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Friends of Products of Two Primes and of Four Times an Odd Prime

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Abstract

Given a positive integer $m = p_1p_2$, p_1, p_2 primes, we give necessary conditions for the existence of an integer $n \neq m$ which has the same abundancy index as m. This work is then applied to give more stringent conditions in the case $m = 33$. Then we give a similar treatment of the cases $m = 4p$, p an odd prime, with application to $m = 20$.

1 Introduction

Definition 1.1. *Let n be a positive integer. The abundancy index $I(n) = \frac{\sigma(n)}{n}$ is the quotient of the divisor function, $\sigma(n) = \sum_{d>0, d|n} d$, and n.*

Let m and n be positive integers and consider only positive primes p.

1. $I(n) \geq 1$, with equality iff $n = 1$.
2. If $m|n$, then $I(m) \leq I(n)$ with equality iff $m = n$.

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3. If $p_1, ..., p_k$ are distinct primes and $e_1, ..., e_k$ are positive integers, then

$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k}\left(\frac{p_j^{e_j+1}-1}{p_j^{e_j}(p_j-1)}\right) = \prod_{j=1}^{k}\left(\frac{\sum_{i=0}^{e_j} p_j^i}{p_j^{e_j}}\right)$$

which follows from the relation for σ

$$\sigma\left(\prod_{j=1}^{k} p_j^{e_j}\right) = \prod_{j=1}^{k}\left(\frac{p_j^{e_j+1}-1}{p_j-1}\right)$$

4. I is weakly multiplicative (if $\gcd(m,n) = 1$, then $I(mn) = I(m)I(n)$).

5. Suppose that $p_1, ..., p_k$ are distinct primes, $q_1, ..., q_k$ are distinct primes, $e_1, ..., e_k$ are positive integers, and for all $j \in \{1, ..., k\}, p_j \leq q_j$. Then

$$I\left(\prod_{j=1}^{k} p_j^{e_j}\right) \geq I\left(\prod_{j=1}^{k} q_j^{e_j}\right)$$

with equality iff $p_j = q_j$, for all $j \in \{1, ..., k\}$.

6. If the distinct prime factors of n are $p_1, ..., p_k$, then

$$\prod_{j=1}^{k} \frac{p_j+1}{p_j} \leq I(n) < \prod_{j=1}^{k} \frac{p_j}{p_j-1}.$$

Proofs of above properties and other related references can be found in [1], [2], and [3].

Definition 1.2. *Positive integers m and n are **friends** if $m \neq n$ and $I(m) = I(n)$.*

Observe that properties 2 and 3 above imply that prime powers have no friends.

Suppose that p is a prime and k is a positive integer. If n is a positive integer and

$$\frac{\sigma(n)}{n} = I(n) = I(p^k) = \frac{p^{k+1}-1}{p^k(p-1)},$$

then $p^k(p-1)\sigma(n) = (p^{k+1}-1)n$. Since p^k and $p^{k+1}-1$ are relatively prime, it follows that $p^k|n$, and then property 2 implies that $p^k = n$.

The next question, which inspired [1], [2], and [3], to which we are indebted, is: if m has exactly two prime divisors, can m have a friend? The simple answer is : yes; $6 = 2 \cdot 3$ is the first of the perfect numbers, which are integers n for which $I(n) = 2$. By a theorem of Euler, 6 is the friend of every $n = 2^{q-1}(2^q - 1)$ when q is a prime and $2^q - 1$ is also a prime — these

are called Mersenne primes — and only of such n. Thus the friends of 6 are all themselves products of two prime powers — a power of 2 times a related Mersenne prime.
It was known in antiquity that integers of this form are perfect. Euler's contribution was to prove the converse, that every even perfect number is of that form. It is still not known if there are infinitely many of them. It is not known if there are any odd perfect numbers, but it is known that any odd perfect number will have many more than two distinct prime factors.

In the process of extracting necessary conditions for an integer n to be a friend of 12, Kim [2] actually found such a friend, 234. This discovery, as well as the form of the even perfect numbers, should excite interest in the friendliness of integers $2^k p$ when k is a positive integer and p is an odd prime.

We will generalize some of the results in [1] and [3] to the case $m = p_1p_2 \neq 6$, $p_1 < p_2$, prime, and then apply our results to the case $m = 33$. Then we will generalize some of the results in [2] to the case $m = 4p$, p an odd prime, $p \neq 7$, and apply these results to the case $m = 20$.

2 Friends of p_1p_2

Throughout this section, $m = p_1p_2$, p_1, p_2 prime, $p_1 < p_2$, $\{p_1, p_2\} \neq \{2, 3\}$. We have $I(m) = \frac{(1+p_1)(1+p_2)}{p_1p_2}$. If $I(n) = \frac{\sigma(n)}{n} = I(m)$, then $p_1p_2\sigma(n) = (1+p_1)(1+p_2)n$. Assume that $I(n) = I(m)$ and $n \neq m$.

Proposition 2.1. *$p_2|n$.*

Proof. Since $p_2 > p_1$ and $\{p_1, p_2\} \neq \{2, 3\}$, $p_2 \geq p_1 + 2 > p_1 + 1$. Therefore, $p_2 \nmid (1+p_1)(1+p_2)$. Thereby, $p_2|n$. □

Proposition 2.2. *All prime factors of n are greater than p_1.*

Proof. Let k be a prime, $k \leq p_1$ and suppose $k|n$. Then $kp_2|n$, which implies $I(n) \geq I(kp_2) \geq I(p_1p_2)$ (by property 5 from section 1) with equality only at $n = p_1p_2$. Thereby, $k \nmid n$. Thereby, all prime factors of n must be greater than p_1. □

Corollary 2.1. *$p_1 \nmid n$.*

Proposition 2.3. *$p_1|(1+p_2)$.*

Proof. Since $p_1 \nmid (1+p_1)n$, we must have $p_1|(1+p_2)$. □

Proposition 2.4. *n has at least 3 distinct prime divisors, except, possibly, when $p_1 = 2$, in which case $n = q^b p_2^a$ is a friend of $m = 2p_2$, where q is a prime, $q \neq p_2$, and a, b are positive integers, if and only if*

1. *$q = 3$,*
2. *$a = 2$, and*
3. *$3^{b+1} - 1 = p_2(1 + p_2)$.*

Remarks: Recall our blanket assumption that if $p_1 = 2$ then $p_2 > 3$. We do not know of any prime $p > 3$ such that for some positive integer b, $n = 3^b p^2$ is a friend of $m = 2p$ — i.e., we know of no such p and b such that $3^{b+1} - 1 = p(1 + p)$ — but we have not been able to prove non-existence.

Proof. To be a friend of m, n must have at least two distinct prime divisors, because a prime power has no friends. If n has only 2 distinct prime divisors, then one of them is p_2, by Proposition 2.1. Suppose that $q \neq p_2$ is a prime such that $n = p_2^a q^b$ is a friend of $m = p_1 p_2$. Then $q > p_1$ by Proposition 2.2.

By property 6 from the Introduction,

$$I(m) = I(n) = \frac{p_1 + 1}{p_1}\frac{p_2 + 1}{p_2} < \frac{q}{q - 1}\frac{p_2}{p_2 - 1}$$

$$\Rightarrow \frac{q}{q - 1} > \frac{p_1 + 1}{p_1}\frac{p_2^2 - 1}{p_2^2}$$

If $q \geq p_1 + 2$, then $\frac{p_1+2}{p_1+1} \geq \frac{q}{q-1} > \frac{p_1+1}{p_1}\frac{p_2^2-1}{p_2^2}$. Thus

$$1 - \frac{1}{(p_1 + 1)^2} = \frac{p_1^2 + 2p_1}{(p_1 + 1)^2} > \frac{p_2^2 - 1}{p_2^2} = 1 - \frac{1}{p_2^2}.$$

As a result

$$p_2 < p_1 + 1, \text{ contrary to our supposition about } p_1 \text{ and } p_2.$$

Therefore, what we have supposed about p_1, p_2 and q imply that $q = p_1 + 1$, which, since p_1 and q are primes, implies that $p_1 = 2$ and $q = 3$.

In the remainder of the proof, let $p_2 = p$. Thus, $m = 2p, n = 3^b p^a$, and

$$I(m) = I(n) = \frac{3}{2}\frac{p + 1}{p} = \frac{3^{b+1} - 1}{2 \cdot 3^b}\frac{p^{a+1} - 1}{p^a(p - 1)}$$

$$\Rightarrow \frac{3^{b+1}}{3^{b+1} - 1} = \frac{p^{a+1} - 1}{p^{a-1}(p^2 - 1)}. \tag{1}$$

If $a = 1$, then this last equation becomes $\frac{3^{b+1}}{3^{b+1}-1} = 1$, clearly false for every positive integer b. Therefore, $a \geq 2$.

Subtracting 1 from both sides of eq. 1, we obtain

$$\frac{1}{3^{b+1}-1} = \frac{p^{a-1}-1}{p^{a-1}(p^2-1)}. \tag{2}$$

Clearly the fraction $\frac{1}{3^{b+1}-1}$ is in lowest terms. If $a-1=2t$ for some integer $t \geq 1$, then

$$\frac{p^{a-1}-1}{p^{a-1}(p^2-1)} = \frac{\sum_{k=0}^{t-1} p^{2k}}{p^{a-1}},$$

also in lowest terms. The two fractions in (2) cannot be equal, because $3^{b+1}-1$ is even and p^{a-1} is odd.

We conclude that $a-1=2t-1$ for some integer $t \geq 1$. Then

$$\frac{p^{a-1}-1}{p^{a-1}(p^2-1)} = \frac{p^{a-2}+\cdots+1}{p^{a-1}(p+1)} = \frac{\sum_{k=0}^{2t-2} p^k}{p^{a-1}(p+1)}.$$

If $t>1$, then $\sum_{k=0}^{2t-2} p^k = (p+1)(\sum_{r=1}^{t-1} p^{2r-1})+1$. Therefore

$$\frac{1}{3^{b+1}-1} = \frac{(p+1)(\sum_{r=1}^{t-1} p^{2r-1})+1}{p^{a-1}(p+1)},$$

and both sides of this equation are fractions in lowest terms. But then they cannot be equal, because

$$(p+1)\sum_{r=1}^{t-1} p^{2r-1}+1 > 1.$$

Therefore, $t=1$, $a=2t=2$, and we have $\frac{1}{3^{b+1}-1} = \frac{p-1}{p(p^2-1)} = \frac{1}{p(p+1)}$.
Consequently, $p(p+1) = 3^{b+1}-1$.

Conversely, if $p>3$ is prime, b is a positive integer, and $p(p+1)=3^{b+1}-1$, then it can be straightforwardly verified that $I(2p) = I(3^b \cdot p^2)$. □

3 An Application to m = 33

$33 = 3 \cdot 11$: thereby if 33 has a friend n, we know that:

- $11|n$ by Proposition 2.1.
- $2 \nmid n, 3 \nmid n$ by Proposition 2.2.
- n has $k \geq 3$ distinct prime factors by Proposition 2.4.

Note $I(33) = \frac{16}{11}$. Assume that $n \neq 33$ is a friend of 33. So $11\sigma(n) = 16n$.

Proposition 3.1. $3 \nmid \sigma(n)$.

Proof. Suppose $3|\sigma(n)$. Then $\sigma(n) = 3x$ for some integer x. Thus, $33x = 16n$; but this would require $3|n$, whereas we know $3 \nmid n$. Therefore our supposition was false and $3 \nmid \sigma(n)$. □

Proposition 3.2. *All prime factors of n congruent to 2 modulo 3 have an even power in the factorization of n.*

Proof. By Proposition 3.1, we know that $3 \nmid \sigma(n)$. Let us consider each prime factor of n congruent to 2 modulo 3. Suppose that $q \equiv 2 \mod 3$ is a prime, e is a positive integer, and $q^e \| n$. Then $\sigma(q^e)|\sigma(n)$. If e is odd, then $\sigma(q^e) = 1 + q + \cdots + q^e \equiv 1 + 2 + \cdots + 2 \equiv \frac{e+1}{2} \cdot 3 \equiv 0 \mod 3$, which would imply that $3|\sigma(n)$. Therefore, e is even. □

Proposition 3.3. *n has $k \geq 4$ distinct prime factors.*

Proof. Suppose n has only 3 prime factors. Then let $n = 11^a \cdot p_2^b \cdot p_3^c$. Let $p_2 < p_3$. Suppose $p_2 \geq 7$; then $I(n) < \frac{7}{6}\frac{11}{10}\frac{13}{12} < \frac{16}{11}$. Therefore, $p_2 = 5$. We will now constrain p_3. We must have $\frac{11}{10}\frac{5}{4}\frac{p_3}{p_3-1} > \frac{16}{11}$. Therefore $p_3 < 19$ and thus $p_3 \in \{7, 13, 17\}$. If $p_3 = 7$, then $I(n) \geq I(5^2 \cdot 7 \cdot 11^2) > \frac{16}{11}$ by Proposition 3.2. Therefore, $p_3 \neq 7$. By the same logic, $p_3 \neq 13$. Therefore, $p_3 = 17$. If the exponent of 5 is 2, then $I(n) \leq I(5^2)\frac{11}{10}\frac{17}{16} < \frac{16}{11}$. Therefore, the exponent of 5 is greater than or equal to 4. However, this causes $I(n) \geq I(5^4 \cdot 11^2 \cdot 17^2) > \frac{16}{11}$. Therefore, $p_3 \neq 17$. Thus $p_2 \neq 5$, and $k > 3$. □

Corollary 3.1. *If n is friend of 33, then $n >$ 209,209.*

Proof. We know that n has at least 4 prime factors, including 11, and excluding 2 and 3. As well, all primes congruent to 2 modulo 3 must be raised to an even power. Therefore, $n \geq 7 \cdot 11^2 \cdot 13 \cdot 19 = 209,209$. □

4 Friends of $4p$

Suppose $m = 4p$, where p is an odd prime. When $p = 7 = 2^3 - 1$, the second Mersenne prime (after 3), we have $m = 2^{3-1}(2^3 - 1)$ is a perfect number, which has many friends, as discussed in the Introduction.

In the rest of this section,p is an odd prime, $p \neq 7$, and $n \neq m = 4p$ is friend of m. Therefore

$$I(m) = \frac{7(1+p)}{4p} = I(n)$$

which implies

$$4p\sigma(n) = 7(1+p)n.$$

Proposition 4.1. $p|n$

Proof. Since $p \neq 7$, p and $7(1+p)$ are relatively prime. Since $p|7(1+p)n$, $p|n$. □

Proposition 4.2. $4 \nmid n$

Proof. Suppose $4|n$. Then, $4p|n$. Therefore, $I(n) \geq I(m)$, with equality only at $m = n$. Therefore, our supposition was false and $4 \nmid n$. □

Proposition 4.3. $7|\sigma(n)$

Proof. Since p is prime and $p \neq 7$, $7 \nmid p$. In addition, $7 \nmid 4$. Therefore, $7|\sigma(n)$. □

We will use k to denote the number of distinct prime factors of n.

Proposition 4.4. $k > 2$

Proof. $k > 1$ because no prime power can have a friend. Suppose n has only two prime divisors. By Proposition 4.1, one of them is p. Let q be the other one. Let a, b be positive integers such that $n = p^a q^b$.

Case 1. $p = 3$. Then $m = 12$, so

$$I(m) = \frac{7.4}{12} = \frac{7}{3} < \frac{3}{2}\frac{q}{q-1} \Rightarrow q < \frac{14}{5} \Rightarrow q = 2.$$

By Proposition 4.2 we then have $n = 2 \cdot p^a = 2 \cdot 3^a$ for some integer $a > 0$. Then

$$\frac{7}{3} = I(n) = \frac{3}{2}\frac{3^a + \cdots + 1}{3^a} \Rightarrow 7 = \frac{3^{a+1} + \cdots + 3}{2 \cdot 3^{a-1}}.$$

There are many reasons why there is no integer a that could satisfy this last equation. We leave it to the reader to finish the proof that this case is impossible.

Case 2. $p = 5$. The argument is similar to that in Case 1:

$$m = 20 \Rightarrow I(m) = \frac{7.6}{20} = \frac{21}{10} = I(n) < \frac{5}{4}\frac{q}{q-1} \Rightarrow q < \frac{42}{17} \Rightarrow q = 2.$$

Again invoking Proposition 4.2, we have $n = 2 \cdot 5^a$ for some integer $a > 0$, whence

$$\frac{21}{10} = I(n) = \frac{3}{2}\frac{5^a + \cdots + 1}{5^a} \Rightarrow 21 = \frac{3(5^a + \cdots + 1)}{5^{a-1}} \Rightarrow a = 1,$$

on the grounds that the fractions on both sides of the equation above are in lowest terms. But then the equation becomes $21 = 3 \cdot 6$, which is not true.

Case 3. $p \geq 11$. (Since p is an odd prime and $p \neq 7$, disposal of this case will finish the proof). The subcase $q = 2$ can be disposed of as above: $q = 2 \Rightarrow n = 2 \cdot p^a$ for some integer $a > 0 \Rightarrow$

$$\frac{7(p+1)}{4p} = I(n) = \frac{3}{2}\frac{p^a + \cdots + 1}{p^a} \Rightarrow 7\frac{p+1}{2} = \frac{3(p^a + \cdots + 1)}{p^{a-1}} \Rightarrow a = 1 \Rightarrow 7 = 6,$$

which is not true. If $q \geq 3$, then

$$I(4p) = \frac{7(p+1)}{4p} = I(n) < \frac{q}{q-1}\frac{p}{p-1} \leq \frac{3}{2}\frac{11}{10} < \frac{7}{4} = I(4) < I(4p).$$

This contradiction concludes the proof.

□

Kim's discovery [2], that $234 = 2 \cdot 3^2 \cdot 13$ is a friend of $12 = 4 \cdot 3$ shows that the conclusion of Proposition 4.4 is "sharp". We strongly suspect that 3 is the only odd prime p not equal to 7 such that there is a friend of $4p$ with exactly 3 distinct prime divisors, but a proof of such a general statement is beyond us, at present. However, we shall verify our conjecture in the case $p = 5$ in the next section.

By the way, Kim proved that any friend of 12 other than 234 would have to have at least five distinct prime divisors, four others besides 3.

5 Application to Friends of 20

Theorem 5.1. *If n is a friend of 20, then n has $k \geq 5$ distinct prime factors, including 2 and 5. As well, 3 and 7 are not factors of n. Lastly, all odd prime factors of n must be raised to an even power in the factorization of n.*

The proof will be the outcome of a series of propositions.

Proposition 5.2. *$2|n$ and $5|n$, but $3 \nmid n$ and $4 \nmid n$.*

Proof. Since $I(20) = \frac{21}{10}$, $10\sigma(n) = 21n$ and therefore $10|n$. Since $10|n$, both $2|n$ and $5|n$ must hold. $4 \nmid n$ by proposition 4.2. If $3|n$, then $30|n$ and by Property 2, $I(n) \geq I(30) = \frac{12}{5} \geq \frac{21}{10} = I(20)$, so $3 \nmid n$. □

Proposition 5.3. *$2 \nmid \sigma(n)$.*

Proof. Suppose $2|\sigma(n)$, then $\sigma(n) = 2x$ for some integer x. Thus, $20x = 21n$ but this would require $4|n$; however we know $4 \nmid n$. Therefore our supposition was false and $2 \nmid \sigma(n)$. □

Proposition 5.4. *All odd prime factors of n have an even power in the factorization of n.*

Proof. By Proposition 5.3, we know that $\sigma(n) \equiv 1 \pmod 2$. Suppose that q is an odd prime, e is a positive integer, and $q^e \| n$. Then $\sigma(q^e) | \sigma(n)$, so $\sigma(q^e)$ is odd. Since $\sigma(q^e) = q^e + \cdots + 1 \equiv e+1 \mod 2$, e must be even. ☐

Proposition 5.5. $7 \nmid n$.

Proof. Suppose $7|n$. Then n must have $2 \cdot 5^2 \cdot 7^2 = 2450$ as a factor. Therefore $I(n) \geq I(2450) \approx 2.16 > \frac{21}{10}$ and thus n would not be a friend of 20. Therefore, $7 \nmid n$. ☐

Proposition 5.6. $k > 2$.

Proof. Application from the general case in Proposition 4.5. ☐

Proposition 5.7. $k > 3$.

Proof. Suppose instead n has only 3 prime factors. Then $I(n) < \frac{3}{2}\frac{5}{4}\frac{11}{10} < \frac{21}{10}$. Therefore, n must have more than 3 prime factors. ☐

Proposition 5.8. $k > 4$.

Proof. Let $n = 2 \cdot 5^a \cdot p_3^b \cdot p_4^c$, where p_3 and p_4 are primes, and $7 < p_3 < p_4$. If $p_3 \geq 19$, then $I(n) \leq \frac{3}{2}\frac{5}{4}\frac{19}{18}\frac{23}{22} \approx 2.07$. Therefore, $p_3 \in \{11, 13, 17\}$. Let us now consider the three cases:

Case 1. $p_3 = 11$. Let us consider all p_4 for which $\frac{3}{2}\frac{5}{4}\frac{11}{10}\frac{p_4}{p_4-1} \geq \frac{21}{10}$. We note that for p_4 to satisfy this condition, $p_4 < 56$. Therefore, $p_4 \in \{13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53\}$. We also must have $I(2 \cdot 5^2 \cdot 11^2 \cdot p_4^2) \leq \frac{21}{10}$. This implies $p_4 \geq 41$. Therefore, $p_4 \in \{41, 43, 47, 53\}$. Suppose the exponent of 5 is 2. Then for all $p_4 \geq 41$, $I(n) < I(2 \cdot 5^2) \cdot \frac{11}{10} \cdot \frac{p_4}{p_4-1} < \frac{21}{10}$. Therefore, the exponent of 5 is at least 4. With that condition, $I(n) \geq I(2 \cdot 5^4 \cdot 11^2 \cdot 53^2) > \frac{21}{10}$. Therefore, there is no possible value of p_4 for which n is a friend. Therefore, $p_3 \neq 11$.

Case 2. $p_3 = 13$. Let us again consider all p_4 for which $\frac{3}{2}\frac{5}{4}\frac{13}{12}\frac{p_4}{p_4-1} > \frac{21}{10}$. We note that for p_4 to satisfy this condition, $p_4 < 31$. Therefore, $p_4 \in \{17, 19, 23, 29\}$. We also must have $I(2 \cdot 5^2 \cdot 13^2 \cdot p_4^2) \leq \frac{21}{10}$. This implies that $p_4 > 23$. Therefore, we must have $p_4 = 29$. Suppose the exponent of 5 is 2. Then $I(n) < I(2 \cdot 5^2) \cdot \frac{13}{12}\frac{29}{28} < \frac{21}{10}$. Therefore, the exponent of 5 is at least 4. Then $I(n) \geq I(2 \cdot 5^4 \cdot 13^2 \cdot 29^2) > \frac{21}{10}$. Therefore, there is no possible value of p_4 for which n is a friend. Therefore $p_3 \neq 13$.

Case 3. $p_3 = 17$. Let us suppose $p_4 \geq 23$, then $I(n) < \frac{3}{2}\frac{5}{4}\frac{17}{16}\frac{23}{22} \approx 2.08$. Therefore, the only option is $p_4 = 19$. Let us now consider $n = 2 \cdot 5^{2e_5} \cdot 17^{2e_{17}} \cdot 19^{2e_{19}}$. Suppose the exponent of 5 is at least 4, then $I(n) \geq I(2 \cdot 5^4 \cdot 17^2 \cdot 19^2) \approx 2.101 > \frac{21}{10}$. Therefore, the exponent of 5 is 2 and we can consider $n = 2 \cdot 5^2 \cdot 17^{2e_{17}} \cdot 19^{2e_{19}}$. $I(n) < \frac{3}{2}\frac{31}{25}\frac{17}{16}\frac{19}{18} \approx 2.086$. Therefore, $I(n) < \frac{21}{10}$ and thus we have reached a contradiction. Therefore, $p_3 \neq 17$.

Case 4. $p_3 = 11$. Let us consider all p_4 for which $\frac{3}{2}\frac{5}{4}\frac{11}{10}\frac{p_4}{p_4-1} \geq \frac{21}{10}$. We note that for p_4 to satisfy this condition $p_4 < 56$. Therefore, $p_4 \in \{13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53\}$. We also must have $I(2 \cdot 5^2 \cdot 11^2 \cdot p_4^2) \leq \frac{21}{10}$. This implies $p_4 \geq 41$. Therefore, $p_4 \in \{41, 43, 47, 53\}$. Suppose the exponent of 5 is 2. Then for all $p_4 \geq 41$, $I(n) < I(2 \cdot 5^2) \cdot \frac{11}{10}\frac{p_4}{p_4-1} < \frac{21}{10}$. Therefore, the exponent of 5 is at least 4. With that condition, $I(n) \geq I(2 \cdot 5^4 \cdot 11^2 \cdot 53^2) > \frac{21}{10}$. Therefore, there is no possible value of p_4 for which n is a friend. Therefore, $p_3 \neq 11$.

Case 5. $p_3 = 13$. Let us again consider all p_4 for which $\frac{3}{2}\frac{5}{4}\frac{13}{12}\frac{p_4}{p_4-1} > \frac{21}{10}$. We note that for p_4 to satisfy this condition, $p_4 < 31$. Therefore, $p_4 \in \{17, 19, 23, 29\}$. We also must have $I(2 \cdot 5^2 \cdot 13^2 \cdot p_4^2) \leq \frac{21}{10}$. This implies that $p_4 > 23$. Therefore, we must have $p_4 = 29$. Suppose the exponent of 5 is 2. Then $I(n) < I(2 \cdot 5^2)\frac{13}{12}\frac{29}{28} < \frac{21}{10}$. Therefore, the exponent of 5 is at least 4. Then $I(n) \geq I(2 \cdot 5^4 \cdot 13^2 \cdot 29^2) > \frac{21}{10}$. Therefore, there is no possible value of p_4 for which n is a friend. Therefore $p_3 \neq 13$.

Case 6. $p_3 = 17$. Let us suppose $p_4 \geq 23$, then $I(n) < \frac{3}{2}\frac{5}{4}\frac{17}{16}\frac{23}{22} \approx 2.08$. Therefore, the only option is $p_4 = 19$. Let us now consider $n = 2 \cdot 5^{2e_5} \cdot 17^{2e_{17}} \cdot 19^{2e_{19}}$. Suppose the exponent of 5 is at least 4, then $I(n) \geq I(2 \cdot 5^4 \cdot 17^2 \cdot 19^2) \approx 2.101 > \frac{21}{10}$. Therefore the exponent of 5 is at least 2 and we can consider $n = 2 \cdot 5^2 \cdot 17^{2e_{17}} \cdot 19^{2e_{19}}$. $I(n) < \frac{3}{2}\frac{31}{25}\frac{17}{16}\frac{19}{18} \approx 2.086$. Therefore, $I(n) < \frac{21}{10}$ and thus we have reached a contradiction. Therefore, $p_3 \neq 17$.

□

Corollary 5.2. *If 20 has a friend n, then $n \geq 295{,}488{,}050$.*

Proof. We know that n has at least 5 prime factors, including 2 and 5, and excluding 3 and 7. As well, all odd primes must be raised to an even power. Therefore, $n \geq 2 \cdot 5^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 = 295{,}488{,}050$. □

Quote. In the words of G. Randolf, "Good friends are hard to find, harder to leave, and impossible to forget."

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