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New Subclass of Analytic Functions Defined by q-Differential Operator with Respect to k-Symmetric Points

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Abstract

In this paper, we introduce new subclass of analytic functions using new q-differential operator. We obtain a sufficient conditions and several coefficient inequalities for functions belong to this subclass. The q-integral representation and convolution conditions are also provided.

1 Introduction

We start by letting $\mathbb{D} = \{z : z \in \mathbb{C}, |z| < 1\}$ the open unit disk in the complex plane $\mathbb{C}$, and $\mathcal{A}$ the class of analytic functions $f(z)$ on $\mathbb{D}$ of the form

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i, \quad (z \in \mathbb{D}). \tag{1.1}$$

The q-calculus has various applications in different branches of mathematics that make the study of geometric function theory interesting and pertinent (see [9, 10]). Some applications of q-operator are studied by Aral and Gupta [6] and Elhaddad et.al ([7, 8]). Many different problems related

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to q-calculus can be seen recently in ([1, 2, 13, 15]).

Now, we give some concepts of q-calculus and definitions of q-difference operator $\partial_q f(z)$ and q-integral, respectively, as:

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z-qz}, & z \in \mathbb{C}\backslash\{0\} \\ 0 \quad , & z = 0 \end{cases}, \tag{1.2}$$

$$\int_0^z f(t) d_q t = (1-q) z \sum_{n=0}^{\infty} q^n f(q^n z). \tag{1.3}$$

As a special case of q-Jackson difference operators and integrals, we have [11]

$$\partial_q \log f(z) = \frac{\log q}{q-1} \frac{\partial_q f(z)}{f(z)} \Rightarrow \int_0^z \frac{\partial_q f(\zeta)}{f(\zeta)} d_q \zeta = \frac{q-1}{\log q} \log f(z).$$

For $f \in \mathcal{A}$, one can see that $\partial_q f(z) = 1 + \sum_{i=2}^{\infty} [i]_q a_i z^{i-1}$ where $[i]_q = \frac{1-q^i}{1-q}$. If $q \to 1^-$, then $\partial_q f(z) \to f^{'}(z)$ and $[i]_q$ tends to i, $(i \in \mathbb{N})$.

For $0 \leq \alpha < 1$, we denote by $\mathcal{S}_q^*(\alpha)$ and $\mathcal{C}_q(\alpha)$ the familiar subclasses of $\mathcal{A}$ consisting of functions which are, respectively, q-starlike of order α and q-convex of order α in $\mathbb{D}$ by

$$\mathcal{S}_q^*(\alpha) = \left\{ f \in \mathcal{A} : \mathbf{Re} \left(\frac{z \partial_q f(z)}{f(z)} \right) > \alpha, z \in \mathbb{D} \right\}.$$

$$\mathcal{C}_q(\alpha) = \left\{ f \in \mathcal{A} : \mathbf{Re} \left(1 + \frac{z \partial_q (\partial_q f(z))}{\partial_q f(z)} \right) > \alpha, z \in \mathbb{D} \right\},$$

These classes are introduced by Seoudy and Aouf [15], then studied by Aldweby and Darus [2].

We note that $\lim_{q \to 1^-} \mathcal{S}_q^*(\alpha) := \mathcal{S}^*(\alpha)$ and $\lim_{q \to 1^-} \mathcal{C}_q(\alpha) := \mathcal{C}(\alpha)$, where $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are, respectively, the class of starlike of order α and convex of order α in $\mathbb{D}$ ([14]).

Sokol [16] introduced $\mathcal{S}_s^{(k)}(\alpha)$ a class of starlike functions of order α w.r.t. k-symmetric points as:

$$\mathcal{S}_s^{(k)}(\alpha) = \left\{ f \in \mathcal{S} : \mathbf{Re} \left(\frac{z f^{'}(z)}{f_k(z)} \right) > \alpha, z \in \mathbb{D}, k \in \mathbb{N} \right\}, \tag{1.4}$$

where $0 \le \alpha < 1$, and $f_k(z)$ defined by

$$f_k(z) = \frac{1}{k}\sum_{v=0}^{k-1} \varepsilon^{-v} f(\varepsilon^v z) \quad (\varepsilon^k = 1, z \in \mathbb{D}), \tag{1.5}$$

where k is a fixed positive integer.

Now, for a function $f \in \mathcal{A}$ we introduce a new q-differential operator: $D^n_{q,\mu,\delta,\kappa,\lambda} f : \mathcal{A} \to \mathcal{A}$ by

$$\begin{aligned}
D^0_{q,\mu,\delta,\kappa,\lambda} f(z) &= f(z) = z + \sum_{i=2}^{\infty} a_i z^i \\
D^1_{q,\mu,\delta,\kappa,\lambda} f(z) &= \Big[1 - (\kappa-\lambda)(\delta-\mu)\Big] f(z) + (\kappa-\lambda)(\delta-\mu) z \partial_q f(z) \\
&= z + \sum_{i=2}^{\infty} \Big[(\kappa-\lambda)(\delta-\mu)([i]_q - 1) + 1\Big] a_i z^i \\
&\ \vdots
\end{aligned}$$

$$D^n_{q,\mu,\delta,\kappa,\lambda} f(z) = D^1_{q,\mu,\delta,\kappa,\lambda}\Big(D^{n-1}_{q,\mu,\delta,\kappa,\lambda} f(z)\Big) = z + \sum_{i=2}^{\infty} \Big[\nabla^{\kappa,\lambda}_{\delta,\mu}[i]_q\Big]^n a_i z^i \tag{1.6}$$

where $\delta, \kappa, \lambda, \mu \ge 0$, $\kappa > \lambda, \delta > \mu$, $n \in \mathbb{N}_0$ and

$$\nabla^{\kappa,\lambda}_{\delta,\mu}[i]_q = (\kappa-\lambda)(\delta-\mu)([i]_q - 1) + 1. \tag{1.7}$$

Remarks: We note that

- When $\kappa = 1$, $\lambda = 0$, $\delta = 1$, $\mu = 0$ and $q \to 1^-$ we get Salagean differential operator introduced in [5].
- When $\beta = 1$, $\kappa = 0$, $\delta = 1$, $\mu = 0$ and $q \to 1^-$ then we get Al-Oboudi differential operator introduced in [4].
- When $q \to 1^-$, we get Ramadan and Darus operator introduced in [13].

Finally, using the q-differential operator defined on (1.6), we introduce new subclass $\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho, \alpha, \beta)$ of analytic functions w.r.to k-symmetric points as:

Definition 1.1. *Let $\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$ denotes the class of functions $f \in \mathcal{A}$ that satisfies the following inequality*

$$\left|\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)} - 1\right| < (1-\rho)\left|\alpha\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)} + (1-\beta)\right|, \quad (1.8)$$

where $0 \leq \rho < 1$, $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$ and f_k defined by (1.5).

It is clear that the q-derivative of $D^n_{q,\mu,\delta,\kappa,\lambda}f(z)$ has the value as in the following equality

$$\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z) = 1 + \sum_{i=2}^{\infty}[i]_q\left(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\right)^n a_i z^{i-1}, \quad (1.9)$$

and applying the operator $D^n_{q,\mu,\delta,\kappa,\lambda}$ on the function $f_k(z)$, we have

$$D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z) = \frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-v}D^n_{q,\mu,\delta,\kappa,\lambda}f(\varepsilon^v z) \quad (\varepsilon^k = 1).$$

$$= z + \sum_{i=2}^{\infty}\left(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\right)^n a_i\psi_i z^i \quad (1.10)$$

where

$$\psi_i = \begin{cases} 1 & ,if \quad i = lk+1 \\ o & ,if \quad i \neq lk+1 \end{cases} \quad (i \geqslant 2, k, l \geqslant 1) \quad (1.11)$$

Throughout this paper, we will assume that $0 \leq \rho < 1$, $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$, $\delta, \kappa, \lambda, \mu \geq 0$, $\kappa > \lambda, \delta > \mu$, $z \in \mathbb{D}$ and $n \in \mathbb{N}_0$.

2 Main Results

With the purpose of ratifying and confirming our results, the following preliminary Lemma are needed.

Lemma 2.1. *[12] Let $-1 \leq C_2 \leq C_1 < D_1 \leq D_2 \leq 1$, then we have*

$$\frac{1+D_1 z}{1+C_1 z} \prec \frac{1+D_2 z}{1+C_2 z}.$$

In the next theorems, we obtain a sufficient conditions for functions $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$.

Theorem 2.2. *For f is given by (1.1) and satisfies the inequality*

$$\sum_{i=2}^{\infty}\Big\{[ik{+}1]_q\Big[1{+}\alpha(1{-}\rho)\Big]{-}\Big[1{-}(1{-}\rho)(1{-}\beta)\Big]\Big\}\Big((\kappa{-}\lambda)\left(\nabla^{\kappa,\lambda}_{\delta,\mu}[ik+1]_q\right)^n\Big|a_{ik+1}\Big|$$

$$+\sum_{\substack{i=2\\ i\neq lk+1}}^{\infty}[i]_q\Big(1+(1-\rho)\alpha\Big)\left(\nabla^{\kappa,\lambda}_{\delta,\mu}[i]_q\right)^n\Big|a_i\Big|<(1-\rho)(1+\alpha-\beta) \qquad (2.12)$$

then $f(z)\in\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$.

Proof. Let $f(z)=z+\sum_{i=2}^{\infty}a_iz^i$, $f_k(z)$ be defined by (1.5), and

$$M=\Big|z\partial_qD^n_{q,\mu,\delta,\kappa,\lambda}f(z){-}D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)\Big|{-}(1{-}\rho)\Big|\alpha z\partial_qD^n_{q,\mu,\delta,\kappa,\lambda}f(z){+}(1{-}\beta)D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)\Big|.$$

To show that $f(z)\in\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$, it suffices to show that $M<0$.

Now,

$$M=\Big|z+\sum_{i=2}^{\infty}[i]_q\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^na_iz^i-\Big(z+\sum_{i=2}^{\infty}\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^na_i\psi_iz^i\Big)\Big|$$

$$-(1{-}\rho)\left|\alpha\Big(z+\sum_{i=2}^{\infty}[i]_q\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^na_iz^i\Big)+(1-\beta)\Big(z+\sum_{i=2}^{\infty}\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^na_i\psi_iz^i\Big)\right|$$

$$\leq\sum_{i=2}^{\infty}\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^n([i]_q{-}\psi_i)|a_i|r^i{-}(1{-}\rho)\left[(\alpha+1-\beta)r-\sum_{i=2}^{\infty}\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^n(\alpha[i]_q+(1-\beta)\psi_i)|a_i|r^i\right]$$

$$\leq\sum_{i=2}^{\infty}\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^n\Big\{([i]_q{-}\psi_i){+}(1{-}\rho)[\alpha[i]_q{+}(1{-}\beta)\psi_i\Big\}|a_i|r^i{-}(1{-}\rho)(1{+}\alpha{-}\beta)r$$

where ψ_i given in (1.11).
Thus, for $|z|=r<1$, we have

$$M<\left(\sum_{i=2}^{\infty}\Big(\nabla^{\kappa,\lambda}_{q,\delta,\mu}[i]_q\Big)^n\Big\{([i]_q-\psi_i)+(1-\rho)[\alpha[i]_q+(1-\beta)\psi_i]\Big\}|a_i|-(1-\rho)(1+\alpha-\beta)\right)r$$

$$M<\sum_{i=2}^{\infty}\Big\{[i]_q\Big(1+(1-\rho)\alpha\Big)+\Big((1-\rho)(1-\beta)-1\Big)\psi_i\Big\}\Big(\nabla_{q,\delta,\mu}^{\kappa,\lambda}[i]_q\Big)^n|a_i|-\Big[(1-\rho)(1+\alpha-\beta)\Big]. \tag{2.13}$$

Substitute ψ_i defined in (1.11) in the inequality (2.13) we get

$$M<\sum_{i=2}^{\infty}\Big\{[ik+1]_q\Big(1+\alpha(1-\rho)\Big)+\Big((1-\rho)(1-\beta)-1\Big)\Big\}\Big((\kappa-\lambda)(\delta-\mu)([ik+1]_q-1)+1\Big)^n\Big|a_{ik+1}\Big|$$

$$-\Big[(1-\rho)(1+\alpha-\beta)\Big]+\sum_{\substack{i=2\\ i\neq lk+1}}^{\infty}[i]_q\Big(1+\alpha(1-\rho)\Big)\Big((\kappa-\lambda)(\delta-\mu)([i]_q-1)+1\Big)^n\Big|a_i\Big|$$

by condition (2.12), implies $M<0$, therefore $f(z)\in\mathcal{C}_{q,\mu,\delta,\kappa,\lambda}^{n,(k)}(\rho,\alpha,\beta)$.

Theorem 2.3. *Let $f\in\mathcal{A}$ then $f\in\mathcal{C}_{q,\mu,\delta,\kappa,\lambda}^{n,(k)}(\rho,\alpha,\beta)$ if and only if*

$$\frac{z\partial_q D_{q,\mu,\delta,\kappa,\lambda}^n f(z)}{D_{q,\mu,\delta,\kappa,\lambda}^n f_k(z)}\prec\frac{1+(1-\rho)(1-\beta)z}{1-(1-\rho)\alpha z}\quad(z\in\mathbb{D}), \tag{2.14}$$

where $\prec$ stands for the subordination (see [3]).

Proof. Let $f(z)\in\mathcal{C}_{q,\mu,\delta,\kappa,\lambda}^{n,(k)}(\rho,\alpha,\beta)$, then by the equality (1.8), we have

$$\left|\frac{z\partial_q D_{q,\mu,\delta,\kappa,\lambda}^n f(z)}{D_{q,\mu,\delta,\kappa,\lambda}^n f_k(z)}-1\right|^2<(1-\rho)^2\left|\alpha\frac{z\partial_q D_{q,\mu,\delta,\kappa,\lambda}^n f(z)}{D_{q,\mu,\delta,\kappa,\lambda}^n f_k(z)}+(1-\beta)\right|^2,$$

by letting $H(z)=\frac{z\partial_q D_{q,\mu,\delta,\kappa,\lambda}^n f(z)}{D_{q,\mu,\delta,\kappa,\lambda}^n f_k(z)}$, therefore

$$|H(z)-1|^2<(1-\rho)^2\,|\alpha H(z)+(1-\beta)|^2.$$

Expanding it we get

$$(1-\alpha^2(1-\rho)^2)\,|H(z)|^2-2(1+(1-\beta)(1-\rho)^2\alpha)\mathbf{Re}\{H(z)\}<(1-\beta)^2(1-\rho)^2-1,$$

If $\rho\neq 0,\alpha\neq 1$, we have

$$|H(z)|^2-2\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\mathbf{Re}\{H(z)\}+\left(\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right)^2$$

$$<\frac{(1-\beta)^2(1-\rho)^2-1}{1-\alpha^2(1-\rho)^2}+\left(\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right)^2,$$

$$\left|H(z)-\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right|^2<\frac{(1-\beta)^2(1-\rho)^2-1}{1-\alpha^2(1-\rho)^2}+\left(\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right)^2,$$

that is,

$$\left|H(z)-\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right|^2<\left(\frac{(1-\beta+\alpha)(1-\rho)}{1-\alpha^2(1-\rho)^2}\right)^2,$$

or equivalent to

$$\left|H(z)-\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right|<\frac{(1-\beta+\alpha)(1-\rho)}{1-\alpha^2(1-\rho)^2}.$$

This tells us that the value of the region of $H(z)=\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}$ is contained in the disk whose center is $\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}$ and of radius $\frac{(1-\beta+\alpha)(1-\rho)}{1-\alpha^2(1-\rho)^2}$. We know the univalent function $\mho=\mathfrak{P}(z)=\frac{1+(1-\rho)(1-\beta)z}{1-(1-\rho)\alpha z}$ maps the unit disk into

$$\left|\mho-\frac{1+(1-\beta)(1-\rho)^2\alpha}{1-\alpha^2(1-\rho)^2}\right|<\frac{(1-\beta+\alpha)(1-\rho)}{1-\alpha^2(1-\rho)^2}.$$

Note that $\mathfrak{P}(0)=H(0)$ and $H(\mathbb{D})\subset\mathfrak{P}(\mathbb{D})$, then we get the following conclusion

$$H(z)\prec\mathfrak{P}(z)=\frac{1+(1-\rho)(1-\beta)z}{1-\alpha(1-\rho)z}.$$

Conversely, let

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}\prec\frac{1+(1-\rho)(1-\beta)z}{1-\alpha(1-\rho)z},$$

or equivalently

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}=\frac{1+(1-\rho)(1-\beta)\mho(z)}{1-\alpha(1-\rho)\mho(z)},\tag{2.15}$$

where $\mho(z)$ is analytic in $\mathbb{D}$, and $\mho(0)=0$, $|\mho(z)|<1$ for all $z\in\mathbb{D}$. By some calculations we can easily obtain

$$\left|\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}-1\right|<(1-\rho)\left|\alpha\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}+(1-\beta)\right|.$$

If $n=0$, $\rho=\beta=0$ and $\alpha=1$

$$\frac{z\partial_q(f(z))}{f_k(z)} \prec \frac{1+z}{1-z},$$

Therefore, the proof of this theorem is complete.

Remark: Since $\mathbf{Re}\left\{\frac{1+(1-\rho)(1-\beta)z}{1-\alpha(1-\rho)z} > 0\right\}$, then from previous theorem, we know that

$$\mathbf{Re}\left\{\frac{z\partial_q(D^n_{q,\mu,\delta,\kappa,\lambda}f(z))}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}\right\} > 0. \tag{2.16}$$

Theorem 2.4. *Let* $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$ *then we have* $D^n_{q,\mu,\delta,\kappa,\lambda}f_k \in \mathcal{S}^*_q(0)$.

Proof. Suppose that $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$, substitute z by $\varepsilon^m z$ in inequality (2.16) where $m=0,1,...,k-1$, then we have

$$\mathbf{Re}\left\{\frac{\varepsilon^m z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(\varepsilon^m z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(\varepsilon^m z)}\right\} > 0,$$

by definition of $f_k(z)$ we know $f_k(\varepsilon^m z) = \varepsilon^m f_k(z)$, and summing up the above inequality for $m=0,1,...,k-1$, we get

$$\mathbf{Re}\left\{\frac{\sum_{m=0}^{k-1} z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(\varepsilon^m z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}\right\} > 0,$$

or equivalently,

$$\mathbf{Re}\left\{\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}\right\} > 0,$$

that is, $D^n_{q,\mu,\delta,\kappa,\lambda}f_k \in \mathcal{S}^*_q(0)$, and the proof is complete.

Similarly as in Theorem2.4, we have $D^n_{q,\mu,\delta,\kappa,\lambda}f_k \in \mathcal{C}_q(0)$ if and only if $D^n_{q,\mu,\delta,\kappa,\lambda}(z\partial_q f_k) \in \mathcal{S}^*_q(0)$.

Theorem 2.5. *Let* $0 \le \rho_1 \le \rho_2 < 1$, $0 \le \alpha_1 \le \alpha_2 \le 1$, $0 \le \beta_1 \le \beta_2 < 1$. *Then we have* $\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho_2,\alpha_2,\beta_2) \subset \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho_1,\alpha_1,\beta_1)$.

Proof. Suppose that $f \in \mathcal{C}^{(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho_2,\alpha_2,\beta_2)$ then f satisfies the equation (2.3), we have

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda}f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)} \prec \frac{1+(1-\rho_2)(1-\beta_2)z}{1-(1-\rho_2)\alpha_2 z}.$$

By using the given assumptions $0 \leq \rho_1 \leq \rho_2 \leq 1$, $0 \leq \alpha_2 \leq \alpha_1 \leq 1$, $0 \leq \beta_1 \leq \beta_2 < 1$, we get
$-1 \leq -(1-\rho_1)\alpha_1 \leq -(1-\rho_2)\alpha_2 < (1-\rho_2)(1-\beta_2) \leq (1-\rho_1)(1-\beta_1) \leq 1.$

Applying Lemma 2.1, we have the final result

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)} \prec \frac{1+(1-\rho_2)(1-\beta_2)z}{1-(1-\rho_2)\alpha_2 z} \prec \frac{1+(1-\rho_1)(1-\beta_1)z}{1-(1-\rho_1)\alpha_1 z}.$$

This means $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho_1, \alpha_1, \beta_1)$, so the proof is complete.

3 Convolution Condition

Definition 3.1. *The convolution or Hadamard product of $f(z)$ given by (1.1) and $\alpha(z) = z + \sum_{i=2}^{\infty} c_i z^i$ is defined by*

$$(f * \alpha)(z) = (\alpha * f)(z) = z + \sum_{i=2}^{\infty} a_i c_i z^i.$$

Theorem 3.1. *A function $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho, \alpha, \beta)$ if and only if*

$$\frac{1}{z}\left\{ f * \left(A(z) - \frac{1+(1-\rho)(1-\beta)e^{i\theta}}{1-(1-\rho)\alpha e^{i\theta}} h(z) \right) \right\} \neq 0, \left(z \in \mathbb{D}, \theta \in [0, 2\pi) \right) \tag{3.17}$$

where

$$A(z) = z + \sum_{i=2}^{\infty} \left(\nabla^{\kappa,\lambda}_{\delta,\mu} [i]_q \right)^n z^i, \quad h(z) = z + \sum_{i=2}^{\infty} \left(\nabla^{\kappa,\lambda}_{\delta,\mu} [i]_q \right)^n \psi_i z^i.$$

Proof. We know that

$$z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f(z) = f(z) * A(z), \tag{3.18}$$

and

$$D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z) = f(z) * h(z). \tag{3.19}$$

Suppose $f \in \mathcal{C}^{(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho, \alpha, \beta)$, then by equation (2.14), we have

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)} \neq \frac{1+(1-\rho)(1-\beta)e^{i\theta}}{1-(1-\rho)\alpha e^{i\theta}}, \qquad \forall z \in \mathbb{D}, \quad \theta \in [0, 2\pi).$$

$$z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f(z).\{1-(1-\rho)\alpha e^{i\theta}\} \neq D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)\{1+(1-\rho)(1-\beta)e^{i\theta}\}. \tag{3.20}$$

Substitute (3.18) and (3.19) into (3.20) to get (3.17), and this completes the proof.

4 The q-Integral Representation

Finally , we give the q-integral representation of functions in the class $\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$.

Theorem 4.1. *Let $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$, then we have*

$$D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z) = z.exp\left\{\frac{\log\ q}{(q-1)k}\sum_{m=0}^{k-1}\int_0^{\varepsilon^m z}\frac{1+(1-\rho)(1-\beta)\mho(t)}{t\Big(1-(1-\rho)\alpha\mho(t)\Big)}d_q t\right\} \tag{4.21}$$

where $\mho(z)$ is analytic with $\mho(0)=0$, $|\mho(z)|<1$ for all $z\in\mathbb{D}$ and f_k is defined in (1.5).

Proof. Consider $f \in \mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$, then f satisfied equation (2.15), we have

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f(z)}{D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)} = \frac{1+(1-\rho)(1-\beta)\mho(z)}{1-(1-\rho)\alpha\mho(z)}, z\in\mathbb{D}.$$

Substituting z by $\varepsilon^m z$ $(m=0,1,...,k-1)$ in the last equation, we get

$$\frac{\varepsilon^{\mu} z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f(\varepsilon^m z)}{D^n_{q,\mu,\delta,\kappa,\lambda} f_k(\varepsilon^m z)} = \frac{1+(1-\rho)(1-\beta)\mho(\varepsilon^m z)}{1-(1-\rho)\alpha\mho(\varepsilon^m z)}, (m=0,1,...,k-1)$$

we know that $f_k(\varepsilon^m z)=\varepsilon^m f_k(z)$, and summing up the above equation for $m=0,1,...,k-1$, we get

$$\frac{z\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)}{D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)} = \frac{1}{k}\sum_{m=0}^{k-1}\frac{1+(1-\rho)(1-\beta)\mho(\varepsilon^m z)}{1-(1-\rho)\alpha\mho(\varepsilon^m z)}. \tag{4.22}$$

From the last equality we have

$$\frac{\partial_q D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)}{D^n_{q,\mu,\delta,\kappa,\lambda} f_k(z)} - \frac{1}{z} = \frac{1}{k}\sum_{m=0}^{k-1}\frac{1+(1-\rho)(1-\beta)\mho(\varepsilon^m z)}{z(1-(1-\rho)\alpha\mho(\varepsilon^m z))}.$$

By applying the q-Jackson integral, we have

$$\frac{q-1}{\log\ q}\log\Big\{\frac{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}{z}\Big\}=\frac{1}{k}\sum_{m=0}^{k-1}\int_0^z\frac{1+(1-\rho)(1-\beta)\mho(\varepsilon^m\vartheta)}{\vartheta(1-(1-\rho)\alpha\mho(\varepsilon^m\vartheta))}d_q\vartheta,$$

such that

$$\log\Big\{\frac{D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)}{z}\Big\}=\frac{\log\ q}{(q-1)k}\sum_{m=0}^{k-1}\int_0^{\varepsilon^m z}\frac{1+(1-\rho)(1-\beta)\mho(t)}{t\Big(1-(1-\rho)\alpha\mho(t)\Big)}d_qt$$

and

$$D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z)=z.exp\Bigg\{\frac{\log\ q}{(q-1)k}\sum_{m=0}^{k-1}\int_0^{\varepsilon^m z}\frac{1+(1-\rho)(1-\beta)\mho(t)}{t\Big(1-(1-\rho)\alpha\mho(t)\Big)}d_qt\Bigg\}.$$

so the proof is complete.

Theorem 4.2. *Let $f\in\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$, then we have*

$$f(z)=\int_0^z exp\Bigg\{\frac{\log\ q}{(q-1)k}\sum_{m=0}^{k-1}\int_0^{\varepsilon^m\vartheta}\frac{1+(1-\rho)(1-\beta)\mho(t)}{t\Big(1-(1-\rho)\alpha\mho(t)\Big)}d_qt\Bigg\}.\frac{1+(1-\rho)(1-\beta)\mho(\vartheta)}{1-(1-\rho)\alpha\mho(\vartheta)}d_q\vartheta \tag{4.23}$$

where $\mho(z)$ is analytic with $\mho(0)=0$, $|\mho(z)|<1$ for all $z\in\mathbb{D}$ and f_k is defined in (1.5).

Proof. Let $f\in\mathcal{C}^{n,(k)}_{q,\mu,\delta,\kappa,\lambda}(\rho,\alpha,\beta)$, then by the equation (2.15), we have

$$z\partial_q(D^n_{q,\mu,\delta,\kappa,\lambda}f(z))=D^n_{q,\mu,\delta,\kappa,\lambda}f_k(z).\frac{1+(1-\rho)(1-\beta)\mho(z)}{1-(1-\rho)\alpha\mho(z)},$$

substitute (4.21) in the last equation, then we get

$$\partial_q(D^n_{q,\mu,\delta,\kappa,\lambda}f(z))$$
$$=exp\Bigg\{\frac{\log\ q}{(q-1)k}\sum_{m=0}^{k-1}\int_0^{\varepsilon^m z}\frac{1+(1-\rho)(1-\beta)\mho(t)}{t\Big(1-(1-\rho)\alpha\mho(t)\Big)}d_qt\Bigg\}.\frac{1+(1-\rho)(1-\beta)\mho(z)}{1-(1-\rho)\alpha\mho(z)}.$$

By applying the q-Jackson integral on both sides, we obtain

$$D^n_{q,\mu,\delta,\kappa,\lambda}f(z)$$
$$=\int_0^z exp\Bigg\{\frac{\log\ q}{(q-1)k}\sum_{m=0}^{k-1}\int_0^{\varepsilon^m\vartheta}\frac{1+(1-\rho)(1-\beta)\mho(t)}{t\Big(1-(1-\rho)\alpha\mho(t)\Big)}d_qt\Bigg\}.\frac{1+(1-\rho)(1-\beta)\mho(\vartheta)}{1-(1-\rho)\alpha\mho(\vartheta)}d_q\vartheta.$$

so the proof is complete.

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