(ab) = \lambda(a(bxb)) = \lambda(a(bx)b) \leq \lambda(a) \vee \lambda(b)$. So $\tilde{f}_\lambda$ is a hybrid subsemigroup of S. □

Theorem 3.4. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda = \tilde{f}_\lambda$ for any hybrid generalized bi-ideal $\tilde{f}_\lambda$ of S and hybrid structure $\tilde{g}_\mu$ of S.

Proof: $(i) \Rightarrow (ii)$ Let $\tilde{f}_\lambda$ be a hybrid generalized bi-ideal and $\tilde{g}_\mu$ a hybrid structure of S. Then by Theorem 3.2, we have $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$. If S is regular and $a \in S$, then there exists $b \in S$ such that $a = aba$. Now, $(\tilde{f} \tilde{\circ} \chi_S(\tilde{g}) \tilde{\circ} \tilde{f})(a) = \bigcup_{a=xy} \{(\tilde{f} \tilde{\circ} \chi_S(\tilde{g}))(x) \cap \tilde{f}(y)\} \supseteq (\tilde{f} \tilde{\circ} \chi_S(\tilde{g}))(ab) \cap \tilde{f}(a) = \bigcup_{ab=pq} \{\tilde{f}(p) \cap \chi_S(\tilde{g})(q)\} \cap \tilde{f}(a)\} \supseteq (\tilde{f}(a) \cap \chi_S(\tilde{g})(b) \cap \tilde{f}(a) \supseteq \tilde{f}(a) \cap U \cap \tilde{f}(a) = \tilde{f}(a)$.

Also
$$\begin{aligned}
(\lambda \tilde{\circ} \chi_S(\mu) \tilde{\circ} \lambda)(a) &= \bigwedge_{a=xy} \bigvee \{(\lambda \tilde{\circ} \chi_S(\mu))(x), \lambda(y)\} \\
&= \bigwedge_{a=xy} \bigvee \{ \bigwedge_{x=pq} \bigvee \{(\lambda(p), \chi_S(\mu)(q)\}, \lambda(y)\} \\
&= \bigwedge_{a=xy} \bigvee \{ \bigwedge_{x=pq} \bigvee \{(\lambda(p), 0\}, \lambda(y)\} \\
&= \bigwedge_{a=pqy} \bigvee \{(\lambda(p), \lambda(y)\} \\
&\leq \bigwedge_{a=pqy} \lambda(pqy) = \lambda(a).
\end{aligned}$$

Thus $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda \gg \tilde{f}_\lambda$ and hence $\tilde{f}_\lambda \odot \chi_S(\tilde{g}_\mu) \odot \tilde{f}_\lambda = \tilde{f}_\lambda$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Let $a \in S$. Then, by Theorem 3.1, $\chi_{B(a)}(\tilde{f}_\lambda)$ is a generalized bi-ideal of S and $\chi_{B(a)}(\tilde{f}_\lambda) = \chi_{B(a)}(\tilde{f}_\lambda) \odot \chi_S(\tilde{g}_\mu) \odot \chi_{B(a)}(\tilde{f}_\lambda)$, $U = \chi_{B(a)}(\tilde{f})(a) = \bigcup_{a=xy} \{(\chi_{B(a)}(\tilde{f}) \tilde{\circ} \chi_S(\tilde{g}))(x) \cap \chi_{B(a)}(\tilde{f})(y)\}$ implies $(\chi_{B(a)}(\tilde{f}) \tilde{\circ} \chi_S(\tilde{g}))(b) = U$ and $\chi_{B(a)}(\tilde{f})(c) = U$ for some $b, c \in S$ with $a = bc$. As $(\chi_{B(a)}(\tilde{f}) \tilde{\circ} \chi_S(\tilde{g}))(b) = U$, we have $(\chi_{B(a)}(\tilde{f})(d) = U$ and $(\chi_S(\tilde{g})(e) = U$ for some $d, e \in S$ with $b = de$ which imply $c, d \in B(a)$ with $a = bv = dec \in B(a)SB(a)$. Therefore S is regular. $\square$

Theorem 3.5. *[3] Let S be a semigroup. then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{f}_\mu \odot \tilde{g}_\lambda$ for every hybrid bi-ideal $\tilde{f}_\mu$ and every hybrid left ideal $\tilde{g}_\lambda$ of S.

Theorem 3.6. *[3] Let S be a semigroup. then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \Cap \tilde{h}_\nu \ll \tilde{f}_\mu \odot \tilde{g}_\lambda \odot \tilde{h}_\nu$ for every hybrid right ideal $\tilde{f}_\mu$, hybrid bi-ideal $\tilde{g}_\lambda$ and every hybrid left ideal of $\tilde{h}_\nu$ of S.

We notice that the above two theorems remains true with a hybrid generalized bi-ideal instead of hybrid bi-ideal, as a consequence of these theorems, we have the following theorem.

Theorem 3.7. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular,

(ii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{f}_\mu \odot \tilde{g}_\lambda$ for every hybrid bi-ideal $\tilde{f}_\mu$ and hybrid left ideal $\tilde{g}_\lambda$ of S,

(iii) $\tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{f}_\mu \odot \tilde{g}_\lambda$ for every hybrid generalized bi-ideal $\tilde{f}_\mu$ and hybrid left ideal $\tilde{g}_\lambda$ of S,

(iv) $\tilde{h}_\gamma \Cap \tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{h}_\gamma \odot \tilde{f}_\mu \odot \tilde{g}_\lambda$ for every hybrid bi-ideal $\tilde{f}_\mu$, hybrid left ideal $\tilde{g}_\lambda$, and hybrid right ideal $\tilde{h}_\gamma$ of S,

(v) $\tilde{h}_\gamma \Cap \tilde{f}_\mu \Cap \tilde{g}_\lambda \ll \tilde{h}_\gamma \odot \tilde{f}_\mu \odot \tilde{g}_\lambda$ for every hybrid generalized bi-ideal $\tilde{f}_\mu$, hybrid left ideal $\tilde{g}_\lambda$, and hybrid right ideal $\tilde{h}_\gamma$ of S.

Following [2], a semigroup S is called left quasiregular if every left ideal of it is globally idempotent. It is well known that a semigroup S is left quasiregular if and only if for each element a of S, there exist x and y in S such that $a = xaya$(Proposition 1.1 of [2]).

Lemma 3.2. *Let S be a semigroup. If S is left quasiregular, then every hybrid generalized bi-ideal of S is a hybrid bi-ideal of S.*

Proof: Let S be a left quasiregular semigroup and $\tilde{f}_\lambda$ a hybrid generalized bi-ideal of S. Then for any $b \in S$, there exist x and y in S such that $b = xbyb$. Now let $a \in S$. Then $\tilde{f}(ab) = \tilde{f}(axbyb) = \tilde{f}(a(xby)b) \supseteq \tilde{f}(a) \cap \tilde{f}(b)$ and $\lambda(ab) = \lambda(axbyb) = \lambda(a(xby)b) \leq \vee\{\lambda(a), \lambda(b)\}$. Thus $\tilde{f}_\lambda$ is a hybrid sub semigroup of S and hence $\tilde{f}_\lambda$ is hybrid bi-ideal of S. $\square$

Theorem 3.8. *Let S be a semigroup. Then S is left quasiregular if and only if $\tilde{f}_\lambda \odot \tilde{f}_\lambda = \tilde{f}_\lambda$ for every hybrid left ideal $\tilde{f}_\lambda$ of S.*

Proof: Assume that S is left quasiregular and let $\tilde{f}_\lambda$ be a hybrid left ideal of S. Then by Theorem 3.12 of [1], we have $\tilde{f}_\lambda \odot \tilde{f}_\lambda \ll \tilde{f}_\lambda$. Let $a \in S$. Then there exist x and y in S such that $a = xaya$. Now, $(\tilde{f}\tilde{\circ}\tilde{f})(a) = \bigcup\limits_{a=pq}\{(\tilde{f})(p) \cap \tilde{f}(q)\} \supseteq \tilde{f}(xa) \cap \tilde{f}(ya) \supseteq \tilde{f}(a) \cap \tilde{f}(a) = \tilde{f}(a)$ and $(\lambda\tilde{\circ}\lambda)(a) = \bigwedge\limits_{a=pq}\bigvee\{\lambda(p), \lambda(q)\} \leq \bigvee\{\lambda(xa), \lambda(ya)\} \leq \bigvee\{\lambda(a), \lambda(a)\} = \lambda(a)$. Thus $\tilde{f}_\lambda \odot \tilde{f}_\lambda \gg \tilde{f}_\lambda$ and hence $\tilde{f}_\lambda \odot \tilde{f}_\lambda = \tilde{f}_\lambda$.

Conversely, assume that every hybrid left ideal of S is idempotent. Let L be any left ideal of S and $a \in S$. Then by Theorem 3.5 of [1], $\chi_L(\tilde{f}_\lambda)$ is a hybrid left ideal of S. By assumption, we have $U = \chi_L(\tilde{f})(a) = (\chi_L(\tilde{f})\tilde{\circ}\chi_L(\tilde{f}))(a) = \bigcup\limits_{a=pq}\{\chi_L(\tilde{f})(p) \cap \chi_L(\tilde{f})(q)\}$, so there exist $x, y \in S$ with $\chi_L(\tilde{f})(x) = U = \chi_L(\tilde{f})(y)$ and $a = xy \in LL$. Thus $L \subseteq LL \subseteq L$ and hence S is left quasiregular. $\square$

Theorem 3.9. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) S is regular and left quasiregular,

(ii) $\tilde{g}_\mu \Cap \tilde{h}_\lambda \Cap \tilde{f}_\gamma \ll \tilde{g}_\mu \odot \tilde{h}_\lambda \odot \tilde{f}_\gamma$ for every hybrid bi-ideal $\tilde{f}_\mu$, hybrid left ideal $\tilde{g}_\lambda$ and every hybrid right ideal $\tilde{h}_\gamma$ of S,

(iii) $\tilde{g}_\mu \Cap \tilde{h}_\lambda \Cap \tilde{f}_\gamma \ll \tilde{g}_\mu \odot \tilde{h}_\lambda \odot \tilde{f}_\gamma$ for every hybrid generalized bi-ideal $\tilde{f}_\mu$, hybrid left ideal $\tilde{g}_\lambda$ and every hybrid right ideal $\tilde{h}_\gamma$ of S.

Proof: $(i) \Rightarrow (ii)$ Assume that S is regular and left quasiregular.

Let $\tilde{f}_\mu$ be a hybrid bi-ideal, $\tilde{g}_\lambda$ a hybrid left ideal and $\tilde{h}_\gamma$ a hybrid right ideal of S. Let $a \in S$. Then there exist $x, y, u, v \in S$ such that $a = xa^2y = uava$ which implies $a = uava = u(xa^2y)va = ((ux)a)((a(yv))a)$. Now, $(\tilde{g}_\mu \tilde{\circ} \tilde{h}_\lambda \tilde{\circ} \tilde{f}_\gamma)(a) = \bigcup\limits_{a=pq} \{\tilde{g}(p) \cap (\tilde{h} \tilde{\circ} \tilde{f})(q)\} \supseteq \tilde{g}((ux)a) \cap (\tilde{h} \tilde{\circ} \tilde{f})((ayv)a) \supseteq \tilde{g}(a) \cap \bigcup\limits_{ayva=pq} \{\tilde{h}(p) \cap (\tilde{f}(q)\} \supseteq \tilde{g}(a) \cap \tilde{h}(a(yu)) \cap \tilde{f}(a) \supseteq \tilde{g}(a) \cap \tilde{h}(a) \cap \tilde{f}(a) = (\tilde{g} \cap \tilde{h} \cap \tilde{f})(a)$.

Also,
$$\begin{aligned}(\mu \tilde{\circ} \lambda \tilde{\circ} \gamma)(a) &= \bigwedge_{a=pq} \bigvee \{\mu(p), (\lambda \tilde{\circ} \gamma)(q)\} \\ &\leq \bigvee \{\mu((ux)a), (\lambda \tilde{\circ} \gamma)((ayv)a) \\ &\leq \bigvee \{\mu(a), \bigwedge_{ayva=pq} \bigvee \{\lambda(p), \gamma(q)\} \\ &\leq \vee\{\mu(a), \vee\{\lambda(a(yu)), \gamma(a)\} \\ &\leq \vee\{\mu(a), \vee\{\lambda(a), \gamma(a)\} = \vee\{\mu(a), \lambda(a), \gamma(a)\} = (\mu \Cap \lambda \Cap \gamma)(a).\end{aligned}$$

Therefore $\tilde{g}_\mu \Cap \tilde{h}_\lambda \Cap \tilde{f}_\gamma \ll \tilde{g}_\mu \odot \tilde{h}_\lambda \odot \tilde{f}_\gamma$.

(ii)$\Rightarrow$ (iii) is trivial and the proof of (iii)$\Rightarrow$ (i) is similar to the proof of Theorem (3.7), (v)$\Rightarrow$ (i). □

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