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Common N-tupled fixed points via (CLR) property and an application to a system of N-integral equations

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Abstract

In this manuscript, we establish results on the existence of common N–tupled fixed points using (CLR) property for two hybrid pair of mappings. We also study existence of a common solution for a system of N-integral equations.

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1 Introduction and Preliminaries

Samet and Vetro [11] proved some coupled fixed point results for multi-valued nonlinear contraction mappings in partially ordered metric spaces. Abbas et al. [13] gave some common coupled fixed (coincidence) point results for hybrid pair of multi-valued mappings. Deshpande and Handa [12] defined occasional w-compatibility and $(E.A)$ property for hybrid pairs of mappings. Rao et al. [4] (see also [3, 9]) initiated (CLR) property in complex valued b -metric spaces. On the other hand, Imdad et al. [1] provided some n-tupled coincidence point results in metric spaces. Recently, Shoaib et al. [8] established results for tripled coincidence and common fixed point for hybrid pair of mappings using (CLR) property. For related results, see [14, 15, 16, 17, 18, 19, 20, 21, 22].

In the present work, we prove some common N-tupled fixed point theorems for hybrid mappings on a metric space. Also, we discuss the existence of a solution for a system of nonlinear N-integral equations.

In the sequel, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the sets of non-negative real numbers, positive integers and non-negative integers, respectively.

Let (Δ, d) be a metric space. Denote by $CB(\Delta)$ the class of all non-empty, bounded and closed subsets of Δ. For $\vartheta \in \Delta$ and $\chi \in CB(\Delta)$, consider

$$d(\vartheta, \chi) = \inf\{d(\vartheta, \zeta) : \zeta \in \chi\}.$$

For $\Delta_1, \Delta_2 \in CB(\Delta)$, define the Hausdorff-Pompieu metric [2] $H : CB(\Delta) \times CB(\Delta) \to \mathbb{R}^+$ as

$$H(\Delta_1, \Delta_2) = \max\{\delta(\Delta_1, \Delta_2), \delta(\Delta_2, \Delta_1)\}$$

where

$$\delta(\Delta_1, \Delta_2) = \sup\{d(r, \Delta_2), r \in \Delta_1\}.$$

Definition 1.1. *[1] Given $F : \Delta^N \to \Delta$. We say that $(x^1, x^2, \cdots, x^N) \in \Delta^N$ is a N-upled fixed point of F if*

$$x^1 = F(x^1, x^2, x^3, \cdots, x^N), x^2 = F(x^2, x^3, x^4, \cdots, x^1), \cdots, x^N = F(x^N, x^1, x^2 \cdot, x^{N-1}).$$

Definition 1.2. *[1] Given $F : \Delta^N \to \Delta$ and $g : \Delta \to \Delta$. We say that $(x^1, x^2, \cdots, x^N) \in \Delta^N$ is a coincidence point of g and F if*

$$gx^1 = F(x^1, x^2, x^3, \cdots, x^N), gx^2 = F(x^2, x^3, x^4, \cdots, x^1), \cdots, gx^N = F(x^N, x^1, x^2, \cdots, x^{N-1}).$$

Denote by $C(g, F)$ the set of N-tupled coincidence point of g and F.

Definition 1.3. *[10] Let $F : \Delta\times\Delta\times\Delta \to CB(\Delta)$ be a multi-valued mapping and g be a self-map on Δ. If $g^2x \in F(gx, gy, gz)$, $g^2y \in F(gy, gz, gx)$, and $g^2z \in F(gz, gx, gy)$ at some point $(x, y, z) \in \Delta^3$, then g is called F-weakly commuting.*

The common limit range property for two single-valued mappings was introduced by Sintunavarat and Kumam [5] (see also [6]). This concept was extended by Abdou [7] to a pair of mappings (one is single-valued and the second is multi-valued) in the class of metric spaces. Leter, this notion has been generalized to the tripled case in [8].

2 Main Results

As a generalization of Definition 1.3, we start with

Definition 2.1. *Given $G : \Delta^N \to CB(\Delta)$. The self-map g on Δ is called G-weakly commuting at $(x^1, x^2, x^3, .., x^N) \in \Delta^N$ if $g^2x^1 \in G(gx^1, gx^2, gx^3, .., gx^N)$, $g^2x^2 \in G(gx^2, gx^3, gx^4, \cdots, gx^1)$ and $g^2x^N \in G(gx^N, gx^1, gx^2, \cdots, gx^{N-1})$.*

Definition 2.2. *Given $F, G : \Delta^N \to CB(\Delta)$. Let $f, g : \Delta \to \Delta$ be two mappings on a metric space (Δ, d). The pairs (F, f) and (G, g) have (CLR_f) and (CLR_g) properties, respectively, if there exist sequences $\{x^1_n\}$, $\{x^2_n\}$,...,$\{x^N_n\}$, $\{u^1_n\}$, $\{u^2_n\}$,...,$\{u^N_n\}$ in Δ, $C^1, C^2, ..., C^N$ and $D^1, D^2, ..., D^N \in CB(\Delta)$ such that for some $z_1, z_2, ..., z_N \in \Delta$, we have*

$$\lim_{n\to\infty} fx^1_n = fz_1 \in C^1 = \lim_{n\to\infty} F(x^1_n, x^2_n, x^3_n, .., x^N_n),$$

$$\lim_{n\to\infty} fx^2_n = fz_2 \in C^2 = \lim_{n\to\infty} F(x^2_n, x^3_n, x^4_n, \cdots, x^1_n),$$

$$\vdots$$

$$\lim_{n\to\infty} fx^N_n = fz_N \in C^N = \lim_{n\to\infty} F(x^N_n, x^1_n, x^2_n, \cdots, x^{N-1}_n),$$

$$\lim_{n\to\infty} gu^1_n = gz_1 \in D^1 = \lim_{n\to\infty} G(u^1_n, u^2_n, u^3_n, .., u^N_n),$$

$$\lim_{n\to\infty} gu^2_n = gz_2 \in D^2 = \lim_{n\to\infty} G(u^2_n, u^3_n, u^4_n, \cdots, u^1_n),$$

$$\vdots$$

$$\lim_{n\to\infty} fu^N_n = gz_N \in D^N = \lim_{n\to\infty} G(u^N_n, u^1_n, u^2_n, \cdots, u^{N-1}_n).$$

Consider the following conditions.

(Ω_1) $f:\Delta\to\Delta$ is F-weakly commuting for some $(z_1,z_2,...,z_N)\in C(f,F)$ with $f^2z_1=fz_1$, $f^2z_2=fz_2$, ...,$f^2z_N=fz_N$;

(Ω_2) $g:\Delta\to\Delta$ is G-weakly commuting for some $(z_1,z_2,...,z_N)\in C(g,G)$ with $g^2z_1=gz_1$, $g^2z_2=gz_2$,..., $g^2z_N=gz_N$.

In all our results, we avoid the completeness of the metric space. It is replaced by the (CLR) property.

Theorem 2.3. *Let d be a metric on Δ. Given $F,G:\Delta^N\to CB(\Delta)$ and $f,g:\Delta\to\Delta$ so that the pairs (F,f) and (G,g) have (CLR_f) and (CLR_g)-properties. Suppose that there exist λ_1,λ_2 and $\lambda_3\in[0,1)$ with $0\le\lambda_1+\lambda_2+\lambda_3<1$ such that*

$$\begin{aligned}H(F(x^1,x^2,x^3,..,x^N),G(u^1,u^2,u^3,\cdots,u^N))\\ +H(F(x^2,x^3,x^4,\cdots,x^1)),G(u^2,u^3,u^4,\cdots,u^1))+\cdots\\ +H(F(x^N,x^1,x^2,\cdots,x^{N-1}),G(u^N,u^1,u^2,\cdots,u^{N-1}))\\ \le\lambda_1\big[d(fx^1,gu^1)+d(fx^2,gu^2)+\cdots+d(fx^N,gu^N)\big]\\ +\lambda_2\big[d(f(x^1),F(x^1,x^2,x^3,\cdots,x^N))+d(f(x^2),F(x^2,x^3,x^4,\cdots,x^1))+\cdots\\ +d(f(x^N),F(x^N,x^1,x^2,\cdots,x^{N-1}))\big]+\lambda_3\big[d(g(u^1),G(u^1,u^2,u^3,\cdots,u^N))\\ +d(g(u^2),G(u^2,u^3,u^4,\cdots,u^1))+\cdots+d(g(u^N),G(u^N,u^1,u^2,\cdots,u^{N-1}))\big],\end{aligned}\tag{2.1}$$

for all $(x^1,x^2,x^3,\cdots,x^N),(u^1,u^2,u^3,\cdots,u^N)\in\Delta^N$. Then

$(\bigwedge_1)$ *f and F have an N-tupled coincidence point.*

$(\bigwedge_2)$ *g and G have an N-tupled coincidence point.*

Moreover,

(a) *if (Ω_1) holds, then F and f have a common N-tupled fixed point;*

(b) *if (Ω_2) holds, then G and g have a common N-tupled fixed point;*

(c) *if (Ω_1) and (Ω_2) hold, then F,G,f and g have a common N-tupled fixed point.*

Proof. The pair (F,f) having the (CLR_f)-property, so there are sequences $\{x_n^1\}$, $\{x_n^2\}$, ...$\{x_n^N\}$ in Δ and $C^1,C^2,...C^N\in CB(\Delta)$ such that for some $z_1,z_2,...z_N\in\Delta$, we have

$$\lim_{n\to\infty}fx_n^1=fz_1\in C^1=\lim_{n\to\infty}F(x_n^1,x_n^2,x_n^3,..,x_n^N),$$

$$\lim_{n\to\infty} fx_n^2 = fz_2 \in C^2 = \lim_{n\to\infty} F(x_n^2, x_n^3, x_n^4, \cdots, x_n^1),$$

$$\vdots$$

$$\lim_{n\to\infty} fx_n^N = fz_N \in C^N = \lim_{n\to\infty} F(x_n^N, x_n^1, x_n^2, \cdots, x_n^{N-1}).$$

Similarly, the pair (G, g) has the (CLR_g)-property, so there are sequences $\{u_n^1\}$, $\{u_n^2\}$, ...$\{u_n^N\}$ in Δ and $D^1, D^2, ...D^N$ in $CB(\Delta)$ such that for such $z_1, z_2, ...z_N$ in Δ, we have

$$\lim_{n\to\infty} gu_n^1 = gz_1 \in D^1 = \lim_{n\to\infty} G(u_n^1, u_n^2, u_n^3, .., u_n^N),$$

$$\lim_{n\to\infty} gu_n^2 = gz_2 \in D^2 = \lim_{n\to\infty} G(u_n^2, u_n^3, u_n^4, \cdots, u_n^1),$$

$$\vdots$$

$$\lim_{n\to\infty} fu_n^N = fz_N \in D^N = \lim_{n\to\infty} G(u_n^N, u_n^1, u_n^2, \cdots, u_n^{N-1}).$$

By inequality (2.1), we get

$$\begin{aligned}
&H(F(x_n^1, x_n^2, x_n^3, .., x_n^N), G(u_n^1, u_n^2, u_n^3, \cdots, u_n^N)) \\
&\quad + H(F(x_n^2, x_n^3, x_n^4, \cdots, x^1)), G(u_n^2, u_n^3, u_n^4, \cdots, u^1)) + \cdots \\
&\quad + H(F(x_n^N, x_n^1, x_n^2, \cdots, x_n^{N-1}), G(u_n^N, u_n^1, u_n^2, \cdots, u_n^{N-1})) \\
&\quad \leq \lambda_1\big[d(fx_n^1, gu_n^1) + d(fx_n^2, gu_n^2) + \cdots + d(fx_n^N, gu^N)\big] \\
&+ \lambda_2\big[d(f(x_n^1), F(x_n^1, x_n^2, x_n^3, \cdots, x_n^N)) + d(f(x_n^2), F(x_n^2, x_n^3, x_n^4, \cdots, x_n^1)) + \cdots \\
&+ d(f(x_n^N), F(x_n^N, x_n^1, x_n^2, \cdots, x_n^{N-1}))\big] + \lambda_3\big[d(g(u_n^1), G(u_n^1, u_n^2, u_n^3, \cdots, u_n^N)) \\
&+ d(g(u_n^2), G(u_n^2, u_n^3, u_n^4, \cdots, u_n^1)) + \cdots + d(g(u_n^N), G(u_n^N, u_n^1, u_n^2, \cdots, u_n^{N-1}))\big].
\end{aligned}$$

Applying $n \to \infty$, we get

$$fz_1 = gz_1, fz_2 = gz_2, ..., fz_N = gz_N. \tag{2.2}$$

Again, from inequality (2.1), one writes

$$\begin{aligned}
&H(F(z_1, z_2, z_3, .., z_N), G(u_n^1, u_n^2, u_n^3, \cdots, u_n^N)) + H(F(z_2, z_3, z_4, \cdots, z_1)), G(u_n^2, u_n^3, u_n^4, \cdots, u_n^1)) + \cdots \\
&\quad + H(F(z_N, z_1, z_2, \cdots, z_{N-1}), G(u_n^N, u_n^1, u_n^2, \cdots, u_n^{N-1})) \\
&\quad \leq \lambda_1\big[d(fz_1, gu_n^1) + d(fz_2, gu_n^2) + \cdots + d(fz_N, gu_n^N)\big] \\
&+ \lambda_2\big[d(f(z_1), F(z_1, z_2, z_3, \cdots, z_N)) + d(f(z_2), F(z_2, z_3, z_4, \cdots, z_1)) + \cdots \\
&+ d(f(z_N), F(z_N, z_2, z_3, \cdots, z_{N-1}))\big] + \lambda_3\big[d(g(u_n^1), G(u_n^1, u_n^2, u_n^3, \cdots, u_n^N)) \\
&+ d(g(u_n^2), G(u_n^2, u_n^3, u_n^4, \cdots, u_n^1)) + \cdots + d(g(u_n^N), G(u_n^N, u_n^2, u_n^3, \cdots, u_n^{N-1}))\big].
\end{aligned}$$

Taking $n \to \infty$ and using (2.2), we have

$$d(fz_1, F(z_1, z_2, z_3, .., z_N)) = 0,$$
$$d(fz_2, F(z_2, z_3, z_4, \cdots, z_1) = 0,$$
$$\vdots$$
$$d(fz_N, F(z_N, z_1, z_2, \cdots, z_{N-1}) = 0.$$

Thus

$$fz_1 \in F(z_1, z_2, z_3, .., z_N),$$
$$fz_2 \in F(z_2, z_3, z_4, \cdots, z_1),$$
$$\vdots$$
$$fz_N \in F(z_N, z_1, z_2, \cdots, z_{N-1}).$$

Hence, $(\bigwedge_1)$ is proved. By equation (2.1), we get

$$\begin{aligned} &H(F(x_n^1, x_n^2, x_n^3, .., x_n^N), G(z_1, z_2, \cdots, z_N)) + H(F(x_n^2, x_n^3, x_n^4, \cdots, x_n^1)), G(z_2, z_3, z_4, \cdots, z_1)) \\ &+ \cdots + H(F(x_n^N, x_n^1, x_n^2, \cdots, x_n^{N-1}), G(z_N, z_1, z_2, \cdots, z_{N-1})) \\ &\le \lambda_1\big[d(fx_n^1, gz_1) + d(fx_n^2, gz_2) + \cdots + d(fx_n^N, gz_N)\big] \\ &+ \lambda_2\big[d(f(x_n^1), F(x_n^1, x_n^2, x_n^3, \cdots, x_n^N)) + d(f(x_n^2), F(x_n^2, x_n^3, x_n^4, \cdots, x_n^1)) + \cdots \\ &+ d(f(x_n^N), F(x_n^N, x_n^1, x_n^2, \cdots, x_n^{N-1}))\big] + \lambda_3\big[d(g(z_1), G(z_1, z_2, \cdots, z_N)) \\ &+ d(g(z_2), G(z_2, z_3, z_4, \cdots, z_1)) + \cdots + d(g(z_N), G(z_N, z_1, z_2, \cdots, z_{N-1}))\big]. \end{aligned}$$

Similarly, at the limit, we obtain that

$$gz_1 \in G(z_1, z_2, \cdots, z_N),$$
$$gz_2 \in G(z_2, z_3, z_4, \cdots, z_1)...$$
$$\vdots$$
$$gz_N \in G(z_N, z_1, z_2, \cdots, z_{N-1}).$$

So $(\bigwedge_2)$ is proved.

In the case that (Ω_1) holds, f is F-weakly commuting. Recall that

$$f^2z_1 \in F(fz_1, fz_2, ..., fz_N),$$

$$f^2z_2 \in F(fz_2, fz_3, \cdots, fz_1)$$

$$\vdots$$

$$f^2z_N \in F(fz_N, fz_1, \cdots, fz_{N-1}).$$

Since $f^2z_1 = fz_1$, $f^2z_2 = fz_2...$, $f^2z_N = fz_N$, $(fz_1, fz_2, ..., fz_N)$ is then a common N-tupled fixed point, so (a) is proved.

A similar argument proves (b) when (Ω_2) holds. Using (2.2), (c) holds immediately.

□

By taking $\lambda_2 = \lambda_3 = 0$ in Theorem 2.3, we deduce the following corollary.

Corollary 2.4. *Let d be a metric on Δ. Given $F, G : \Delta^N \to CB(\Delta)$ and $f, g : \Delta \to \Delta$ so that the pairs (F, f) and (G, g) have (CLR_f) and (CLR_g)-properties. Suppose that there exists $0 \le \lambda_1 < 1$ such that*

$$\begin{aligned} H(F(x^1, x^2, x^3, .., x^N), G(u^1, u^2, u^3, \cdots, u^N)) + H(F(x^2, x^3, x^4, \cdots, x^1)), G(u^2, u^3, u^4, \cdots, u^N)) \\ + \cdots + H(F(x^N, x^1, x^2, \cdots, x^{N-1}), G(u^N, u^1, u^2, \cdots, u^{N-1})) \\ \le \lambda_1 \big[d(fx^1, gu^1) + d(fx^2, gu^2) + \cdots + d(fx^N, gu^N)\big], \end{aligned}$$

for all $(x^1, x^2, x^3, \cdots, x^N), (u^1, u^2, u^3, \cdots, u^N) \in \Delta^N$. Then

$(\bigwedge_1)$ *f and F have an N-tupled coincidence point.*

$(\bigwedge_2)$ *g and G have an N-tupled coincidence point.*

Moreover,

- (a) *if (Ω_1) holds, then F and f have a common N-tupled fixed point;*
- (b) *if (Ω_2) holds, then G and g have a common N-tupled fixed point;*
- (c) *if (Ω_1) and (Ω_2) hold, then F, G, f and g have a common N-tupled fixed point.*

By taking $g = f$ and $G = F$ in Theorem 2.3, we state

Corollary 2.5. *Let d be a metric on Δ. Given $F : \Delta^N \to CB(\Delta)$ and $f : \Delta \to \Delta$ so that the pair (F, f) has the (CLR_f) property. Suppose that there exist λ_1, λ_2 and $\lambda_3 \in [0, 1)$ with $0 \leq \lambda_1 + \lambda_2 + \lambda_3 < 1$ such that*

$$\begin{aligned} &H(F(x^1, x^2, x^3, .., x^N), F(u^1, u^2, u^3, \cdots, u^N)) \\ &\quad + H(F(x^2, x^3, x^4, \cdots, x^1)), F(u^2, u^3, u^4, \cdots, u^1)) + \cdots \\ &\quad + H(F(x^N, x^1, x^2, \cdots, x^{N-1}), F(u^N, u^1, u^2, \cdots, u^{N-1})) \\ &\quad \leq \lambda_1 \big[d(fx^1, fu^1) + d(fx^2, fu^2) + \cdots + d(fx^n, fu^n)\big] \\ &+ \lambda_2 \big[d(f(x^1), F(x^1, x^2, x^3, \cdots, x^N)) + d(f(x^2), F(x^2, x^3, x^4, \cdots, x^1)) + \cdots \\ &+ d(f(x^N), F(x^n, x^1, x^2, \cdots, x^{N-1}))\big] + \lambda_3 \big[d(f(u^1), F(u^1, u^2, u^3, \cdots, u^N)) \\ &+ d(f(u^2), F(u^2, u^3, u^4, \cdots, u^1)) + \cdots + d(f(u^N), F(u^n, u^1, u^2, \cdots, u^{N-1}))\big], \end{aligned}$$

for all $(x^1, x^2, x^3, \cdots, x^N)$ and $(u^1, u^2, u^3, \cdots, u^N) \in \Delta^N$. Then f and F have an N-tupled coincidence point. Moreover, if (Ω_1) holds, then F and f have a common N-tupled fixed point.

By taking $gx = fx = x$ in Theorem 2.3, we have

Corollary 2.6. *Let d be a metric on Δ . Given $F, G : \Delta^N \to CB(\Delta)$. Suppose that there exist λ_1, λ_2 and $\lambda_3 \in [0, 1)$ with $0 \leq \lambda_1 + \lambda_2 + \lambda_3 < 1$ such that*

$$\begin{aligned} &H(F(x^1, x^2, x^3, .., x^N), G(u^1, u^2, u^3, \cdots, u^N)) \\ &\quad + H(F(x^2, x^3, x^4, \cdots, x^1)), G(u^2, u^3, u^4, \cdots, u^1)) + \cdots \\ &\quad + H(F(x^N, x^1, x^2, \cdots, x^{N-1}), G(u^N, u^1, u^2, \cdots, u^{N-1})) \\ &\quad \leq \lambda_1 \big[d(x^1, u^1) + d(x^2, u^2) + \cdots + d(x^N, u^N)\big] \\ &+ \lambda_2 \big[d(x^1, F(x^1, x^2, x^3, \cdots, x^N)) + d(x^2, F(x^2, x^3, x^4, \cdots, x^1)) + \cdots \\ &+ d(x^N, F(x^N, x^1, x^2, \cdots, x^{N-1}))\big] + \lambda_3 \big[d(u^1, G(u^1, u^2, u^3, \cdots, u^N)) \\ &+ d(u^2, G(u^2, u^3, u^4, \cdots, u^1)) + \cdots + d(u^N, G(u^N, u^1, u^2, \cdots, u^{N-1}))\big], \end{aligned}$$

for all $(x^1, x^2, x^3, \cdots, x^N), (u^1, u^2, u^3, \cdots, u^N) \in \Delta^N$. Then F and G have a common N–tupled fixed point.

By taking $gx = fx = x$ and $G = F$ and $\lambda_2 = \lambda_3 = 0$ in Theorem 2.3, we obtain following:

Corollary 2.7. *Let (Δ, d) be a metric space. Given $F : \Delta^n \to CB(\Delta)$. Suppose that there exists $0 \leq \lambda < 1$ such that*

$$\begin{aligned} H(F(x^1, x^2, x^3, .., x^n), F(u^1, u^2, u^3, \cdots, u^N)) \\ + H(F(x^2, x^3, x^4, \cdots, x^1)), F(u^2, u^3, u^4, \cdots, u^1)) + \cdots \\ + H(F(x^N, x^1, x^2, \cdots, x^{N-1}), F(u^N, u^1, u^2, \cdots, u^{N-1})) \\ \leq \lambda\big[d(x^1, u^1) + d(x^2, u^2) + \cdots + d(x^N, u^N)\big], \end{aligned}$$

for all $(x^1, x^2, x^3, \cdots, x^N), (u^1, u^2, u^3, \cdots, u^N) \in \Delta^N$. Then F has an N-tupled fixed point.

3 Application

In this section, we give an existence theorem for the solution of a system of N- integral equations. Let $\Delta = C[\zeta_1, \zeta_2]$ be the set of all real continuous functions on $[\zeta_1, \zeta_2]$ where $(\zeta_1 < \zeta_2)$. Define

$$d(u, v) = \max |u(t_1) - v(t_1)|, \quad \text{for all } u, v \in \Delta, t_1 \in [\zeta_1, \zeta_2].$$

Consider the following nonlinear system of N-integral equations:

$$\left.\begin{aligned} x^1(r_1) &= K_1(r_1) + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, x^1(t_1)) + f_2(r_1, x^2(t_1)) + \cdots + f_N(r_1, x^N(t_1))\big]dt_1, \\ x^2(r_1) &= K_1(r_1) + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, x^2(t_1)) + f_2(r_1, x^3(t_1)) + \cdots + f_N(r_1, x^1(t_1))\big]dt_1, \\ &\cdots \\ x^N(r_1) &= K_1(r_1) + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, x^N(t_1)) + f_2(r_1, x^1(t_1)) + \cdots + f_N(r_1, x^{N-1}(t_1))\big]dt_1. \end{aligned}\right\}, \tag{3.1}$$

Theorem 3.1. *Suppose that*

(i) $f_1, f_2, \cdots f_N : [\zeta_1, \zeta_2] \times \Delta \to \mathbb{R}$ *are N- continuous functions;*

(ii) $K_1 : [\zeta_1, \zeta_2] \to \mathbb{R}$ *and* $G^* : [\zeta_1, \zeta_2] \times [\zeta_1, \zeta_2] \to \mathbb{R}^+$ *are continuous functions;*

(iii) *There exist* $0 < L_i < 10 (i = 1, 2, 3, \cdots N)$ *such that for all* $x, y \in \Delta$,

$$|f_1(r_1, x(t_1)) - f_1(r_1, y(t_1))| \leq L_1|x - y|,$$
$$|f_2(r_1, x(t_1)) - f_2(r_1, y(t_1))| \leq L_2|x - y|,$$
$$\cdots, |f_N(r_1, x(t_1)) - f_N(r_1, y(t_1))| \leq L_N|x - y|;$$

(iv)

$$\int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)dt_1 \leq \frac{1}{L^2}, \tag{3.2}$$

where $L^2 = \max\{L_1, L_2, ...L_N\}$.

Then the system (3.1) of integral equations has a unique common solution in $\Delta = C[\zeta_1, \zeta_2]$.

Proof. Define $F : \Delta^N \to \Delta$ by

$$\left.\begin{aligned} F(x^1, x^2, \cdots x^N)(r_1) = k(r_1) + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, x^1(t_1)) + f_2(r_1, x^2(t_1)) \\ + \cdots + f_N(r_1, x^N(t_1))\big]dt_1. \end{aligned}\right\}$$

We have

$$\Big|d(F(x^1, x^2, \cdots x^N)(r_1)) - F(u^1, u^2, \cdots u^N)(r_1)\Big|$$
$$= \Big|k(r_1) + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, x^1(t_1)) + f_2(r_1, x^2(t_1)) + \cdots + f_N(r_1, x^N(t_1))\big]dt_1$$
$$-k(r_1) + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, u^1(t_1)) + f_2(r_1, u^2(t_1)) + \cdots + f_N(r_1, u^N(t_1))\big]dt_1\Big|$$

$$= \Big| \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_1(r_1, x^1(t_1)) - f_1(r_1, x^1(t_1))\big] + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_2(r_1, x^2(t_1)) - f_2(r_1, u^2(t_1))\big]$$
$$+ \cdots + \int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)\big[f_N(r_1, x^N(t_1)) - f_N(r_1, u^N(t_1))\big]\big]dt_1\Big|$$

$$\leq L_1\Big|\max|x^1(t_1) - x^1(t_1)|\Big| + L_2\Big|\max|x^2(t_1)) - u^2(t_1)|\Big| +$$
$$\cdots + L_n\Big|\max|x^N(t_1)) - u^N(t_1)|\Big|\Big(\int_{\zeta_1}^{\zeta_2} G^*(r_1, t_1)dt_1\Big)$$

$$\leq L^2\left[\left|\max|x^1(t_1)-x^1(t_1)|\right|+\left|\max|x^2(t_1))-u^2(t_1)|\right|+\cdots+\left|\max|x^N(t_1))-u^N(t_1)|\right|\right]\frac{1}{L}$$

$$\leq\ L\left[d(x^1(t_1),x^1(t_1))\ +\ d(x^2(t_1)),u^2(t_1))\ +\ \cdots\ +\ d(x^N(t_1)),u^N(t_1))\right].$$

It follows that

$$\begin{aligned}&d((F(x^1,x^2,\cdots x^N)(r_1)),F(u^1,u^2,\cdots u^N)(r_1))\\&\leq L\left[d(x^1(t_1),u^1(t_1))+d(x^2(t_1)),u^2(t_1))+\cdots+d(x^N(t_1)),u^N(t_1))\right].\end{aligned}\tag{3.3}$$

By similar arguments, we get

$$\begin{aligned}&d((F(x^2,x^3,x^4,\cdots,x^1)(r_1)),F(u^2,u^3,u^4,\cdots,u^1)(r_1))\\&\quad\leq L\left[d(x^2(t_1),u^2(t_1))+d(x^3(t_1)),u^3(t_1))+\cdots+d(x^1(t_1)),u^1(t_1))\right]\end{aligned}\tag{3.4}$$

...

$$\begin{aligned}&d((F(x^N,x^1,x^2,\cdots,x^{N-1})(r_1)),F(u^N,u^1,u^2,\cdots,u^{N-1})(r_1))\\&\leq L\left[d(x^n(t_1),u^n(t_1))+d(x^1(t_1)),u^1(t_1))+\cdots+d(x^{N-1}(t_1)),u^{N-1}(t_1))\right].\end{aligned}\tag{3.5}$$

By (3.3), (3.4) and (3.5), we have

$$\begin{aligned}&d(F(x^1,x^2,\cdots x^N),F(u^1,u^2,\cdots u^n))+d((F(x^2,x^3,x^4,\cdots,x^1)(r_1)),F(u^2,u^3,u^4,\cdots,u^1)(r_1))\\&\quad+\cdots+d((F(x^N,x^1,x^2,\cdots,x^{N-1})(r_1)),F(u^N,u^1,u^2,\cdots,u^{N-1})(r_1))\\&\qquad\leq L[d(x^1(t_1),u^1(t_1))+d(x^2(t_1)),u^2(t_1))+\cdots+d(x^N(t_1)),u^N(t_1))].\end{aligned}$$

Using Corollary 2.7, there exists $x\in\Delta$ such that $F(x,x,\cdots.x)=x$, i.e., x is a common solution of the equations (3.1).

□

Lemma 3.2. *[**?**] Let (Δ,d) be a metric space. For all $\Delta_1,\Delta_2\in CB(\Delta)$, we have $d(\xi,\Delta_2)\leq H(\Delta_1,\Delta_2)$, for each $\xi\in\Delta_1$.*

Lemma 3.3. *[2] Let* (Δ, d) *be a metric space and* $\Delta_1, \Delta_2 \in CB(\Delta)$*. For all* $\gamma > 1$ *and* $\kappa_1 \in \Delta_1$*, there exists* $\kappa_2(\kappa_1) \in \Delta_2$ *such that* $d(\kappa_1, \kappa_2) \leq \gamma H(\Delta_1, \Delta_2)$*.*

In [2], Lemma 3.3 also holds for $\gamma \geq 1$.

Lemma 3.4. *[2] Let* (Δ, d) *be a metric space and* $\Delta_1, \Delta_2 \in CB(\Delta)$*. For all* $\gamma \geq 1$ *and* $\kappa_1 \in \Delta_1$*, there exists* $\kappa_2(\kappa_1) \in \Delta_2$ *such that* $d(\kappa_1, \kappa_2) \leq \gamma H(\Delta_1, \Delta_2)$*.*

The following is a consequence of Lemma 3.4.

Lemma 3.5. *[2] Let* Δ_1 *and* Δ_2 *be two non-empty compact subsets of a metric space* (Δ, d) *and* $\lambda : \Delta_1 \to CB(\Delta)$ *be a multi-valued mapping. Let* $\gamma \geq 1$*, then for* $\kappa_1, \kappa_2 \in \Delta_1$ *and* $\xi_1 \in \lambda\kappa_1$*, there exists* $\xi_2 \in \lambda\kappa_2$ *such that* $d(\xi_1, \xi_2) \leq \gamma H(\lambda\kappa_1, \lambda\kappa_2)$*.*

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