International Journal of Mathematics and
Computer Science, **14**(2019), no. 4, 879–888

On Some Refinements of Hardy-type Integral inequalities

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(Received March 22, 2019, Accepted July 9, 2019)

Abstract

In this paper, some new refinements of Hardy's inequality in integral form are obtained. Further improvements are also presented.

1 Introduction

In [3], Hardy proved the following integral inequality:

Theorem 1.1. *Let f be a non-negative p-integrable function defined on $(0,\infty)$ and $p<\infty$. If f is integrable over the interval $(0,x)$, for each positive x, then the inequality below is valid:*

$$\int_0^\infty \left(\frac{1}{x}\left[\int_0^x f(t)\ dt\right]^p\right)\ dx \ \leq\ C_p \int_0^\infty f^p(x)\ dx, \tag{1.1}$$

where the constant $C_p=\left(\frac{p}{p-1}\right)^p$ is the best possible.

After its introduction and proof, earlier researchers in this area focused attention on obtaining a simplified proof of (1.1). For example, Hardy himself

Key words and phrases: Best possible; Convexity, Holder's Inequality.
AMS (MOS) Subject Classifications: Primary: 26D15. Secondary: 35A23, 47A63.
ISSN 1814-0432, 2019, http://ijmcs.future-in-tech.net

in [?] proved that for $p > 1, k \neq 1$ and a function F defined on $\mathbb{R}_+ = (0, \infty)$ by

$$F(x) = \begin{cases} \int_0^x f(t)dt & if\ k > 1 \\ \int_0^\infty f(t)dt & if\ k < 1, \end{cases}$$

$$\int_0^\infty x^{-k} F^p(x)\ dx \leq \left[\frac{p}{k-1}\right]^p \int_0^\infty x^{p-k} f^p(x)\ dx \tag{1.2}$$

In another development, Levison [8] proved that (1.2) holds for $0 < a < b < \infty$. In [2], the direct and simple way of obtaining Hardy inequality via the convexity argument was initiated. This concept was rediscovered by Kaijser 'et. al. in [6], where he proved that inequality (1.2) is just a special case of Hardy-Knopp-type inequality given in the theorem below:

Theorem 1.2. *Let φ be a positive and convex function on $(0, \infty)$. Then,*

$$\int_0^\infty \varphi\left(\frac{1}{x}\int_0^x h(t)\ dt\right)^p \frac{dx}{x} \leq \int_0^\infty \varphi\left(h^p(x)\right)\ \frac{dx}{x} \tag{1.3}$$

The purpose of this work is to give some Hardy-type inequalities which are refinements and extension of some earlier work in the literature (See [11], [10], [9], [7], [5] and [1] for similar work on Hardy-type inequalities).

2 Preliminary Lemmas

The following lemmas are needed in the proof of the main results.

Lemma 2.1. *Let p and q be constants such that for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. For $0 \leq f \leq \infty$ and $\int_0^\beta f(t)\ dt < \infty$, there exists a real number $x_0 \in (0, \beta)$ such that for every $x \in (x_0, \beta)$,*

$$\int_0^x f(t)\ dt \leq \left(\frac{p}{p-1}\right)^{\frac{1}{q}} x^{(\frac{1}{q})^2} \left(\int_0^x t^{\frac{1}{q}}\ f^p\ dt\right)^{\frac{1}{p}} \tag{2.1}$$

Proof. For every $x \in (0,\beta)$ using Holder's inequality,

$$\begin{aligned}
\int_0^x f(t)\,dt &= \int_0^x t^{\frac{-1}{pq}}\ f(t)\ t^{\frac{1}{pq}} \\
&\le \left(\int_0^x t^{\frac{-1}{p}}\right)^{\frac{1}{q}} \left(\int_0^x t^{\frac{1}{q}} f^p(t)\,dt\right)^{\frac{1}{p}} \\
&= \left(\frac{p-1}{p}\right)^{\frac{-1}{q}} \left(x^{\frac{1}{q}}\right)^{\frac{1}{q}} \left(\int_0^x t^{\frac{1}{q}}\ f^p(t)\,dt\right)^{\frac{1}{p}} \\
&= \left(\frac{p}{p-1}\right)^{\frac{1}{q}} x^{\frac{1}{q^2}} \left(\int_0^x t^{\frac{1}{q}}\ f^p(t)\,dt\right)^{\frac{1}{p}}
\end{aligned}$$

□

Lemma 2.2. *Let $f \ge 0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, and $0<\int_0^\infty f^p(t)\,dt<\infty$. If $\alpha \ge 0$, $x_0 \in (a,\infty)$, then for all $x > x_0$,*

$$\int_\alpha^x f(t)\,dt \le \left(\int_\alpha^x t^{\frac{1}{q}} f^p(t)dt\right)^{\frac{1}{p}} \left(\frac{p}{p-1}\right)^{\frac{1}{q}} \left(x^{\frac{1}{q}}-\alpha^{\frac{1}{q}}\right)^{\frac{1}{q}} \tag{2.2}$$

Proof. The left hand side of (2.1) can be expressed as

$$\int_\alpha^x f(t)\,dt = \int_\alpha^x t^{\frac{1}{pq}}\ f(t)\ t^{\frac{-1}{pq}}\,dt$$

and, by Holder's inequality, we have

$$\begin{aligned}
\int_\alpha^x f(t)\,dt &\le \left(\int_\alpha^x t^{\frac{1}{q}} f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_\alpha^x t^{-\frac{1}{p}}\,dt\right)^{\frac{1}{q}} \\
&\le \left(\frac{p}{p-1}\right)^{\frac{1}{q}} \left(\int_\alpha^x t^{\frac{1}{q}} f^p(t)\,dt\right)^{\frac{1}{p}} \left(x^{\frac{1}{q}}-\alpha^{\frac{1}{q}}\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence

$$\int_\alpha^x f(t)\,dt \le \left(\frac{p}{p-1}\right)^{\frac{1}{q}} \left(x^{\frac{1}{q}}-\alpha^{\frac{1}{q}}\right)^{\frac{1}{q}} \left(\int_\alpha^x t^{\frac{1}{q}} f^p(t)dt\right)^{\frac{1}{p}}.$$

□

3 Main Results

The main results are presented in the theorems below:

Theorem 3.1.
Let p and q be constants such that $\frac{1}{p}+\frac{1}{q}=1$ for $p>1$. Let $0<\alpha<\beta<\infty$, $0\le f<\infty$, $0<\int_\alpha^\beta f^p(t)\,dt<\infty$, $F(x):=\int_\alpha^x f(t)\,dt$. Then

$$\int_\alpha^\beta \left(x^{-1}\int_\alpha^x f(t)\,dt\right)^p dx \;\le \left(\frac{p}{p-1}\right)^p \frac{[\alpha^{\frac{1}{q}}-\beta^{\frac{1}{q}}]^p}{\beta^{\frac{p}{q}}}\int_\alpha^\beta f^p(t)\,dt \tag{3.1}$$

Proof. Applying 2.2 on the left hand side of (3.1), we have

$$\begin{aligned}
\int_\alpha^\beta \left(x^{-1}\int_\alpha^x f(t)dt\right)^p dx &\le \int_\alpha^\beta \left[x^{-1}\left(\frac{p}{p-1}\right)^{\frac{1}{q}}\left(x^{\frac{1}{q}}-\alpha^{\frac{1}{q}}\right)^{\frac{1}{q}}\left(\int_\alpha^x t^{\frac{1}{q}}f^p(t)\,dt\right)^{\frac{1}{p}}\right]^p dx\\
&= \int_\alpha^\alpha x^{-p}\left(\frac{p}{p-1}\right)^{\frac{p}{q}}\left(x^{\frac{1}{q}}-a^{\frac{1}{q}}\right)^{\frac{p}{q}}\left[\int_\alpha^x t^{\frac{1}{q}}f^p(t)\,dt\right]dx\\
&= \left(\frac{p}{p-1}\right)^{\frac{p}{q}}\int_\alpha^\beta x^{-p}\left(x^{\frac{1}{q}}-\alpha^{\frac{1}{q}}\right)^{\frac{p}{q}}\int_\alpha^x t^{\frac{1}{q}}f^p(t)\,dt.dx
\end{aligned}$$

By further simplification and applying Fubini's theorem we get

$$\begin{aligned}
\int_\alpha^\beta \left(x^{-1}\int_\alpha^x f(t)dt\right)^p dx &\le \left(\frac{p}{p-1}\right)^{\frac{p}{q}}\int_\alpha^\beta\left(\int_t^\beta x^{-p+\frac{p}{q^2}}\left[1-\left(\frac{\alpha}{x}\right)^{\frac{1}{q}}\right]^{\frac{p}{q}}dx\right)t^{\frac{1}{q}}f^p(t)\,dt\\
&= \left(\frac{p}{p-1}\right)^{\frac{p}{q}}\int_\alpha^\beta\left(\int_t^\beta x^{-1-\frac{1}{q}}\left[1-\left(\frac{\alpha}{\beta}\right)^{\frac{1}{q}}\right]^{\frac{p}{q}}dx\right)t^{\frac{1}{q}}f^p(t)\,dt\\
&= q^{\frac{p}{q}}.\left(-\frac{1}{q}\right)^{-1}\int_a^b\left(x^{-\frac{1}{q}}|_t^b\left[1-\left(\frac{a}{b}\right)^{\frac{1}{q}}\right]^{\frac{p}{q}}\right)t^{\frac{1}{q}}f^p(t)\,dt\\
&= q^{\frac{p}{q}+1}\int_a^b\left(t^{-\frac{1}{q}}-b^{-\frac{1}{q}}\left[1-\left(\frac{a}{b}\right)^{\frac{1}{q}}\right]^{\frac{p}{q}}\right)t^{\frac{1}{q}}f^p dt\\
&= q^p\left(1-\left(\frac{a}{b}\right)^{\frac{1}{q}}\right)^{p-1}\int_a^b\left[1-\left(\frac{a}{b}\right)^{\frac{1}{q}}\right]f^p dt\\
&= \left(\frac{p}{p-1}\right)^p\left(1-\left(\frac{a}{b}\right)^{\frac{1}{q}}\right)^p\int_a^b f^p(t)\,dt
\end{aligned}$$

□

This is the required inequality.

Corollary 3.2.
If in particular $a = 0$ and $b \to \infty$, then (3.1) becomes

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\ dt\right) dx \ \le\ \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\ dt$$

which is the classical Hardy's integral inequality (1.1). Hence (3.1) is a generalization of Hardy's integral inequality (1.1).

Theorem 3.3.
Let p, r, and q be constants such that $\frac{1}{p}+\frac{1}{q}=1,\ r \ge p > 1,\ 0 < \beta < \infty$. If $f \ge 0,\ 0 < \int_a^b x^{-r+p}\ f^p(x)\ dx < \infty$ and $b \in (0,\infty)$ then,

$$\int_0^\beta x^{-r}\left(\int_0^x f(t)dt\right)^p dx \le \frac{(\frac{p}{p-1})^p}{(r-p)q+1}\int_0^\beta \kappa(t,\beta,q)\ f^p(t)\ dt \qquad (3.2)$$

where

$$\kappa(t,\beta,q) = 1 - \left(\frac{t}{\beta}\right)^{-r+p+\frac{1}{q}}$$

Proof. Applying (2.1), we have

$$\begin{aligned}
\int_0^\beta x^{-r}\left(\int_0^x f(t)dt\right)^p dx \ &\le\ \int_0^\beta x^{-r}\left[\left(\frac{p}{p-1}\right)^{\frac{1}{q}} x^{\frac{1}{q^2}}\left(\int_0^x f^p(t)\ dt\right)^{\frac{1}{p}}\right]^p \\
&=\ \left(\frac{p}{p-1}\right)^{\frac{p}{q}}\int_0^\beta x^{-r+\frac{p}{q^2}}\ \int_0^x f^p(t)\ dt.\ dx \\
&\le\ \left(\frac{p}{p-1}\right)^{p-1}\int_0^\beta\left(\int_t^\beta x^{-r+p-\frac{1}{q}}\ dx\right) t^{\frac{1}{q}}\ f^p(t)\ dt \\
&=\ \frac{\left(\frac{p}{p-1}\right)^{p-1}}{-r+(p-\frac{1}{q})}\int_0^\beta\left[\beta^{-r+p-\frac{1}{q}} -\ t^{-r+p-\frac{1}{q}}\right] t^{\frac{1}{q}}\ f^p(t)\ dt \\
&=\ \frac{\left(\frac{p}{p-1}\right)^{p}}{(r-p)q+1}\int_0^\beta\left[1-\ \left(\frac{t}{\beta}\right)^{r-p+\frac{1}{q}}\right] t^{\frac{1}{q}}\ f^p(t)\ dt \\
&=\ \frac{\left(\frac{p}{p-1}\right)^{p}}{(r-p)q+1}\int_0^\beta \kappa(t,\beta,q)\ t^{-r+p}\ f^p(t)\ dt
\end{aligned}$$

□

Remark: (i) If in particular $r = p$, inequality 3.2 reduces to

$$\int_0^\beta \left(x^{-1} \int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\beta \left(\frac{\beta^{\frac{1}{q}} - t^{\frac{1}{q}}}{\beta^{\frac{1}{q}}}\right) f^p(t)\, dt \qquad (3.3)$$

(ii). If $b = \infty$, then we obtain

$$\int_0^\infty x^{-r} \left(\int_0^x f(t)dt\right)^p dx \leq \frac{\left(\frac{p}{p-1}\right)^p}{(r-p)q+1} \int_0^\infty t^{-r+p} f^p(t)\, dt \qquad (3.4)$$

the proof being similar to that of (3.2).

Applying inequality (2.1), we obtain

$$\begin{aligned}
\int_0^\infty x^{-r} \left(\int_0^x f(t)dt\right)^p dx &\leq \int_0^\infty x^{-r} \left(\left(\frac{p}{p-1}\right)^{\frac{1}{q}} x^{\frac{1}{q^2}} \left(\int_0^x t^{\frac{1}{q}} f^p(t)dt\right)^{\frac{1}{p}}\right)^p dx \\
&= \left(\frac{p}{p-1}\right)^{\frac{p}{q}} \int_0^\beta x^{-r+\frac{p}{q^2}} \int_0^x t^{\frac{1}{q}} f^p(t)\, dt\ .\ dx \\
&\leq \left(\frac{p}{p-1}\right)^{\frac{p}{q}} \int_0^\beta \left(\int_t^\beta x^{-r+p-\frac{1}{q}}\, dx\right) t^{\frac{1}{q}} f^p(t)\, dt \\
&= \left(\frac{p}{p-1}\right)^p \int_0^x t^{-r+p} f^p(t)\, dt
\end{aligned}$$

in particular when $r = p$, (3.4) above reduces to the classical Hardy inequality.

Theorem 3.4.
Let γ be a nonnegative real number, $f \geq 0$ and $g > 0$ on $[a, b] \subseteq (0, \infty)$, such that $0 \leq a < x < b$ and $\frac{(x-a)}{g(x)}$ is non-increasing, $F(x) = \int_a^x f(t)dt$, for all $x \in [a, b]$. Then

$$\int_a^b \left(\frac{F(x)}{g(x)}\right)^p dx \ \leq\ \left(\frac{p}{p-1}\right)^p \int_a^b (x-a)^p \frac{f(x)^p}{g(x)^p}\ .\ \gamma\, dx, \qquad (3.5)$$

where γ is defined as

$$1 - \frac{(x-a)^{1-\frac{1}{p}}}{(b-a)^{1-\frac{1}{p}}}.$$

Proof: By Holder's inequality and since $\frac{(x-a)}{g(x)}$ is non-increasing on [a,b]) we have

$$\begin{aligned}
\int_a^b \left(\frac{F(x)}{g(x)}\right)^p dx &= \int_a^b g^{-p}(x)\ (F(x))^p\, dx \\
&= \int_a^b g^{-p}(x)\ \left(\int_a^x f(t)\ dt\right)^p\ dx \\
&\leq \int_a^b g^{-p}(x) \left[\left(\int_a^x (t-a)^{1-\frac{1}{p}} f^p(t)\ dt\right)^{\frac{1}{p}} \left(\int_a^x (t-a)^{-\frac{1}{p}} dt\right)^{1-\frac{1}{p}}\right]^p dx \\
&= \int_a^b g^{-p}(x) \left(\int_a^x (t-a)^{1-\frac{1}{p}} f^p(t)\ dt\right) \left(\int_a^x (t-a)^{-\frac{1}{p}} dt\right)^{p-1} dx.
\end{aligned}$$

Evaluating the far right integral, we now have

$$\begin{aligned}
\int_a^b \left(\frac{F(x)}{g(x)}\right)^p dx &\leq \int_a^b g^{-p}(x) \left(\int_a^x (x-a)^{1-\frac{1}{p}} f^p(t)\ dt\right) \left(\frac{(t-a)^{1-\frac{1}{p}}}{1-\frac{1}{p}}\right)^{p-1} dx \\
&= \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b dx\ \int_a^x (t-a)^{(1-\frac{1}{p})(p-1)}\ g^{-p}(x)(t-a)^{1-\frac{1}{p}} f^p(t)\ dt.
\end{aligned}$$

By Fubini's theorem we have

$$\int_a^b \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b dt \int_t^b (x-a)^{p+\frac{1}{p}-2} g^{-p}(x)\ (t-a)^{1-\frac{1}{p}}\ f^p(t)\ dx$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b dt \int_t^b (x-a)^{\frac{1}{p}-2} \ (x-a)^p \ g^{-p}(x) \ (t-a)^{1-\frac{1}{p}} \ f^p(t) \ dx$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b dt \int_t^b (x-a)^{\frac{1}{p}-2} \ \left(\frac{x-a}{g(x)}\right)^p \ (t-a)^{1-\frac{1}{p}} \ f^p(t) \ dx$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b \left(\frac{x-a}{g(x)}\right)^p \ (t-a)^{1-\frac{1}{p}} \ f^p(t) \ dt \int_t^b (x-a)^{\frac{1}{p}-2} \ dx$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b \left(\frac{x-a}{g(x)}\right)^p \ (t-a)^{1-\frac{1}{p}} \ f^p(t) \ dt \ . \frac{1}{\left(\frac{1}{p}-1\right)} \left[(b-a)^{\frac{1}{p}-1} \ - \ (t-a)^{\frac{1}{p}-1}\right]$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p-1}} \int_a^b \left(\frac{t-a}{g(t)}\right)^p \ \frac{-1}{\left(1-\frac{1}{p}\right)} \left[(b-a)^{\frac{1}{p}-1} \ - \ (t-a)^{\frac{1}{p}-1}\right] \ (t-a)^{1-\frac{1}{p}} \ f^p(t) \ dt$$

$$= \frac{-1}{\left(1-\frac{1}{p}\right)^{p}} \int_a^b \left(\frac{t-a}{g(t)}\right)^p \ \left[(b-a)^{\frac{1}{p}-1} \ (t-a)^{1-\frac{1}{p}} \ - \ 1\right] \ f^p(t) \ dt$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p}} \int_a^b \left(\frac{t-a}{g(t)}\right)^p \ \left[1-(t-a)^{1-\frac{1}{p}} \ . \ (b-a)^{-(1-\frac{1}{p})}\right] \ f^p(t) \ dt$$

$$= \frac{1}{\left(1-\frac{1}{p}\right)^{p}} \int_a^b \left(\frac{t-a}{g(t)}\right)^p \ \left[1-\frac{(t-a)^{1-\frac{1}{p}}}{(b-a)^{1-\frac{1}{p}}}\right] \ f^p(t) \ dt$$

and if $t = x$

$$= \left(\frac{p}{p-1}\right)^p \int_a^b (x-a)^p \ \left(\frac{f(x)}{g(x)}\right)^p .\gamma \ dt$$

This proves the theorem.
The following corollaries hold true:

Corollary 3.5.
Let $f > 0$ and $g > 0$ on $[a, b]$ such that $\frac{(x-a)}{g(x)}$ is non increasing, $\gamma = 1$ and $F(x) = \int_a^x f(t)dt$ for $p > 1$. Then,

$$\int_a^b \left(\frac{F(x)}{g(x)}\right)^p \ dx \ \leq \ \left(\frac{p}{p-1}\right)^p \int_a^b \left(\frac{x-a}{g(x)}\right)^p f^p(x) \ dx \qquad (3.6)$$

Proof: Similar to that of Theorem (4.3) when $\gamma = 1$.

Corollary 3.6.
Let f be a nonnegative function on $[a,b] \subseteq (0,\infty)$ and $F(x) = \int_a^x f(t)dt$, Then for $p > 1$,

$$\int_a^b \left(\frac{F(x)}{x-a}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx \tag{3.7}$$

Proof: The proof is obvious if $g(x) = x - a,\ \gamma = 1,\ t = x$, and $0 < a,\ \ x \in [a,b]$.
This implies that, for $0 < a < \gamma$, the inequality below is true

$$\int_a^b \left(\frac{F(x)}{x}\right)^p dx \leq \int_a^b \left(\frac{F(x)}{x-a}\right)^p \leq \left(\frac{p}{p-1}\right)^p \int_a^b f^p(x)dx \tag{3.8}$$

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