International Journal of Mathematics and
Computer Science, **14**(2019), no. 4, 809–819

Order Graph: A new representation of finite groups

Bilal N. Al-Hasanat, Ahmad S. Al-Hasanat

Department of Mathematics
Al Hussein Bin Talal University
Ma'an, Jordan

email: bilal_hasanat@yahoo.com, hasanat85@yahoo.com

(Received April 25, 2019, Accepted June 15, 2019)

Abstract

There are many ways to associate a graph with a finite group. Such an association yields many of the group properties. The link established between a graph and a group is usually determined by the definition of the adjacent vertices. In this research, the orders of the elements will take a place in the graph creation. The order graph of a finite group is the directed graph whose vertices are the elements of the group order classes, and for two distinct vertices x and y there is an arc from x to y if and only if x divides y. This paper will cover the creation of the order graph in general and then concentrate on some groups.

1 Introduction

Graph theory is a helpful tool in studying groups properties. This link was introduced by A. Cayley, in 1878, in which a graph represents a finite group. This graph is associated with a group G and a subset A of G. The set of vertices of this graph is the set of elements of G and two vertices x and y are adjacent if and only if there exists $a \in A$ such that $y = ax$. The represented graph is called the Cayley graph of G. There are some other well-known graphs associated with finite groups such as the power graph which has been

Key words and phrases: Element order, Order classes, Graph, Weighted graph.
AMS (MOS) Subject Classifications: 20F99, 20E99, 20C99.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

introduced in [1]. It is the graph of a group G which has all elements of G as vertices and two distinct vertices are adjacent if and only if one of them is a power of the other. Some properties of the power graph were studied in [5]. Another case which has been used by many authors [3, 4, 2] is the commuting graph. For a nonempty subset H of a finite group G, the commuting graph on H has H as its vertices set and two distinct vertices are adjacent if and only if they commute.

The main aim of this paper is to introduce a new type of graphs associated with finite groups. This graph is called the order graph, where the order classes of the group are used in this association. Next, we will introduce the most important properties of such graphs when the associated groups are the prime order classes groups and the dihedral group of order 2^n , $n \geq 3$.

2 Basic definitions and notations

A graph can be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. In such a diagram one is mainly interested in whether or not two given points are joined by a line. This can be written formally using the next definition.

Definition 2.1. *A graph Γ is an ordered triple (V, E, ψ) consisting of a nonempty set V of vertices, a set E of edges and an incidence function ψ that associates with each edge of Γ a pair of vertices (not necessarily distinct) of Γ.*

Let u and v be two vertices and let c be an edge so that $\psi(c) = (u, v)$ or $\psi(c) = (v, u)$. Then we say that c joins u and v. A finite graph is a graph with finite set of vertices and finite set of edges. In this paper all graphs are finite.

To keep this research self contained, we list some basic concepts of graphs in the following definitions.

Definition 2.2. *Let $\Gamma(V, E, \psi)$ be a graph. Then*

- *The order of the graph Γ is the number of its vertices; i.e., $|V|$. The size of the graph Γ is the number of its edges; i.e., $|E|$.*

- *The graph Γ is said to be a directed graph (digraph) if when replacing the edges by directed edges (arrows, arcs) each connects an ordered pair of vertices. For the arrow c in E, with $\psi(c) = (u, v)$, we say that, c points from u (origin) and points to v (terminal).*

- *A subgraph* $\Upsilon(V^*, E^*, \psi^*)$ *of a graph* $\Gamma(V, E, \psi)$ *is a graph in which* $V^* \subseteq V$, $E^* \subseteq E$ *and* ψ^* *is the restriction of* ψ *on* V^*.

- *An edge* $c \in E$ *is called a loop if* $\psi(c) = (v, v)$ *for* $v \in V$.

- Γ *has multiple edges, if there exist two distinct edges* c *and* d *in* E*, such that* $\psi(c) = \psi(d) = (u, v)$ *for* $u, v \in V$.

- *The graph* Γ *is a simple graph if and only if it has no loops or multiple edges.*

- *Let* $v \in V$. *Then the degree of* v*, denoted by* $deg(v)$*, is the number of edges incident with it. If* Γ *is a digraph, then the indegree of* v *is the number of head ends adjacent to* v*, denoted by* $deg^-(v)$*, and the outdegree of* v *is the number of tail ends adjacent to* v*, denoted by* $deg^+(v)$. *If* $deg^-(v) = 0$ *and* $deg^+(v) = |E|$*, then* v *is called a sink. If* $deg^-(v) = |E|$ *and* $deg^+(v) = 0$*, then* v *is called a source.*

- *A path* P *in the graph* Γ *is a sequence of edges which connect a sequence of distinct vertices. Consequently, in a digraph* Γ*, a directed path is a sequence of arrows all of the same direction that connect a sequence of distinct vertices.*

- *A cycle in* Γ *is a path starting and ending in the same vertex.*

- Γ *is said to be connected graph if and only if there is a path between any two distinct vertices. Otherwise,* Γ *is disconnected graph.*

- *If removing a vertex* v *and all edges incident with it from* Γ *reduces to a disconnected graph, then* v *is called a cut vertex. A cut edge (bridge) is defined similarly by removing an edge.*

- *The graph* Γ *is called a tree graph if and only if* Γ *is a simple connected graph with no cycle. If* $deg(v) > 1$ ($deg(v) = 1$) *in a tree, then* v *is called a root (a leaf), respectively.*

The set of all possible orders for the elements of a finite group G is called the order classes of G, and denoted by $OC(G)$. This can be written as, $OC(G) = \{k \in \mathbb{N} \mid k = o(x)\ ,\ x \in G\}$.

3 The Order Graph

Let G be a finite group. Then $1 \leq o(x) \leq |G|$. Since $1 \in OC(G)$, the associated graph Γ from any finite group G, with $OC(G)$ is the set of vertices having at least one vertex.

We are now in a position to define our new associated graph from a finite group.

Definition 3.1. *The* **order graph** *(O-graph) of a finite group G is a digraph whose set of vertices is $OC(G)$ and for any two distinct vertices x and y there is an arc from x to y if and only if x divides y.*

The previous definition asserts that, the O-graph is a simple connected digraph with no loops. In addition, for an O-graph $\Gamma(V, E, \psi)$ or $\Gamma(V, E)$, associated to a group G, we have $1 \in OC(G)$. Thus $|V| = |OC(G)| \geq 1$. Moreover 1 will be linked to all other vertices. Therefore, if the group G is a non-trivial group, then $|OC(G)| > 1$ and consequently $|E| \geq 1$.

Example 3.1. Let $G = \mathbb{Z}_{10}$, the cyclic group of order 10. Then the O-graph of G is the graph $\Gamma(V, E, \psi)$ with $V = \{1, 2, 5, 10\}$ and $E = \{c_1, c_2, c_3, c_4, c_5\}$ where $\psi(c_1) = (1, 2)$, $\psi(c_2) = (1, 5)$, $\psi(c_3) = (1, 10)$, $\psi(c_4) = (2, 10)$, and $\psi(c_5) = (5, 10)$:

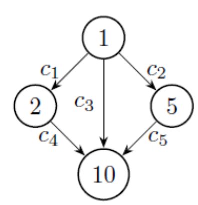

Figure 1: The O-graph of $\mathbb{Z}_{10}$.

Remark 3.2. *In view of the above example, the O-graph of $\mathbb{Z}_n$ is a connected graph with no cut vertices or bridges. Also, note that the vertex 1 is a source vertex and n is a sink.*

In an O-graph $\Gamma(V, E)$ corresponding to a non-trivial finite group G, the arrow $(x, y) \in E$ if and only if $x|y$. This means that the outdegree of x is the number of the multiples of x in $OC(G) = V$. Similarly, the indegree of x is the number of the divisors of x in V. Clearly, since $1 \in V$ and $1|x$

for all $x \in V$, $deg^+(1) = |V|$ and $deg^-(1) = 0$. Therefore, 1 is the source of all O-graph Γ associated to any finite group. For all $x \in V \setminus \{1\}$, we have $deg^+(x) < |V|$ and $deg^-(x) \geq 1$.

Obviously, the O-graph associated of a group G has no cycles. It corresponds to the O-graph definition, where for any two distinct vertices x and y the arrow $(x, y) \in E$ implies that $x|y$ and $y > x$. So, for any path $x_1, e_1, x_2, \cdots, x_{k-1}, e_{k-1}, x_k$ in the O-graph Γ, we have $x_k > x_{k-1} > \cdots > x_2 > x_1$. Then x_k does not divide x_1. As a result, this path is not cycle.

Remark 3.3. *The O-graph associated of a finite group is a directed acyclic graph (DAG).*

Isomorphic groups have many common properties. This asserts that the O-graphs of isomorphic groups are isomorphic. Actually, the isomorphic O-graph associated to a finite group does not in general identify isomorphic groups; that is, the O-graph gives only the form of the order classes of the corresponding group (only the set of all available orders). The formal order classes can not uniquely identify the group structure:

Example 3.2. Consider the cyclic group $\mathbb{Z}_8$ with the order classes $\{1, 2, 4, 8\}$. which are also those of D_{16}. Therefore, these groups have the same O-graph, although they are non-isomorphic groups:

The O-graph of $\mathbb{Z}_8$ The O-graph of D_{16}

Figure 2: Same O-graphs associated of non-isomorphic groups.

Another formalization of the order classes may give certain classification of a given finite group. Let G be a finite group, with $OC(G) = \{k \mid k = o(x)\ ,\ x \in G\}$ and let $O(k) = \{x \in G \mid o(x) = k\}$. One can represent the order classes of the group G by

$$O(G) = \{[k, |O(k)|] \mid k \in OC(G)\}$$

Therefore, we can re-establish the O-graph of a group G using the new formalization of its order classes. This is done by assigning weights to their arrows. These weights are $|O(y)|$ for each arrows of the form $(1, y)$ and 0 for all other arrows. The O-graph that obtains is called the weighted O-graph and is denoted by WO-graph. If we apply this formalization on the graphs in Example 3.2, we get the graphs shown in Figure 3.

The WO-graph of $\mathbb{Z}_8$ The WO-graph of D_{16}

Figure 3: WO-graphs associated of non-isomorphic groups

Clearly, in a WO-graph of a group G, one finds that the sum of all weights equal $|G| - 1$. This describes the associated group of the O-graph more precisely than without weights.

For a subgroup H of a finite group G, the order classes of H is $OC(H) = \{k \in \mathbb{N} \mid k = o(x)\ ,\ x \in H\} \subseteq OC(G)$. This easily shows that the O-graph of a subgroup H of a finite group G is a subgraph of the O-graph of G.

4 The O-graph of POC-group

The notion of POC-group (Prime Order Classes group) was introduced by Al-Hasanat in [6]. To keep this paper self-contained, we give the definition and the main axioms of this notion.

Definition 4.1. *[6] A POC-group G is a finite group in which each $i \in OC(G)$ is a prime factor of $|G|$.*

This type of groups has distinct cases:

Remark 4.2. *[6] A group G of order $n = p^k$ (p is prime) is a POC-group if and only if $G \cong \mathbb{Z}_p^k$.*

Theorem 4.3. *[6] A group G of order $n = 2p$ (p is an odd prime) is a POC-group with $OC(G) = \{1, 2, p\}$ if and only if $G \cong D_n$.*

The main property of a POC-group is that $OC(G)$ contains only primes, which will restrict the adjacently on the set of vertices of the associated O-graph. Therefore, the O-graphs of these groups have certain properties.

Let G be a POC-group with $OC(G) = \{1, p_1, \cdots, p_k\}$, where p_i is prime for $i = 1, 2, \cdots, k$. Then, the O-graph associated to G is the digraph $\Gamma(V, E)$ with the set of vertices being $V = \{1, p_1, \cdots, p_k\}$. Therefore, the edges set E contains only arrows of the form $(1, p_i)$ for $i = 1, 2, \cdots, k$. Hence,

Theorem 4.4. *If the group G is a POC-group and $OC(G) = \{1, p_1, p_2, \cdots, p_k\}$ is its order classes, then the O-graph of G is a directed rooted tree with the vertex 1 being the root of k leaves.*

Proof. Let G be a POC-group and let $OC(G) = \{1, p_1, p_2, \cdots, p_k\}$ be its order classes. Using Definition 3.1, the O-graph of G is the digraph Γ of vertices set $\{1, p_1, p_2, \cdots, p_k\}$ and set of edges E. Since $1|x$ for all $x \in V$, then $\{(1, x) \mid x \in V \setminus \{1\}\} \subseteq E$. Since V contains primes only, the arrow $(x, y) \in E$ if and only if $x = 1$. Therefore, $E \subseteq \{(1, x) \mid x \in V \setminus \{1\}\}$; that is, $E = \{(1, x) \mid x \in V \setminus \{1\}\}$. This shows that the root vertex of Γ is 1, where any vertex $x \neq 1$ has $deg^-(x) = 1$ and $deg^+(x) = 0$. This completes the proof. □

Remark 4.5. *In the O-graph Γ of a POC-group G, we have $deg^-(1) = |V| - 1 = |OC(G)| - 1$ and $deg^+(v \neq 1) = 1$. Moreover, 1 is the only cut vertex and all arrows are bridges; that is, the number of bridges is $deg^-(1)$.*

Example 4.1. The dihedral group $G = D_{22}$ is a POC-group and the order classes of G is $OC(G) = \{1, 2, 11\}$. So, the O-graph of G is $\Gamma(V, E)$, where $V = \{1, 2, 11\}$ and $E = \{(1, 2), (1, 11)\}$. It is easy to show that $O(G) = \{[1, 1], [2, 11], [11, 10]\}$, which identifies the WO-graph of G. See Figure 4.

The O-graph of D_{22} The WO-graph of D_{22}

Figure 4: The O-graph of D_{22}

5 The O-graph of dihedral groups of order 2^n

This paper also deals with dihedral groups of order a power of 2, which have some important and unique properties. This makes the classification of this sort of groups easier. For more about this family of groups we refer the reader to [7, 8].

The order classes of these groups were given precisely in [8].

Proposition 5.1. *[7] If $G = D_{2m}$, then the set of all possible orders is*

$$OC(G) = \{j \in \mathbb{N} \mid 2 < j < m\,,\ m \equiv 0(mod\, j)\} \cup \{1, 2, m\}.$$

Setting $2m = 2^n$ gives the dihedral group of order a power of 2. The main aim of this paper is to associate the O-graph of a certain group. To achieve this, the order classes of this group should be handled. This explains the next steps.

The following result is a direct consequence of Proposition 5.1.

Corollary 5.2. *Let $G = D_{2^n}$ for $n \geq 3$. Then the order classes of G is:*

$$OC(G) = \{2^k \mid k = 0, 1, \cdots, n-1\}.$$

Proof. Let $G = D_{2^n}$ for $n \geq 3$, using Proposition 5.1, the order classes of G is:

$$OC(G) = \left\{j \in \mathbb{N} \mid 2 < j < 2^{n-1},\ 2^{n-1} \equiv 0(mod\, j)\right\} \cup \{1, 2, 2^{n-1}\}.$$

Since, $2^{n-1} \equiv 0(mod\, 2^k)$ and $2 < 2^k < 2^{n-1}$ for all $k = 2, \cdots, n-2$, it follows that $OC(G) = \{2^2, 2^3, \cdots, 2^{n-2}\} \cup \{1, 2, 2^{n-1}\}$ and the claim follows. □

This result declare that the group D_{2^n} has n terms in its order classes. The next results will describe the O-graph associated of this group.

Corollary 5.3. *If $G = D_{2^n}$, $n \geq 3$, then the O-graph associated of G has n vertices.*

Proof. Let $G = D_{2^n}$, $n \geq 3$. Then, $OC(G) = \{2^k \mid k = 0, 1, \cdots, n-1\}$ by Corollary 5.2. The O-graph of G has the set of vertices $V = OC(G)$. Thus, $|V| = |OC(G)| = \left|\{2^k \mid k = 0, 1, \cdots, n-1\}\right| = n$. □

Let G is the dihedral group of order 2^n. Since $2^k|2^{k+1}$ for each $k = 0, 1, \cdots, n-2$, each vertex is adjoint to all other vertices in the corresponding order class (regardless of the arrow direction). There are exactly 2^{n-2} arrows from $1 = 2^0$ to all of the other vertices. Similarly, the vertex 2^1 adjacent to $2^2, 2^3, \cdots, 2^{n-1}$.

Theorem 5.4. *The O-graph of $G = D_{2^n}$, $n \geq 3$ is a reducible tournament graph on n vertices with source $v = 1$ and sink $u = 2^{n-1}$.*

Proof. Let G be a dihedral group of order 2^n , $n \geq 3$. Then the O-graph $\Gamma(V, E)$ associated to G has n vertices. Let $v \in V$. Then $v = 2^m$ for $m \in \{0, 1, 2, \cdots, n-1\}$. There is an arrow from this vertex v to another vertex $z \in V$ if and only if $v|z$. So $z = 2^{m+i}$ for any $i \in \{1, 2, \cdots, n-m-1\}$. Thus, a vertex $v = 2^m$ has $deg^+(v) = n - m - 1$ for $m = 0, 1, \cdots, n-1$. This implies that the set of all outdegrees of Γ is $\{0, 1, 2, ..., n-1\}$. Hence, $|E| = \sum_{i=0}^{n-1} i = \frac{1}{2}n(n-1) = \binom{n}{2}$. This shows that the underlying graph of Γ is a complete graph of n vertices, and since any O-graph is DAG graph, Γ is a graph that assigns a direction for each edge in the corresponding complete graph.

Finally, we need to show that Γ is reducible. It is enough to show that there is a non-empty subset of vertices X, such that $A(X) = \{u \in V \mid (v, u) \in E \text{ , for } v \in X\} \subseteq X$ (see Section 3 in [9]). So let $X = \{2^{n-2}, 2^{n-1}\} \subseteq V$. Then $A(X) = \{2^{n-1}\} \subseteq X$. □

The above theorem enables us to construct the O-graphs of D_{2^n} for large n. We illustrate this in the following example.

Example 5.1. The O-graph Γ of $D_{2^{10}}$ can be established first by determining the set of vertices V which is $V = \{2^k \mid k = 0, 1, \cdots, 9\}$. Then configuring the adjacent vertices which determined the set of arrows E as $|E| = \frac{1}{2}n(n-1) = 45$ arrows. See Figure 5 for the corresponding graph Γ.

Remark 5.5. *The O-graph of D_{2^n} with the set of vertices $V = \{2^0, 2^1, \cdots, 2^{n-1}\}$ has no cut vertices or bridges. Also, $deg^-(v) + deg^+(v) = n-1$ for all $v \in V$. In addition, $deg^-(v) = deg^+(u)$ and $deg^+(v) = deg^-(u)$ if and only if $u \cdot v = 2^{n-1}$.*

For the cyclic group $G = \mathbb{Z}_{p^n}$ (p is prime and $n \geq 3$), the order classes is $OC(G) = \{p^k \mid k = 0, 1, 2, \cdots, n\}$. Then, the O-graph of G is of order $n+1$ and size $\frac{1}{2}n(n+1)$.

Corollary 5.6. *The O-graph of the cyclic group $\mathbb{Z}_{p^n}$ with prime p and $n \geq 3$ is isomorphic to the O-graph of $D_{2^{n+1}}$.*

Proof. It is enough to show that these groups have isomorphic order classes. Define the map $\phi : OC(\mathbb{Z}_{p^n}) \longrightarrow OC(D_{2^{n+1}})$ by $\phi(p^k) = 2^k$ for $k = 0, 1, \cdots, n$. Clearly, ϕ is an isomorphism. Then, $OC(\mathbb{Z}_{p^n}) = \{p^k \mid k =$

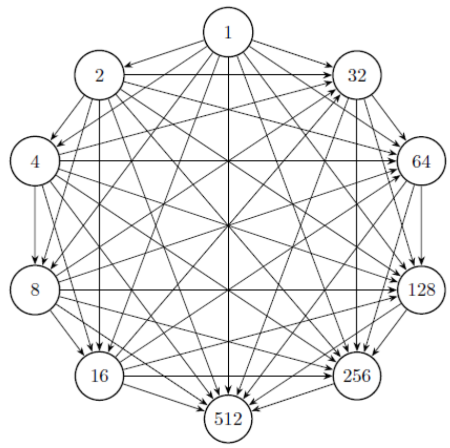

Figure 5: The O-graph of $D_{2^{10}}$

$0, 1, \cdots, n\} \cong \{2^k \mid k = 0, 1, \cdots, n\} = OC\left(D_{2^{n+1}}\right)$ (Corollary 5.2). This implies that these groups associate isomorphic O-graphs. $\square$

Note that, the WO-graphs of these groups are non-isomorphic. For the weights sum in both graphs are distinct. More precisely, $\mathbb{Z}_{2^n}$ is a normal subgroup of $D_{2^{n+1}}$ of index 2, and the weights sum of WO-graph which associated of a finite group G is $|G| - 1$.

References

[1] A. V. Kelarev, S. J. Quinn, A combinatorial property and power graphs of groups, Contributions to General Algebra, **12**, no. 58, (2000), 3–6.

[2] C. Bates, D. Bundy, S. Perkins, P. Rowley, Commuting involution graphs for symmetric groups, J. Algebra, **266**, (2003), 133–153

[3] Y. Seqev, On finite homomorphic images of the multiplicative group of a division algebra, Ann. of Math., **149**, (1999), 219–251.

[4] Y. Seqev, G. Seitz, Anisotropic groups of type An and the commuting graph of finite simple groups, Pacific J. Math., **202**, (2002), 125–225.

[5] A. R. Moghaddamfar, S. Rahbarivan, W. J. Shi, Certain properties of the power graph associated with a finite group, Journal of Algebra and its Applications, **13**, no. 7, (2014).

[6] Bilal N. Al-Hasanat, Mohanad N. Al-Hasanat, Asad A. Al-Hasanat, Groups with Prime Order Classes, International Journal of Pure and Applied Mathematics, **116,** no. 3, (2017), 695–699.

[7] Bilal Al-Hasanat, Azhana Ahmad, Hajar Sulaiman, Order classes of dihedral groups, AIP Conference Proceedings, **1605,** no. 1, (2014), 551–556.

[8] Bilal N. Al-Hasanat, Othman A. Almatroud, Mousa S. Ababneh, Dihedral Groups of Order 2^{m+1}, International Journal of Applied Mathematics, **26,** no. 1, (2013), 1–7.

[9] John W. Moon, Topics on Tournaments, Holt, Rinehart and Winston, New York, 1968.