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Prime bi-ideals in ordered ternary semirings

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Abstract

In this paper, we study the notions of prime and semi prime bi-ideals in ordered ternary semirings and we obtain equivalent conditions for a bi-ideal to be a prime bi-ideal.

1 Introduction

Throughout this paper S will denote an ordered ternary semiring with zero. For basic terminology and notations for ternary semiring we refer the reader to [1] and [2], for ordered ternary semiring we refer him/her to [4].

A non-empty set S together with a binary operation, called addition and ternary multiplication (juxtaposition), is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions:

i) $(abc)de = a(bcd)e = ab(cde)$
ii) $(a+b)cd = acd + bcd$
iii) $a(b+c)d = abd + acd$
iv) $ab(c+d) = abc + abd$, for all $a, b, c, d, e \in S$[2].

Following [4], an ordered ternary semiring is an algebraic structure $(S, +, ., \leq)$ such that $(S, +, .)$ is a ternary semiring, $(S, \leq)$ is a partially ordered set and the relation $\leq$ is compatible to the operation $+$ and . on S.

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Let S be an ordered ternary semiring. If there exists an element $0 \in S$ such that $0 + x = x$, $0xy = x0y = xy0 = 0$ and $0 \leq x$ for all $x, y \in S$, then 0 is called the zero element of the ternary semiring S.

Following [3], an ordered ternary semiring S is called regular if for every $a \in S,\ a \leq axa$ for some $x \in S$.

A ternary subsemiring B of S is called a bi-ideal of S if $BSBSB \subseteq B$ and $b \in B,\ a \in S,\ a \leq b$ implies $a \in B$. An ordered ternary semiring S is called B-simple if it has no non-zero proper bi-ideals.

For subsets A, B and C of S, $(A : B)_l = \{x \in S/xSB \subseteq A\}$, $(A : B)_r = \{x \in S/BSx \subseteq A\}$ and $(A : B, C)_m = \{x \in S/BSxSC \subseteq A\}$. We denote by $< a >_b$ the bi-ideal generated by a.

A proper bi-ideal P of S is called prime if $ABC \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for any three bi-ideals A, B and C of S. A proper bi-ideal Q of S is called semiprime if $A^3 \subseteq Q$ implies $A \subseteq Q$. Following [1], a proper bi-ideal P of S is called weakly prime if $P \subset A, P \subset B$ and $P \subset C$, then $ABC \nsubseteq P$ for any bi-ideals A, B and C of S.

2 Main Results

Lemma 2.1. *Let S be an ordered ternary semiring. If P is an additive subsemigroup of S, then $((P : A)_l]$, $((P : A)_r]$ and $((P : A, B)_m]$ are bi-ideals of S for any subsets A, B and C of S.* □

Proposition 2.2. *Let S be an ordered ternary semiring. Then S is regular and B-simple if and only if $(xSx] = S$ for any $0 \neq x \in S$.*

Proof: Let S be a regular and B-simple ternary semiring. Then for any $0 \neq x \in S$, there exists $y \in S$ such that $x \leq xyx \in xSx$ which implies $(xSx] \neq 0$. Since $(xSx]$ is a bi-ideal of S and S is B-simple, we have $(xSx] = S$.

Conversely, let $x(\neq 0) \in S$. Then, by assumption, $S = (xSx]$ which implies there exists $y \in S$ such that $x \leq xyx$ and so S is regular. Let A be a non-zero bi-ideal of S and let $a(\neq 0) \in A$. Then, by assumption, $S = (aSa] \subseteq (aaSaa] \subseteq (a(aaS)aaa] \subseteq (ASASA] \subseteq (A] = A$. Thus $A = S$ and hence S is B-simple. □

Lemma 2.3. *A ternary subsemiring B of a regular ordered ternary semiring S is a bi-ideal of S if and only if $B = (BSB]$.*

Proof: If $B = (BSB]$, then it is easy to see that B is a bi-ideal of S. Conversely, suppose that B is a bi-ideal of a regular ternary semiring

S. Let $b \in B$. Then there exists $x \in S$ such that $b \leq bxb$. This implies that $b \in (BSB]$ and hence $B \subseteq (BSB]$. Again, $(BSB] \subseteq ((BSB]SB] \subseteq (BSBSB] \subseteq (B] = B$.

Proposition 2.4. *Let S be an ordered ternary semiring and $a \in S$. Then the principal bi-ideal generated by a is given by $< a >_b = (\{na + ma^3 + \sum as_1as_2a \;/ s_1, s_2 \in S \text{ and } n, m \in \mathbb{Z}_0^+\}]$, where $\sum$ denotes the finite sum.*

Proof: Let $a \in S$ and $T = (\{na+ma^3+\sum as_1as_2a \;/ s_1, s_2 \in S \; and \; n, m \in \mathbb{Z}_0^+\}]$, where $\sum$ denotes the finite sum. Then T is a bi-ideal of S and contained in $< a >_b$. But $< a >_b$ is the smallest bi-ideal of S containing a. Thus $< a >_b = T$. Hence $< a >_b = (\{na + ma^3 + \sum as_1as_2a \;/ s_1, s_2 \in S \; and \; n, m \in \mathbb{Z}_0^+\}]$. □

Proposition 2.5. *Let S be a regular ordered ternary semiring and $a \in S$. Then $< a >_b = (aSa]$.*

Proof. Clearly $< a >_b \subseteq (aSa]$. Now $(aSa] \subseteq (< a >_b S < a >_b] = (< a >_b] =< a >_b$ by Lemma 2.3. Hence $< a >_b = (aSa]$. □

Proposition 2.6. *Let S be a regular ordered ternary semiring. Then the following conditions are equivalent:*
i) $(xSxSx] = (xSS] = (SSx] = (SxS] = (xSx]$ for any $x \in S$,
ii) Every bi-ideal of S is an ideal of S.

Proof. i)⇒ ii) It is trivial.
ii)⇒ i) Let $x \in S$. Clearly $(xSx] \subseteq (SSx]$. Since S is regular and every bi-ideal is an ideal of S, we have $(SSx] \subseteq (SS(xSx)] \subseteq (SS(xSx]] \subseteq (xSx]$. Thus $(SSx] = (xSx]$. In a similar manner, we can get $(SSx] = (xSx] = (xSS]$. Also $(SxS] \subseteq (S(xSx]S] \subseteq (xSx]$ and $(xSx] \subseteq (x(SxS]x] \subseteq (SxS]$. Thus $(SxS] = (xSx]$. Clearly $(xSxSx] \subseteq (xSx]$. Since S is regular, we have $(xSx] \subseteq (xSxSx]$. Thus $(xSxSx] = (xSx]$. Hence $(xSxSx] = (xSS] = (SSx] = (SxS] = (xSx]$ for any $x \in S$. □

Proposition 2.7. *Let S be an ordered ternary semiring. Then the following conditions are equivalent:*
i) S is regular and every bi-ideal is an ideal of S,
ii) $< xyz >_b =< x >_b \cap < y >_b \cap < z >_b$ for any $x, y, z \in S$.

Proof. i)⇒ ii) Let $x, y, z \in S$ and let $t \in < xyz >_b$. Then $t \in (xyzSxyz] \subseteq ((xSS]S(xSS]]$. By Proposition 2.6, we have $(xSS] = (xSxSx] =< x >_b$. Then $t \in (< x >_b S < x >_b] \subseteq (< x >_b] =< x >_b$. Similarly, we have

$t \in < y >_b$ and $t \in < z >_b$. Thus $< xyz >_b \subseteq < x >_b \cap < y >_b \cap < z >_b$. Let $t \in < x >_b \cap < y >_b \cap < z >_b$. Then $t \in < t >_b = (tSt] = (tStSt] \subseteq (xSS]S(ySS]S(zSS] \subseteq (x(SyS]](zSS] \subseteq (x(ySS]](zSS] = (xy(SSz]SS] \subseteq (xyzSS] \subseteq (< xyz >_b SS] \subseteq (< xyz >_b] = < xyz >_b$. Thus $t \in < xyz >_b$. Hence $< xyz >_b = < x >_b \cap < y >_b \cap < z >_b$ for any $x, y, z \in S$.

ii)$\Rightarrow$ i) By assumption, for any $x \in S$, we have $< x^3 >_b = < x >_b$. Then $x \leq nx^3 + mx^5 + x^3 s_1 x^3 s_2 x^3 \in xSx$ for some $n, m \in Z_0^+$ and $s_1, s_2 \in S$. Thus S is regular.

Let $a \in S$ and let $x \in (aSS]$. Then $x \leq \sum_{i=1}^{n} a s_i s_i^{'}$. Now $< a s_i s_i^{'} >_b = < a >_b \cap < s_i >_b \cap < s_i^{'} >_b = < a >_b \cap < s_i >_b \cap < a >_b \cap < s_i^{'} >_b \cap < a >_b = < a s_i a s_i^{'} a >_b \subseteq (aSaSa]$ for each i. Thus $(aSS] \subseteq (aSaSa]$. Obviously $(aSaSa] \subseteq (aSS]$. Hence $(aSS] = (aSaSa]$. In a similar way, $(SSa] = (SaS] = (aSS]$ holds. By Proposition 2.6, we have every bi-ideal of S is an ideal of S. □

Proposition 2.8. *Let S be an ordered ternary semiring and let P be a bi-ideal of S. If P is prime, then P is a left or right or lateral ideal of S.*

Proof. Suppose that P is a prime bi-ideal of S. Clearly $(PSS](SPS](SSP] \subseteq (PSPSP] \subseteq P$. By assumption, we have $(PSS] \subseteq P$ or $(SPS] \subseteq P$ or $(SSP] \subseteq P$. Hence P is a left or a right or a lateral ideal of S. □

Lemma 2.9. *Let S be a commutative ordered ternary semiring and P be a bi-ideal of S. Then P is prime if and only if $xyz \in P$ implies $x \in P$ or $y \in P$ or $z \in P$.*

Proof. Let P be a prime bi-ideal and $xyz \in P$. By Proposition 2.8, we have P is an ideal of S. Let $t \in < x >_b < y >_b < z >_b$. Then

$$t \leq \sum_{i=1}^{n} [nx + mx^3 + x s_i x s_i^{'} x][n_1 y + m_1 y^3 + y t_i y t_i^{'} y][n_2 z + m_2 z^3 + z u_i z u_i^{'} z]$$

where $n, n_1, n_2, m, m_1, m_2 \in Z$ and $s_i, s_i^{'}, t_i, t_i^{'}, u_i, u_i^{'} \in S$. Since $xyz \in P$ and P is an ideal of S, we have $t \in P$. Thus $< x >_b < y >_b < z >_b \subseteq P$. By assumption, we have $x \in P$ or $y \in P$ or $z \in P$. The converse is trivial. □

Proposition 2.10. *Let S be a regular commutative ordered ternary semiring and let P be a bi-ideal of S. Then the following conditions are equivalent*

i) P is prime bi-ideal,

ii) $abc \in P$ implies $a \in P$ or $b \in P$ or $c \in P$,

iii) P is weakly prime.

Proof. $i) \Rightarrow ii)$ It follows from Lemma 2.9

$ii) \Rightarrow i)$ and $i) \Rightarrow iii)$ are trivial.

$iii) \Rightarrow ii)$ Let $a, b, c \in S$ such that $abc \in P$. Suppose $a, b, c \notin P$. Then $(P+SSa]$; $(P+SSb]$ and $(P+SSc]$ are bi-ideals of S and $P \subset (P+SSa]$; $P \subset (P+SSb]$ and $P \subset (P+SSc]$. By assumption, we have $(P+SSa](P+SSb](P+SSc] \nsubseteq P$. But $(P+SSa](P+SSb](P+SSc] \subseteq (PPP+PSSbP+SSaPP+SSaSSbP+PPSSc+PSSbSSc+SSaPSSc+SSaSSbSSc] \subseteq (P+S(abc)S] \subseteq (P+SPS] \subseteq P$ by Lemma 2.3. Thus $(P+SSa](P+SSb](P+SSc] \subseteq P$, a contradiction. Hence $a \in P$ or $b \in P$ or $c \in P$. □

Proposition 2.11. *Let S be an ordered ternary semiring and P be a bi-ideal of S. Then P is prime if and only if $(xSxSx](ySySy](zSzSz] \subseteq P$ implies $x \in P$ or $y \in P$ or $z \in P$.*

Proof. Let P be a prime bi-ideal of S and let $(xSxSx](ySySy](zSzSz] \subseteq P$ for some $x, y, z \in S$. Since $(xSxSx], (ySySy]$ and $(zSzSz]$ are bi-ideals of S, we have $xSxSx \subseteq P$, or $ySySy \subseteq P$ or $zSzSz \subseteq P$.

If $xSxSx \subseteq P$, then by Lemma 2.1 we have $< x >_b S < x >_b S < x >_b \subseteq P$. By Corollary 3.20 of [2], we have $x \in P$. In a similar manner we have $y \in P$ or $z \in P$. The converse is clear. □

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