International Journal of Mathematics and
Computer Science, **14**(2019), no. 4, 833–847

On a Subclass of Harmonic Univalent Functions Involving a New Operator Containing q-Mittag-Leffler Function

Suhila Elhaddad[1], **Huda Aldweby**[2], **Maslina Darus**[1]

[1]Centre of Modelling and Data Science
Faculty of Science and Technology
Universiti Kebangsaan Malaysia

[2]Department of Mathematics
Faculty of Science
AL Asmaraya Islamic University, Libya

email: suhila.e@yahoo.com, hu.aldweby@gmail.com, maslina@ukm.edu.my

(Received May 31, 2019, July 1, 2019)

Abstract

This article attempts to define a new differential operator involving q-Mittag-Leffler function. Thus, a new starlike class of complex-valued harmonic univalent functions is defined by means of the aforementioned differential operator. In addition to this, different properties and characteristics were considered in the study for this class. Some of these properties include a necessary and sufficient coefficient, growth bounds and neighborhoods.

Key words and phrases: Harmonic univalent functions , q-Mittag-Leffler function, differential operator, Growth bounds.
AMS (MOS) Subject Classification: 30C45.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

1 Introduction and preliminaries

Let $\mathcal{S}_H$ denote the class of functions $f = h + \bar{g}$ that are harmonic ,univalent, normalized and sense-preserving within the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, where h and g within the class $\mathcal{A}$ of all analytic functions in $\mathcal{U}$ having the following form (See Clunie and Sheil-Small [19]):

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1, \tag{1.1}$$

where h is called the analytic part and g is called the co-analytic part of f. Observe that, if the co-analytic part of the members of the class $\mathcal{S}_H$ is zero, then $\mathcal{S}_H$ reduces to the class $\mathcal{S}$ of univalent functions .

The $\mathcal{S}_H$ class, along with geometric subclasses of $\mathcal{S}_H$, was first examined in the early 1980s by Clunie and Sheil-Small [19], whose introduction of several properties for $\mathcal{S}_H$ inspired further work on $\mathcal{S}_H$ subclasses, (e.g. [1], [6], [7], [19], [31], [32] and [40]). These functions are significant because of their application in researching minimal surfaces and a range of applied mathematical problems.

Quantum calculus, or q-calculus is the subject of a range of research based on the multiple applications identified for it across different mathematical fields, in addition to significance to theoretical physics. The application of q-calculus was initiated by Jackson [27],[28] first explored q-calculus applications, systematically developing q-derivative and q-integral. In addition the Baskakov Durrmeyer operators q-analogue was put forward in [11],[12] and [13], on the basis of the beta function q-analogue. Q-calculus has also been generalised for complex operators with significant results in q-Gauss-Weierstrass and q-Picard singular integral operators, as [8] and [10] discuss. In addition, fractional q-derivative and fractional q-integral operators, among other q-calculus operators, have been applied in [1], [2], [3], [4], [21] [23] and [24]. On this basis, the derivation of q-analogues for operators within analytic functions may have potential significance. Aral et al. [14] gives a thorough discussion of q-analysis as applied within operator theory.

This study begins with definitions of the principal terms used and in-depth concepts for the applications of q-calculus used. In this report, it is assumed that $0 < q < 1$. Definitions are first given for fractional q-calculus operators in a complex-valued function $f(z)$, as follows:

Definition 1.1. *Let* $0 < q < 1$ *and define the q-number* $[n]_q$ *by*

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & (n \in \mathbb{C}) \\ \sum_{k=0}^{m-1} q^k = 1 + q + q^2 + ... + q^{m-1} & (n = m \in \mathbb{N}). \end{cases}$$

Definition 1.2. *(see [27],[28]) The q-derivative (or the q-difference) operator* D_q *of a function* f *is defined by*

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z} & (z \neq 0) \\ f'(z) & (z = 0). \end{cases} \tag{1.2}$$

In case $f(z) = z^n$ for $n \in \mathbb{N}_0 = \{1, 2, 3, ...\}$, the q-derivative of $f(z)$ is given by

$$D_q z^n = \frac{z^n - (zq)^n}{z(1-q)} = [n]_q z^{n-1},$$

where $[n]_q$ defined in Definition 1.1.

We note from Definition 1.2 that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(zq) - f(z)}{(q-1)z} = f'(z).$$

Recent attention has been drawn to Mittag-Leffer function research, as this kind of function can be widely applied across engineering, chemical and biological sciences, physics and in applied science. Various factors in applying such functions are evident within chaotic, stochastic and dynamic systems, fractional differential equations, and distribution of statistics. The geometric characteristics such as convexity, close-to-convexity and starlikeness, of the functions investigated here have been broadly examined by many authors, and direct applications from such functions can be seen for a number of fractional calculus tools, including significant work by [9], [16], [18], [20], [23], [26], [33], [34] , [38] and [39]. The Mittag-Leffer function $E_\sigma(z)$ is named after the Swedish mathematician who proposed it (as shown in [30]). It can be defined as:

$$E_\sigma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + 1)}. \tag{1.3}$$

The initial two parametric generalisations for the function shown in (1.3) were given by Wiman [41],[42]. It is defined in the following way:

$$E_{\sigma,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + \delta)},$$

where $\sigma, \delta \in \mathbb{C}$, $Re(\sigma)>0$ and $Re(\delta)>0$.

In 2014 Sharma and Jain [37] introduced the q-analogue of generalized Mittag-Leffler function $E^{\gamma}_{\sigma,\delta}(z;q)(\sigma,\delta,\gamma \in \mathbb{C}, Re(\sigma)>0, Re(\delta)>0, Re(\gamma)>0)$, which is defined by

$$E^{\gamma}_{\sigma,\delta}(z;q) = \sum_{n=0}^{\infty} \frac{(q^{\gamma};q)_n}{(q;q)_n} . \frac{z^n}{\Gamma_q(\sigma n + \delta)}, \qquad (|q|<1)$$

where $\Gamma_q(z)$ is the q-gamma function and $\lim_{q\to 1} \Gamma_q(z) = \Gamma(z)$.

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by (see [25])

$$(\lambda, q)_n = \begin{cases} (1-\lambda)(1-\lambda q)...(1-\lambda q^{n-1}), & n = 1,2,3,..., \\ 1, & n = 0. \end{cases}$$

Further, the q-gamma function $\Gamma_q(z)$ satisfies the functional equation (see [15], [25])

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z) = [z]_q\Gamma_q(z)$$

Also,

$$(q^{\lambda}, q)_n = \frac{(1-q)^n \Gamma_q(\lambda+n)}{\Gamma_q(\lambda)}. \qquad (n>0)$$

We define the function $Q^{\gamma}_{\sigma,\delta}(z)$ by

$$\begin{aligned} Q^{\gamma}_{\sigma,\delta}(z) &= z\Gamma_q(\delta)E^{\gamma}_{\sigma,\delta}(z;q) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q(\delta)(q^{\gamma};q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} z^n. \end{aligned}$$

Now, for $f \in \mathcal{A}$ we define the following differential operator: $D^{\gamma,m}_{\lambda,q}(\sigma,\delta)f : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$D^{\gamma,0}_{\lambda,q}(\sigma,\delta)f(z) = f(z) * Q^{\gamma}_{\sigma,\delta}(z), \tag{1.4}$$

$$D^{\gamma,1}_{\lambda,q}(\sigma,\delta)f(z) = (1-\lambda)(f(z) * Q^{\gamma}_{\sigma,\delta}(z)) + \lambda z D_q(f(z) * Q^{\gamma}_{\sigma,\delta}(z)) \tag{1.5}$$

$$\vdots$$

$$D^{\gamma,m}_{\lambda,q}(\sigma,\delta)f(z) = D^{\gamma,1}_{\lambda,q}(D^{\gamma,m-1}_{\lambda,q}(\sigma,\delta)f(z)) \tag{1.6}$$

If $f \in \mathcal{A}$ then from (1.5) and (1.6) we see that

$$D^{\gamma,m}_{\lambda,q}(\sigma,\delta)f(z) = z + \sum_{n=2}^{\infty}[1+([n]_q-1)\lambda]^m \frac{\Gamma_q(\delta)(q^\gamma;q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} a_n z^n. \tag{1.7}$$

Note that

- If $q \to 1$ and $\gamma = 1$, we obtain the operator in [22].
- If $q \to 1$, $\sigma = 0$, $\gamma = 1$ and $\delta = 1$, we obtain Al-Oboudi operator [5].
- If $q \to 1$, $\sigma = 0$, $\gamma = 1$, $\delta = 1$ and $\lambda = 1$, we obtain Sălăgean operator [36].
- If $q \to 1$, $m = 0$ and $\gamma = 1$, we obtain $\mathbb{E}_{\sigma,\delta}(z)$ [38].

In this paper, we define the operator $D^{\gamma,m}_{\lambda,q}(\sigma,\delta)f(z)$ in (1.7) of harmonic function $f = h + \overline{g}$ as

$$D^{\gamma,m}_{\lambda,q}(\sigma,\delta)f(z) = D^{\gamma,m}_{\lambda,q}(\sigma,\delta)h(z) + \overline{D^{\gamma,m}_{\lambda,q}(\sigma,\delta)g(z)} \qquad z \in \mathcal{U},$$

where

$$D^{\gamma,m}_{\lambda,q}(\sigma,\delta)h(z) = z + \sum_{n=2}^{\infty}[1+([n]_q-1)\lambda]^m \frac{\Gamma_q(\delta)(q^\gamma;q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} a_n z^n,$$

$$D^{\gamma,m}_{\lambda,q}(\sigma,\delta)g(z) = \sum_{n=1}^{\infty}[1+([n]_q-1)\lambda]^m \frac{\Gamma_q(\delta)(q^\gamma;q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} b_n z^n,$$

for $m \in \mathbb{N}_0, \lambda \geq 0$.

Using the operator $D^{\gamma,m}_{\lambda,q}(\sigma,\delta)f(z)$, we introduce the class of harmonic univalent functions as illustrated below.

Definition 1.3. *For* $0 \leq \vartheta < 1$ *, the function* $f = h + \overline{g}$ *is in the class* $S^{\gamma,m}_H(\lambda,\sigma,\delta,q,\vartheta)$ *if satisfy the inequality*

$$Re\left\{\frac{zD_q(D^{\gamma,m}_{\lambda,q}(\sigma,\delta)h(z)) - \overline{zD_q(D^{\gamma,m}_{\lambda,q}(\sigma,\delta)g(z))}}{D^{\gamma,m}_{\lambda,q}(\sigma,\delta)h(z) + \overline{D^{\gamma,m}_{\lambda,q}(\sigma,\delta)g(z)}}\right\} \geq \vartheta \qquad |z| = r < 1. \tag{1.8}$$

Note that $S_H^{1,0}(0,0,1,q,\vartheta) = S_H(\vartheta)$ is the class of sense-preserving harmonic univalent functions which are starlike of order ϑ in $\mathcal{U}$ defined by Jahangiri [29].

Let $S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$ denote the subclass of $S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$ consisting of harmonic functions $f = h + \overline{g}$, where h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z_n, \quad |b_1|<1.$$

2 Main Results

In our first result, we start with a sufficient coefficient condition for functions f in $S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$.

Theorem 2.1. *Let $f = h + \overline{g}$, where $h(z)$ and $g(z)$ are defined by(1.1). If*

$$\sum_{n=2}^{\infty} \left[\frac{[n]_q - \vartheta}{1-\vartheta} |a_n| + \frac{[n]_q + \vartheta}{1-\vartheta} |b_n| \right] \Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta) \leq 1 - \frac{1+\vartheta}{1-\vartheta} |b_1|, \tag{2.1}$$

where $a_1 = 1, 0 \leq \vartheta<1$ and $\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)$ given by

$$\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta) = \left| [1 + ([n]_q - 1)\lambda]^m \frac{\Gamma_q(\delta)(q^\gamma; q)_{n-1}}{\Gamma_q(\sigma(n-1)+\delta)(q;q)_{n-1}} \right|, \tag{2.2}$$

then f is sense-preserving, harmonic, univalent in $\mathcal{U}$, and $f \in S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$

Proof. If $|z_1| \neq |z_2|<q$, then

$$\left| \frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)} \right| \geq 1 - \left| \frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)} \right| = 1 - \left| \frac{\sum_{n=1}^{\infty} b_n (z_1^n - z_2^n)}{(z_1 - z_2) + \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)} \right|$$

$$>1 - \frac{\sum_{n=1}^{\infty} [n]_q |b_n|}{1 - \sum_{n=2}^{\infty} [n]_q |a_n|} \geq 1 - \frac{\sum_{n=1}^{\infty} [([n]_q + \vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta) / (1-\vartheta)] |b_n|}{1 - \sum_{n=2}^{\infty} [([n]_q - \vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta) / (1-\vartheta)] |a_n|} \geq 0,$$

which proves the univalence. Observe that, f is sense-preserving in $\mathcal{U}$, because

$$|D_q h(z)| \geq \left(1 - \sum_{n=2}^{\infty} [n]_q |a_n| |z|^{n-1} \right) > \left(1 - \sum_{n=2}^{\infty} \frac{([n]_q - \vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\vartheta} |a_n| \right)$$

$$\geq \left(\sum_{n=1}^{\infty} \frac{([n]_q + \vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\vartheta} |b_n| \right) > \left(\sum_{n=1}^{\infty} \frac{([n]_q + \vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\vartheta} |b_n| |z|^{n-1} \right)$$

$$\geq \sum_{n=1}^{\infty} [n]_q |b_n| |z|^{n-1} \geq |D_q g(z)|.$$

Then we have $\lim\limits_{q\to 1}[|D_q h(z)| \geq |D_q g(z)|] = [|h'(z)| \geq |g'(z)|]$.

Now, we show that $f \in S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$. From (1.8), we can write

$$Re\left\{\frac{zD_q(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)h(z)) - \overline{zD_q(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)g(z))}}{D_{\lambda,q}^{\gamma,m}(\sigma,\delta)h(z) + \overline{D_{\lambda,q}^{\gamma,m}(\sigma,\delta)g(z)}}\right\} = Re\left\{\frac{C(z)}{E(z)}\right\},$$

where

$$\begin{aligned} C(z) &= zD_q(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)h(z)) - \overline{zD_q(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)g(z))} \\ &= z + \sum_{n=2}^{\infty}[n]_q\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n z^n - \sum_{n=1}^{\infty}[n]_q\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\overline{b_n}\overline{z^n}, \end{aligned}$$

and

$$\begin{aligned} E(z) &= D_{\lambda,q}^{\gamma,m}(\sigma,\delta)h(z) + \overline{D_{\lambda,q}^{\gamma,m}(\sigma,\delta)g(z)} \\ &= z + \sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_n z^n + \sum_{n=1}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\overline{b_n}\overline{z^n}. \end{aligned}$$

Using the fact that $Re(\omega) \geq \vartheta$ if and only if $|1-\vartheta+\omega| \geq |1+\vartheta-\omega|$,it suffices to show that

$$|C(z) + (1-\vartheta)E(z)| - |C(z) - (1+\vartheta)E(z)| \geq 0. \tag{2.3}$$

Replacing for $C(z)$ and $E(z)$ in (2.3), we get

$$
\begin{aligned}
&|C(z)+(1-\vartheta)E(z)|-|C(z)-(1+\vartheta)E(z)|\\
&=\left|(2-\vartheta)z+\sum_{n=2}^{\infty}([n]_q-\vartheta+1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_nz^n-\sum_{n=1}^{\infty}([n]_q+\vartheta-1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\overline{b_nz^n}\right|\\
&-\left|-\vartheta z+\sum_{n=2}^{\infty}([n]_q-\vartheta-1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)a_nz^n-\sum_{n=1}^{\infty}([n]_q+\vartheta+1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\overline{b_nz^n}\right|\\
&\geq(2-\vartheta)|z|-\sum_{n=2}^{\infty}([n]_q-\vartheta+1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)|a_n||z|^n-\sum_{n=1}^{\infty}([n]_q+\vartheta-1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)|b_n||z|^n\\
&-\vartheta|z|-\sum_{n=2}^{\infty}([n]_q-\vartheta-1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)|a_n||z|^n-\sum_{n=1}^{\infty}([n]_q+\vartheta+1)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)|b_n||z|^n\\
&\qquad\geq 2(1-\vartheta)|z|\left\{1-\sum_{n=2}^{\infty}\frac{([n]_q-\vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\vartheta}|a_n||z|^{n-1}\right.\\
&\qquad\qquad\left.-\sum_{n=1}^{\infty}\frac{([n]_q+\vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}{1-\vartheta}|b_n||a_n||z|^{n-1}\right\}\\
&=2(1-\vartheta)|z|\left\{1-\frac{1+\vartheta}{1-\vartheta}|b_1|-\left(\sum_{n=2}^{\infty}\left[\frac{[n]_q-\vartheta}{1-\vartheta}|a_n|+\frac{[n]_q+\vartheta}{1-\vartheta}|b_n|\right]\right)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\right\}.
\end{aligned}
$$

By using the enquiringly (2.1), we see that the last expression is non-negative. This implies that $f\in S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$.

Now, the necessary and sufficient condition for a function belongs to the class $S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$ is obtained.

Theorem 2.2. *Let $f=h+\overline{g}$. Then $f\in S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$ if and only if*

$$
\sum_{n=1}^{\infty}\left[\frac{[n]_q-\vartheta}{1-\vartheta}|a_n|+\frac{[n]_q+\vartheta}{1-\vartheta}|b_n|\right]\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\leq 1-\frac{1+\vartheta}{1-\vartheta}|b_1|, \tag{2.4}
$$

where $a_1=1, 0\leq\vartheta<1$ and $\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)$ given by (2.2).

Proof. Since $S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)\subseteq S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$, we only need to prove the only if part of the theorem. To this purpose, for functions $f\in S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$, we notice that (1.8) is equivalent to

$$
Re\left\{\frac{zD_q(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)h(z))-\overline{zD_q(D_{\lambda,q}^{\gamma,m}(\sigma,\delta)g(z))}}{D_{\lambda,q}^{\gamma,m}(\sigma,\delta)h(z)+\overline{D_{\lambda,q}^{\gamma,m}(\sigma,\delta)g(z)}}-\vartheta\right\}\geq 0.
$$

That is

$$Re\left[\frac{(1-\vartheta)z-\sum_{n=2}^{\infty}([n]_q-\vartheta)\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)|a_n|z^n-\sum_{n=1}^{\infty}([n]_q+\vartheta)\overline{\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}|b_n|\overline{z}^n}{z-\sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)|a_n|z^n+\sum_{n=1}^{\infty}\overline{\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)}|b_n|\overline{z}^n}\right] \geq 0. \quad (2.5)$$

The above condition must hold for all values of z in $\mathcal{U}$. Upon selecting the values of z on the positive real axis where $0 \leq z = r<1$, we should have

$$\frac{(1-\vartheta)-(1+\vartheta)b_1-\left(\sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\left[([n]_q-\vartheta)|a_n|+([n]_q+\vartheta)|b_n|\right]r^{n-1}\right)}{1+|b_1|+\sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)[|a_n|+|b_n|]r^{n-1}} \geq 0 \quad (2.6)$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0,1)$ for which the quotient of (2.6) is negative. Then, $f \in S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$ and the proof is complete .

Next, we determine the growth bounds for functions $f \in S_{\overline{H}}^{\gamma,m}(\lambda,\alpha,\beta,q,\vartheta)$.

Theorem 2.3. *If* $f \in S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$ *then*

$$|f(z)| \leq (1+|b_1|)r+\frac{\Gamma_q(\sigma+\delta)}{[\gamma]_q[1+\lambda]^m\Gamma_q(\delta)}\left(\frac{1-\vartheta}{1+q-\vartheta}-\frac{1+\vartheta}{1+q-\vartheta}|b_1|\right)r^2, \quad |z|=r<1,$$

and

$$|f(z)| \geq (1-|b_1|)r-\frac{\Gamma_q(\sigma+\delta)}{[\gamma]_q[1+\lambda]^m\Gamma_q(\delta)}\left(\frac{1-\vartheta}{1+q-\vartheta}-\frac{1+\vartheta}{1+q-\vartheta}|b_1|\right)r^2, \quad |z|=r<1.$$

Proof. The left-hand inequality was proved where as the proof for the right hand Inequality will be omitted for being similar. Let $f \in S_{\overline{H}}^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$.

Taking the absolute value of f, we obtain

$$\begin{aligned}
|f(z)| &= \left| z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n \right| \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2 \\
&= (1-|b_1|)r - \frac{(1-\vartheta)\Gamma_q(\sigma+\delta)}{(1+q-\vartheta)[\gamma]_q[1+\lambda]^m \Gamma_q(\delta)} \\
&\times \sum_{n=2}^{\infty} \left(\frac{(1+q)-\vartheta}{1-\vartheta} |a_n| + \frac{(1+q)-\vartheta}{1-\vartheta} |b_n| \right) \frac{[\gamma]_q[1+\lambda]^m \Gamma_q(\delta)}{\Gamma_q(\sigma+\delta)} r^2 \\
&\geq (1-|b_1|)r - \frac{(1-\vartheta)\Gamma_q(\sigma+\delta)}{(1+q-\vartheta)[\gamma]_q[1+\lambda]^m \Gamma_q(\delta)} \left(1 - \frac{1+\vartheta}{1-\vartheta} |b_1| \right) r^2 \\
&= (1-|b_1|)r - \frac{\Gamma_q(\sigma+\delta)}{[\gamma]_q[1+\lambda]^m \Gamma_q(\delta)} \left(\frac{1-\vartheta}{(1+q)-\vartheta} - \frac{1+\vartheta}{(1+q)-\vartheta} |b_1| \right) r^2.
\end{aligned}$$

Thus, the proof is complete.

Subsequent to work performed by Avici and Zotkiewicz [17] and Ruscheweyh [35], the q-neighborhood of a function $f \in S_H$, $\nu \geq 0$ is defined herewith as follows:

$$N_\nu^q(f) = \left\{ F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n} : \sum_{n=2}^{\infty} [n]_q (|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leq \nu \right\}.$$

In our case, let us define the generalised $q-\nu$-neighborhood of f to be the set

$$\begin{aligned}
N_\nu^q(f) = \Bigg\{ F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n} : \sum_{n=2}^{\infty} \Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta) [([n]_q - \vartheta)|a_n - A_n| \\
+([n]_q + \vartheta)|b_n - B_n|] + (1+\vartheta)|b_1 - B_1| \leq (1-\vartheta)\nu \Bigg\}.
\end{aligned}$$

Theorem 2.4. *Let $f \in \mathcal{S}_H$. If satisfies the conditions*

$$\sum_{n=2}^{\infty} [n]_q \Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta) \Big[([n]_q - \vartheta)|a_n| + ([n]_q + \vartheta)|b_n| \Big] \leq (1-\vartheta) - (1+\vartheta)|b_1|, \quad 0 \leq \vartheta < 1 \tag{2.7}$$

and

$$\nu \leq \frac{q-\vartheta}{[2]_q-\vartheta}\left(1-\frac{1+\vartheta}{1-\vartheta}|b_1|\right)$$

then $N_\nu^q(f) \subset S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$

Proof. Let f satisfies (2.7) and

$$F(z)=z+\overline{B_1 z}+\sum_{n=2}^{\infty}\left(A_n z^n+\overline{B_n z^n}\right)$$

which belongs to $N_\nu^q(f)$, we observe that

$$(1+\vartheta)|B_1|+\sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\Big[([n]_q-\vartheta)|A_n|+([n]_q+\vartheta)|B_n|\Big]$$

$$\leq (1+\vartheta)|B_1-b_1|+(1+\vartheta)|b_1|+\sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\Big[([n]_q-\vartheta)|A_n-a_n|+([n]_q+\vartheta)|B_n-b_n|\Big]$$

$$+\sum_{n=2}^{\infty}\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\Big[([n]_q-\vartheta)|a_n|+([n]_q+\vartheta)|b_n|\Big]$$

$$\leq (1-\vartheta)\nu+(1+\vartheta)|b_1|+\frac{1}{[2]_q-\vartheta}\sum_{n=2}^{\infty}[n]_q\Theta_{\lambda,q}^{\gamma,m}(\sigma,\delta)\Big[([n]_q-\vartheta)|A_n|+([n]_q+\vartheta)|B_n|\Big].$$

$$\leq (1-\vartheta)\nu+(1+\vartheta)|b_1|+\frac{1}{[2]_q-\vartheta}\left[(1-\vartheta)-(1+\vartheta)|b_1|\right]\leq 1-\vartheta.$$

Hence for $\nu \leq \dfrac{q-\vartheta}{[2]_q-\vartheta}\left(1-\dfrac{1+\vartheta}{1-\vartheta}|b_1|\right)$, then $F(z) \in S_H^{\gamma,m}(\lambda,\sigma,\delta,q,\vartheta)$.

References

[1] H. Aldweby, M. Darus, A Subclass of harmonic univalent functions associated with q-Analogue of Dziok-Srivatava Operator, *ISRN Mathematical Analysis*, **2013** (2013), Article ID 382312, 1-6.

[2] H. Aldweby, M. Darus, Some subordination results on q-analogue of Ruscheweyh differential operator, *Abstract and Applied Analysis*, **2014** (2014), Article ID 958563, 1-6.

[3] A. Alsoboh, M. Darus, On Fekete-Szego problem associated with q-derivative operator, *J. Phys.: Conf. Ser.*, **1212,** (2019), 012003.

[4] A. Alsoboh, M. Darus, New subclass of analytic functions defined by q-differential operator with respect to k-symmetric points, *International Journal of Mathematics and Computer Science*, **14** no. 4, (2019), 761–773.

[5] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, *Internat. J. Math. Math. Sci.*,**27,** (2004), 1429–1436.

[6] K. Al Shaqsi, M. Darus, On subclass of harmonic starlike functions with respect to k-symmetric points, *International Mathematical Forum*, *2*(57,) (2007), 27992805.

[7] K. Al-Shaqsi, M. Darus, On harmonic univalent functions with respect to k-symmetric points, *International Journal of Contemporary Mathematical Sciences,* **3,** no. 3, (2008), 111-118.

[8] G. A. Anastassiou, S. G. Gal, Geometric and approximation properties of generalized singular integrals in the unit disk, *J. Korean Math. Soc.,* **43,** no. 2, (2006), 425–443.

[9] M. K. Aouf, T. M. Seoudy, Subclasses of p-valent functions involving a new operator containing the generalized MittagLeffler function, *Mediterr. J. Math.*, **15,** no. 4, (2018), 1–19.

[10] A. Aral, On the generalized Picard and Gauss Weierstrass singular integrals, *J. Comput. Anal. Appl.,* **8,** no. 3, (2006), 246-261.

[11] A. Aral, V. Gupta, On q-Baskakov type operators. *Demonstratio Math.*, **42,** no. 1, (2009), 109-122.

[12] A. Aral, V. Gupta, Generalized q-Baskakov operators, *Math. Slovaca.,* **61,** no. 4, (2011), 619-634.

[13] A. Aral, V. Gupta, On the Durrmeyer type modification of the q-Baskakov type operators, *Nonlinear Analytic Theory,* **72,** (2010), 1171-1180.

[14] A. Aral, V. Gupta, R. P. Agarwal, Applications of q-Calculus in Operator Theory, Springer, New York, 2013.

[15] R. Askey, The q-gamma and q-beta functions, *Applicable Analysis*, **8,** no. 2, (1978), 125–141.

[16] A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, *Filomat*, **30,** (2016), 2075–2081.

[17] Y. Avici, E. Zotkiewicz, On harmonic univalent mappings, *Ann. Univ. Mariae Curie-Skodowska*, **44,** (1990), 1–7.

[18] D. Bansal, J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, *Complex Var. Elliptic Eq.,* **61,** no. 3, (2016), 338-350.

[19] J. Clunie, T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fen. Ser. A I Math.*, **9,** (1984), 3–25.

[20] S. Elhaddad, H. Aldweby, M. Darus, Majorization properties for subclass of analytic p-valent functions associated with generalized differential operator involving Mittag-Leffler function, *Nonlinear Functional Analysis and Applications*, **23,** no. 4,(2018), 743–753.

[21] S. Elhaddad, H. Aldweby, M. Darus, Some properties on a class of harmonic univalent functions defined by q-analogue of Ruscheweyh operator, *Journal of Mathematical Analysis*, **9,** no. 2, (2018), 28–35.

[22] S. Elhaddad, H. Aldweby, M. Darus, Neighborhoods of certain classes of analytic functions defined by a generalized differential operator involving Mittag-Leffler function, *Acta Universitatis Apulensis*, **18,**(55) (2018), 1–10.

[23] S. Elhaddad, M. Darus, On Meromorphic Functions Defined by a New Operator Containing the MittagLeffler Function, *Symmetry*, **11,** no. 2, (2019):210.

[24] S. Elhaddad, M. Darus, On Fekete-Szegö problems for a certain subclass defined by q-analogue of Ruscheweyh operator, *J. Phys.: Conf. Ser.,* **1212,** (2019), 012002.

[25] G. Gasper, M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[26] I. S. Gupta, L. Debnath, Some properties of the Mittag-Leffler functions, *Integral Transform. Spec. Funct.*, **18,** no. 5, (2007), 329-336.

[27] F. H. Jackson, On q- definite integrals, *The Quarterly Journal of Pure and Applied Mathematics*, **41,** (1910),193–203.

[28] F. H. Jackson, On $q-$ functions and a certain difference operator, *Transactions of the Royal Society of Edinburgh*, **46,** no. 2, (1908), 253–281.

[29] J.M. Jahangiri, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.*, **235** (1999) 470-477.

[30] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *CR Acad. Sci. Paris* **137(2)** (1903), 554-558.

[31] A. Oshah, M. Darus, A subclass of harmonic univalent functions associated with generalized fractional differential operator, *Bulletin Calcutta Mathematical Society*,**107,** (2015), 205–218.

[32] S. Porwal, M. K. Aouf, On a new subclass of harmonic univalent functions defined by fractional calculus operator, *Journal of Fractional Calculus and Applications*, **4,**no. 10, (2013), 1–12.

[33] D. Răducanu, Third-Order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions, *Mediterr. J. Math.*, **14,** (2017), 18 pages.

[34] D. Răducanu, Partial sums of normalized Mittag-Leffler functions, *An. Şt.Univ. Ovidius Constanţ,* **25,**no. 2, (2017), 8 pages.

[35] S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.,* **81,** (1981), 521–527.

[36] G. S. Sălăgean, Subclasses of univalent functions, Complex Analysis Fifth Romanian-Finnish Sem., Part. 1(Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin (1983), 367–372.

[37] S. K. Sharma, R. Jain, On some properties of generalized q-Mittag-Leffler function, *Mathematica Aeterna*, **4,**no. 6, (2014), 613–619.

[38] H. M. Srivastava, B. A. Frasin, V. Pescar, Univalence of integral operators involving Mittag- Leffler functions, *Appl. Math. Inf. Sci.*, **11,** no. 3, (2017), 635–641.

[39] H. M. Srivastava, Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, *Appl. Math. Comput.*, **211,** (2009), 198–210.

[40] B. A. Stephen, P. Nirmaladevi, T.V. Sudharsan, K. G. Subramanian, A class of harmonic multivalent functions and a coefficient inequality, *Far East J. Math. Sci.*, **32,** (2009), 55–68.

[41] A. Wiman, Über den Fundamental satz in der Theorie der Funktionen $E_\alpha(x)$, *Acta Mathematica,* **29,** no. 1, (1905), 191–201.

[42] A. Wiman, Über die Nullstellun der Funktionen $E_\alpha(x)$, *Acta Mathematica,* **29,** no. 1, (1905), 217-234.