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New Fixed Point Theorems in G-metric Spaces

Gehad M. Abd-Elhamed[1,2]

[1]Department of Mathematics
College of Girls
Ain Shams University
Cairo, Egypt

[2]Department of Mathematics
College of Science and Humanities studies
Sattam Bin Abdul-Aziz University
Al-Kharj, Saudi Arabia

email: gehadmahfood@gmail.com

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Abstract

We prove new theorems for generalized contractions in G-metric spaces. Our results extend some known results in the literature.

1 Introduction

Mustafa and Sims [9] introduced the concept of G-metric spaces as a generalization of a metric space. Since then, several interesting results for various contractive conditions in G-metric spaces have been acquired (see ([1]-[4],[6]-[17]).

Moradlou et. al. [7] and Aggarwal et. al. [2] proved some fixed point theorems for generalized contractions in G-Metric spaces. Our results extend those as well as those of Edelstein [5] and Suzuki [18].

We recall some basic definitions and results of G-metric spaces. For details on the following notions we refer the reader to [5].

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Definition 1.1. *Let X be a non empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:*

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$, for all x, $y \in X$, with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$, for all x, y, $z \in X$, with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all x, y, z, $a \in X$, (triangle inequality).
Then the function G is called a generalized metric, and the pair (X, G) is called a G-metric space.

Proposition 1.2. *Let (X, G)be a G-metric space. Then for any x, y, z and $a \in X$,*

(1) if $G(x, y, z) = 0$, then $x = y = z$,
(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(3) $G(x, y, y) \leq 2G(y, x, x)$,
(4) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(5) $G(x, y, z) \leq \frac{2}{3} \left(G(x, y, a) + G(x, a, z) + G(a, y, z)\right)$,
(6) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 1.3. *Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. we say that (x_n) is G-convergent to x if for any $\varepsilon > 0$, there exists $n_0 \in N$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq n_0$.*

Proposition 1.4. *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) (x_n), is G-convergent to x,
(2) $G(x_n, x_n, x) \longrightarrow 0$, as $n \longrightarrow \infty$,
(3) $G(x_n, x, x) \longrightarrow 0$, as $n \longrightarrow \infty$,
(4) $G(x_m, x_n, x) \longrightarrow 0$, as $m, n \longrightarrow \infty$.

Definition 1.5. *Let (X, G) be a G-metric space. A sequence (x_n) is called G-Cauchy if given $\varepsilon > 0$, there is $n_o \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq n_o$.*

Definition 1.6. *Let (X, G) and (X^*, G^*) be G-metric spaces and let $f : (X, G) \longrightarrow (X^*, G^*)$ be a function. Then f is said to be $G-$continuous at a point $a \in X$, if given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X, G(a, x, y) < \delta$ implies $G^*(f(a), f(x), f(y)) < \varepsilon$.*

Proposition 1.7. *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is continuous in all variables.*

Definition 1.8. *A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Definition 1.9. *A G-metric space (X, G) is said to be a compact G-metric space if it is G-complete and G-totally bounded.*

2 Main results

Theorem 2.1. *Let (X, G) be a complete G-metric space and T be a mapping on X. Suppose that there exist $r \in [0, 1)$, $(b + c) \in (0, \frac{1}{2}), a \in [0, 1]$,*

$$((a + b + c)r^2 + r) \leq \frac{1}{2}, \textit{if} \;\; r \in [\frac{1}{6}, \frac{1}{\sqrt{6}}],$$

and

$$a + (2a + b + c)r \leq 1, \textit{if} \quad r \in [\frac{1}{\sqrt{6}}, 1)$$

such that

$$aG(\zeta, T\zeta, T\zeta) + bG(\nu, T\zeta, T\zeta) + cG(z, T\zeta, T\zeta) \leq G(\zeta, \nu, z)$$

implies

$$G(T\zeta, T\nu, Tz) \leq rM(\zeta, \nu, z), \; \textit{for all} \; \zeta, \nu, z \in X,$$

where $M(\zeta, \nu, z) = \max[G(\zeta, \nu, z), G(\zeta, T\zeta, T\zeta), G(\nu, T\nu, T\nu), G(z, Tz, Tz)]$. Then there exists a unique fixed point q of T. Moreover, $\lim_n T^n\zeta = q$ for all $\zeta \in X$ and T is G-continuous at q.

Proof. Since $aG(\zeta, T\zeta, T\zeta) + bG(T\zeta, T\zeta, T\zeta) + cG(T\zeta, T\zeta, T\zeta) = aG(\zeta, T\zeta, T\zeta) \leq G(\zeta, T\zeta, T\zeta)$ holds for every $\zeta \in X$, by hypothesis, we get

$$G(T\zeta, T^2\zeta, T^2\zeta) \leq rM(\zeta, T\zeta, T\zeta),$$

where

$$M(\zeta, T\zeta, T\zeta) = \max[G(\zeta, T\zeta, T\zeta), G(T\zeta, T^2\zeta, T^2\zeta)].$$

If $M(\zeta, T\zeta, T\zeta) = G(T\zeta.T^2\zeta, T^2\zeta)$, then $G(T\zeta, T^2\zeta, T^2\zeta) \leq rG(T\zeta x, T^2\zeta, T^2\zeta)$. Thus $M(\zeta, T\zeta, T\zeta) = G(\zeta, T\zeta, T\zeta)$, and

$$G(T\zeta, T^2\zeta, T^2\zeta) \leq rG(\zeta, T\zeta, T\zeta), \forall \; \zeta \in X. \tag{2.1}$$

Define a sequence (u_n) in X by $u_n = Tu_{n-1}, ..., u_n = T^n u$, $u \in X$. Then

$$G(u_n, u_{n+1}, u_{n+1}) = G(T^n u, T^{n+1}u, T^{n+1}u) \leq rG(T^{n-1}u, T^n u, T^n u) \leq ...$$

$$\leq r^n G(u, Tu, Tu).$$

By a repeated application of the triangle inequality and (2.1)

$$G(u_m, u_m, u_n) \leq G(u_{n+1}, u_{n+1}, u_n) + G(u_{n+2}, u_{n+2}, u_{n+1}) + ...$$

$$+G(u_{m-1}, u_{m-1}, u_{m-2}) + G(u_m, u_m, u_{m-1}),$$

$$\leq (r^n + r^{n+1} + ... + r^{m-1})G(u, Tu, Tu),$$

$$G(u_m, u_m, u_n) \leq (\sum_{i=n}^{m-1} r^i)G(u, Tu, Tu) = \frac{r^n}{1-r}G(u, Tu, Tu).$$

Then $\lim G(u_m, u_m, u_n) = 0$, as $n, m \to \infty$. For $n, m, l \in N$, (G5) implies that

$$G(u_n, u_m, u_l) \leq G(u_n, u_m, u_m) + G(u_m, u_m, u_l).$$

Letting $n, m, l \to \infty$, we have $G(u_n, u_m, u_l) \to 0$. So (u_n) is a G-Cauchy sequences as (u_n) converges to some point q in X. Since $\lim_{n\to\infty} G(u_n, Tu_n, Tu_n) = 0$,

$$\lim_{n\to\infty} G(\zeta, Tu_n, Tu_n) = \lim_{n\to\infty} G(\zeta, u_n, u_n) = G(\zeta, q, q),$$

there exist a positive integer k such that

$$G(u_n, q, q) \leq \frac{1}{6}G(\zeta, q, q), G(u_n, u_n, q) \leq \frac{1}{12}G(\zeta, q, q) \tag{2.2}$$

$$G(\zeta, u_{n+1}, u_{n+1}) \leq \frac{1}{6}G(\zeta, q, q), \tag{2.3}$$

for all $n \geq k$. By the triangle inequality, Proposition (1.2), and inequalities (2.2) and (2.3), we get

$$aG(u_n, u_{n+1}, u_{n+1}) + (b+c)G(\zeta, u_{n+1}, u_{n+1}) < G(u_n, u_{n+1}, u_{n+1}) + \frac{1}{2}G(\zeta, u_{n+1}, u_{n+1})$$

$$\leq G(u_n, q, q) + G(q, u_{n+1}, u_{n+1}) + \frac{1}{12}G(\zeta, q, q) \leq \frac{2}{6}G(\zeta, q, q) =$$

$$\frac{2}{5}[G(\zeta, q, q) - \frac{1}{6}G(\zeta, q, q)] < \frac{2}{5}G(\zeta, u_n, u_n) < \frac{4}{5}G(u_n, \zeta, \zeta) < G(u_n, \zeta, \zeta).$$

By hypothesis, $G(Tu_n, T\zeta, T\zeta) \leq rM(u_n, \zeta, \zeta)$. Letting n tend to ∞, we get $G(q, T\zeta, T\zeta x) \leq rM(q, \zeta, \zeta)$, where

$$M(q, \zeta, \zeta) = \max[G(q, \zeta, \zeta), G(\zeta, T\zeta, T\zeta)],\ \forall\ \zeta \in X.$$

If $M(q, \zeta, \zeta) = G(\zeta, T\zeta, T\zeta)$, then $G(q, T\zeta, T\zeta) \leq rG(\zeta, T\zeta, T\zeta)$, and $G(T^{j+1}q, T^{j+1}q, q) \leq rG(T^{j+1}q, T^{j+1}q, q)$, which is a contradiction. Thus

$$G(q, T\zeta, T\zeta) \leq rG(q, \zeta, \zeta). \tag{2.4}$$

$$\begin{aligned} G(u_{j+1}, u_{j+1}, q) = G(T^{j+1}q, T^{j+1}q, q) \leq rG(T^jq, T^jq, q) \leq \\ r^2G(T^{j-1}q, T^{j-1}q, q) \leq ... \leq r^jG(Tq, Tq, q). \end{aligned} \tag{2.5}$$

Now,

(1) If $0 \leq r \leq \frac{1}{6}$, then, using the rectangle inequality, Proposition(1.2) and inequalities (2.1) and (2.4), we have

$$G(Tq, q, q) \leq G(Tq, T^2q, T^2q)+G(T^2q, q, q) \leq G(Tq, T^2q, T^2q)+2G(q, T^2q, T^2q)$$

$$\leq 2rG(q, Tq, Tq)+rG(q, Tq, Tq) \leq 3rG(q, Tq, Tq) \leq 6rG(q, q, Tq) \leq G(q, q, Tq),$$

a contradiction.

(2) If $\frac{1}{6} \leq r \leq \frac{1}{\sqrt{6}}$ and

$$aG(T^2q, T^3q, T^3q) + (b + c)G(q, T^3q, T^3q) > G(T^2q, q, q),$$

then

$$\begin{aligned} G(Tq, q, q) &\leq G(Tq, T^2q, T^2q) + G(T^2q, q, q) \\ &< rG(q, Tq, Tq) + [aG(T^2q, T^3q, T^3q) + (b + c)G(q, T^3q, T^3q)] \\ &< rG(q, Tq, Tq) + ar^2G(q, Tq, Tq) + (b + c)r^2G(q, Tq, Tq) \end{aligned}$$

$$< ((a+b+c)r^2+r)G(q, Tq, Tq) < 2((a+b+c)r^2+r)G(q, q, Tq) \leq G(q, q, Tq),$$

which is a contradiction. So,

$$aG(T^2q, T^3q, T^3q) + (b + c)G(q, T^3q, T^3q) \leq G(T^2q, q, q)$$

$$G(Tq, q, q) \leq G(Tq, T^3q, T^3q)+G(T^3q, q, q) \leq rG(q, T^2q, T^2q)+2G(T^3q, T^3q, q)$$

$$\leq r^2G(q, Tq, Tq)+2r^2G(q, Tq, Tq) = 3r^2G(q, Tq, Tq) \leq 6r^2G(Tq, q, q) \leq G(Tq, q, q).$$

(3) If $\frac{1}{\sqrt{6}} \leq r \leq 1$, then there exists an integer λ such that

$$aG(u_n, u_{n+1}, u_{n+1})+(b+c)G(q, u_{n+1}, u_{n+1}) > G(u_n, q, q) \text{ for all } n \geq \lambda. \tag{2.1}$$

Then

$$G(u_n,q,q) < aG(u_n,u_{n+1},u_{n+1}) + 2(b+c)G(u_{n+1},q,q)$$

$$< aG(u_n,u_{n+1},u_{n+1})+2(b+c)[aG(u_{n+1},u_{n+2},u_{n+2})+(b+c)G(w,u_{n+2},u_{n+2})]$$

$$< (a+2r(b+c)a)G(u_n,u_{n+1},u_{n+1}) + 4(b+c)^2G(u_{n+2},q,q)$$

$$< (a+2r(b+c)a)G(u_n,u_{n+1},u_{n+1}) + (2(b+c))^2[aG(u_{n+2},u_{n+3},u_{n+3})+ (b+c)G(q,u_{n+3},u_{n+3})]$$

$$< (a+2r(b+c)a)G(u_n,u_{n+1},u_{n+1}) + (2(b+c))^2r^2aG(u_n,u_{n+1},u_{n+1})+ (2(b+c))^3G(q,q,u_{n+3})$$

$$< [a+2(b+c)ar+(2(b+c))^2r^2a+ ... +a(2(b+c))^{p-1}r^{p-1}]G(u_n,u_{n+1},u_{n+1}) + (2(b+c))^pG(q,q,u_{n+p})$$

$$< a\frac{1-(2(b+c)r)^p}{1-2(b+c)r}G(u_n,u_{n+1},u_{n+1}) + (2(b+c))^pG(q,q,u_{n+p}),$$

for all $n \geq \lambda, p \geq 1$. Letting $p \to \infty$ and taking $d = 2(b+c)$, we obtain

$$G(u_n,q,q) < \frac{a}{1-dr}G(u_n,u_{n+1},u_{n+1}), \text{ for all } n \geq \lambda.$$

Thus,

$$G(u_{n+1},q,q) < \frac{a}{1-dr}G(u_{n+1},u_{n+2},u_{n+2}) < \frac{ar}{1-dr}G(u_n,u_{n+1},u_{n+1}), \text{ for all } n \geq \lambda,$$

so,

$$G(u_n,u_{n+1},u_{n+1}) \leq G(u_n,q,q) + G(q,u_{n+1},u_{n+1})$$

$$< \frac{a}{1-dr}G(u_n,u_{n+1},u_{n+1})+2G(q,q,u_{n+1}) < [\frac{a}{1-dr}+\frac{2ar}{1-dr}]G(u_n,u_{n+1},u_{n+1})$$

$$\leq G(u_n,u_{n+1},u_{n+1}), \text{for all } n \geq \lambda,$$

a contradiction. So there exists a subsequence (u_{n_k}) of (u_n) such that

$$aG(u_{n_k},u_{n_{k+1}},u_{n_{k+1}}) + dG(q,u_{n_{k+1}},u_{n_{k+1}}) \leq G(u_{n_k},q,q), \text{ for all } k \geq 1.$$

By hypothesis, $G(Tu_{n_k},Tq,Tq) \leq rM(u_{n_k},q,q)$, for all $k \geq 1$, where

$$M(u_{n_k},q,q) = \max[G(u_{n_k},q,q), G(u_{n_k},Tu_{n_k},Tu_{n_k}), G(q,Tq,Tq)].$$

By taking $k \to \infty$, we obtain $G(q, Tq, Tq) \leq rG(q, Tq, Tq)$, so $G(q, Tq, Tq) = 0$, that is $Tq = q$ which is a contradiction. Thus there exists an integer j such that $T^j q = q$. By (2.1), we get

$$G(q, Tq, Tq) = G(T^j q, T^{j+1} q, T^{j+1} q) \leq r^j G(q, Tq, Tq),$$

so $G(q, Tq, Tq) = 0$; that is, $Tq = q$. Now suppose that ν is another fixed point of T. Then

$$aG(\nu, T\nu, T\nu) + (b + c)G(q, T\nu, T\nu) \leq G(q, \nu, \nu),$$

implies $G(Tq, T\nu, T\nu) \leq rM(q, \nu, \nu) = rG(q, \nu, \nu)$. Hence $G(q, \nu, \nu) = 0$, which is a contradiction. To see that T is G-continuous at a fixed point q. Let (ν_n) be a sequence such that $\lim_{n\to\infty} \nu_n = q$. Then

$$aG(q, Tq, Tq) + dG(\nu_n, Tq, Tq) < \frac{1}{2}G(\nu_n, Tq, Tq) < G(\nu_n, \nu_n, q)$$

By hypothesis, we get

$$G(T\nu_n, T\nu_n, Tq) \leq rM(\nu_n, \nu_n, q),$$

where

$$M(\nu_n, \nu_n, q) = \max[G(\nu_n, \nu_n, q), G(\nu_n, T\nu_n, T\nu_n), G(q, Tq, Tq)]$$

$$= \max[G(\nu_n, \nu_n, q), G(\nu_n, q, q) + G(q, T\nu_n, T\nu_n)].$$

If $M(\nu_n, \nu_n, q) = G(\nu_n, q, q) + G(q, T\nu_n, T\nu_n)$, we get

$$G(T\nu_n, T\nu_n, Tw) \leq r(G(\nu_n, q, q) + G(q, T\nu_n, T\nu_n))$$

$$G(T\nu_n, T\nu_n, q) \leq \frac{r}{1 - r}G(\nu_n, q, q). \tag{2.6}$$

If $M(\nu_n, \nu_n, q) = G(\nu_n, \nu_n, q)$, then

$$G(T\nu_n, T\nu_n, q) \leq rG(\nu_n, q, q). \tag{2.7}$$

In each of the inequalities (2.6) and (2.8), take the limit as $n \to \infty$ to see that $G(T\nu_n, T\nu_n, q) \to 0$. So, by Proposition(1.4), the sequence $(T\nu_n)$ is G-convergent to $q = Tq$. Consequently, T is G-continuous at q. □

We now give a fixed point theorem on compact G-metric spaces.

Theorem 2.2. *Let (X,G) be a compact G-metric space and let T be a mapping on X. Assume that*

$$aG(\zeta, T\zeta, T\zeta) + bG(\nu, T\zeta, T\zeta) + cG(z, T\zeta, T\zeta) < G(\zeta, \nu, z)$$

implies

$$G(T\zeta, T\nu, Tz) < M(\zeta, \nu, z), \textit{for all } \zeta, \nu, z \in X,$$

where $M(\zeta, \nu, z) = \max[G(\zeta, \nu, z), G(\zeta, T\zeta, T\zeta), G(\nu, T\nu, T\nu), G(z, Tz, Tz)]$ *and* $a > 0, b > 0, c > 0$, $3a + 2(b + c) < 1, 2(b + c) < 1$. *Then T has a unique fixed point.*

Proof. If we consider $\beta = \inf\{G(\zeta, T\zeta, T\zeta) : \zeta \in X\}$, then there exists a sequence (ζ_n) in X such that $\lim_{n\to\infty} G(\zeta_n, T\zeta_n, T\zeta_n) = \beta$. Since X is compact G- metric space, there exists $v, q \in X$ such that a sequence (ζ_n) G-converges to $v \in X$, and $(T\zeta_n)$ G-converges to q$\in X$. We assume $\beta > 0$. Hence, by the continuity of the function G, we have

$$\beta = \lim_{n\to\infty} G(\zeta_n, T\zeta_n, T\zeta_n) = G(v, q, q) = \lim_{n\to\infty} G(\zeta_n, q, q).$$

Since

$$\lim_{n\to\infty}[aG(\zeta_n, T\zeta_n, T\zeta_n) + (b + c)G(q, T\zeta_n, T\zeta_n)] = a\beta < \lim_{n\to\infty} G(\zeta_n, q, q) = \beta,$$

we can choose a positive integer N such that

$$aG(\zeta_n, T\zeta_n, T\zeta_n) + (b + c)G(q, T\zeta_n, T\zeta_n) < G(\zeta_n, q, q), \text{ for all } n \geq N.$$

By hypothesis, $G(T\zeta_n, Tq, Tq) < M(\zeta_n, q, q)$, holds for $n \geq N$, where

$$M(\zeta_n, q, q) = \max[G(\zeta_n, q, q), G(\zeta_n, T\zeta_n, T\zeta_n), G(q, Tq, Tq)],$$

this implies
$G(q, Tq, Tq) = \lim_{n\to\infty} G(T\zeta_n, Tq, Tq) < \lim_{n\to\infty} M(\zeta_n, q, q) = \max[\beta, G(q, Tq, Tq)]$, if $\max[\beta, G(q, Tq, Tq)] = G(q, Tq, Tq)$, which is impossible. As a result, $G(q, Tq, Tq) < \beta$. From the definition of β we obtain $\beta = G(q, Tq, Tq)$. Since

$$aG(q, Tq, Tq) + (b + c)G(Tq, Tq, Tq) < G(q, Tq, Tq),$$

we have $G(Tq, T^2q, T^2q) < M(q, Tq, Tq) = \max[G(q, Tq, Tq), G(Tq, T^2q, T^2q)] = G(q, Tq, Tq) = \beta$, which contradicts the definition of β. Therefore $\beta = 0$, and

$$\lim_{n\to\infty} G(\zeta_n, q, q) = \lim_{n\to\infty} G(\zeta_n, T\zeta_n, T\zeta_n) = \lim_{n\to\infty} G(v, T\zeta_n, T\zeta_n) = G(v, q, q) = 0,$$

so $v = q$. Thus $\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} T\zeta_n$. Next, we show that T has a fixed point. Suppose, on the contrary, that T does not have a fixed point. Since

$$aG(\zeta_n, T\zeta_n, T\zeta_n) + (b+c)G(T\zeta_n, T\zeta_n, T\zeta_n) < G(\zeta_n, T\zeta_n, T\zeta_n), \text{ for all } n \geq 1$$

we get
$G(T\zeta_n, T^2\zeta_n, T^2\zeta_n) < M(\zeta_n, T\zeta_n, T\zeta_n) = \max[G(\zeta_n, T\zeta_n, T\zeta_n), G(T\zeta_n, T^2\zeta_n, T^2\zeta_n)] = G(\zeta_n, T\zeta_n, T\zeta_n)$. By using the triangle inequality, we have

$$G(q, T^2\zeta_n, T^2\zeta_n) \leq G(q, T\zeta_n, T\zeta_n) + G(T\zeta_n, T^2\zeta_n, T^2\zeta_n) < G(q, T\zeta_n, T\zeta_n) + G(\zeta_n, T\zeta_n, T\zeta_n)$$

$$\lim_{n\to\infty} G(q, T^2\zeta_n, T^2\zeta_n) < \lim_{n\to\infty} (G(q, T\zeta_n, T\zeta_n) + G(\zeta_n, T\zeta_n, T\zeta_n)) = 0.$$

Thus $(T^2\zeta_n)$ is G-convergent to q. By induction, we obtain

$$G(T^p\zeta_n, T^{p+1}\zeta_n, T^{p+1}\zeta_n) \leq G(T^{p-1}\zeta_n, T^p\zeta_n, T^p\zeta_n) \leq ... \leq G(\zeta_n, T\zeta_n, T\zeta_n)$$

$$G(q, T^p\zeta_n, T^p\zeta_n) \leq G(q, T^{p-1}\zeta_n, T^{p-1}\zeta_n) + G(T^{p-1}\zeta_n, T^p\zeta_n, T^p v_n)$$
$$\lim_{n\to\infty} G(q, T^p\zeta_n, T^p\zeta_n) < \lim_{n\to\infty} (G(q, T^{p-1}\zeta_n, T^{p-1}\zeta_n) + G(v, T\zeta_n, T\zeta_n))$$

because $\lim_{n\to\infty} T^p\zeta_n = q$, for all $p \geq 1$. If there exists an integer $p \geq 1$ and a subsequence (ζ_{n_k}) of (ζ_n) such that

$$aG(T^{p-1}\zeta_{n_k}, T^p\zeta_{n_k}, T^p\zeta_{n_k}) + (b+c)G(q, T^p\zeta_{n_k}, T^p\zeta_{n_k}) < G(T^{p-1}\zeta_{n_k}, q, q), \text{ for all } k \geq 1.$$

By hypothesis, we have

$$G(T^p\zeta_{n_k}, Tq, Tq) < M(T^{p-1}\zeta_{n_k}, q, q) =$$

$$\max[G(T^{p-1}\zeta_{n_k}, q, qw), G(T^{p-1}\zeta_{n_k}, T^p\zeta_{n_k}, T^p x_{n_k}), G(q, Tq, Tq)].$$

Taking the limit as $k \to \infty, G(q, Tq, Tq) = 0$, $q = Tq$ which is a contradiction. Hence, we can assume that for every $m \geq 1$, there exists an integer $n(m) \geq 1$ such that

$$aG(T^{m-1}\zeta_{n_k}, T^m\zeta_n, T^m\zeta_n) + (b+c)G(q, T^m x_n, T^m x_n) \geq G(T^{m-1}\zeta_n, q, q), \text{ for all } n \geq n(m) \tag{2.11}$$

Put $\gamma = \max[n(1), n(2), ..., n(p)]$. Then by inequality (2.11) we have

$$G(\zeta_n, q, q) \leq aG(\zeta_n, T\zeta_n, T\zeta_n) + (b+c)G(q, T\zeta_n, T\zeta_n) \leq aG(\zeta_n, T\zeta_n, T\zeta_n) + 2(b+c)G(q, q, T\zeta_n)$$

$$\leq aG(\zeta_n, T\zeta_n, T\zeta_n) + 2(b+c)(aG(T\zeta_n, T^2\zeta_n, T^2\zeta_n) + (b+c)G(q, T^2\zeta_n, T^2\zeta_n))$$

$$\leq aG(\zeta_n, T\zeta_n, T\zeta_n) + 2a(b+c)G(T\zeta_n, T^2\zeta_n, T^2\zeta_n) + (2(b+c))^2G(q,q,T^2\zeta_n)$$
$$\leq aG(\zeta_n, T\zeta_n, T\zeta_n) + 2a(b+c)G(T\zeta_n, T^2\zeta_n, T^2\zeta_n)+$$
$$(2(b+c))^2(aG(T^2\zeta_n, T^3\zeta_n, T^3\zeta_n) + (b+c)G(q, T^3\zeta_n, T^3\zeta_n)) \leq ...$$
$$\leq aG(\zeta_n, T\zeta_n, T\zeta_n) + 2a(b+c)G(T\zeta_n, T^2\zeta_n, T^2\zeta_n) + ...+$$
$$a(2(b+c))^{p-1}G(T^{p-1}\zeta_n, T^p\zeta_n, T^p\zeta_n) + (2(b+c))^pG(q,q,T^2\zeta_n)$$
$$\leq a(1+2(b+c)+(2(b+c))^2+...+(2(b+c))^{p-1})G(\zeta_n, T\zeta_n, T\zeta_n)+(2(b+c))^pG(q,q,T^p\zeta_n)$$
$$G(\zeta_n, q, q) \leq \frac{a(1-d^p)}{1-d}G(\zeta_n, T\zeta_n, T\zeta_n) + d^pG(q,q,T^p\zeta_n), \text{ where } d = 2(b+c). \tag{2.12}$$

Using the triangle inequality and Proposition (1.2), we get

$$G(T^p\zeta_n, q, q) \leq G(T^p\zeta_n, \zeta_n, \zeta_n)+G(\zeta_n, q, q) \leq 2G(T^p\zeta_n, T^p\zeta_n, \zeta_n)+G(\zeta_n, q, q)$$
$$\leq 2(G(\zeta_n, T\zeta_n, T\zeta_n)+G(T\zeta_n, T^2\zeta_n, T^2\zeta_n)+...+G(T^{p-1}\zeta_n, T^p\zeta_n, T^p\zeta_n))+G(\zeta_n, q, q)$$
$$\leq 2pG(\zeta_n, T\zeta_n, T\zeta_n) + G(v, q, q) \tag{2.13}$$

Substituting (2.13) into (2.12), we get

$$G(\zeta_n, q, q) \leq \frac{a(1-d^p)}{1-d}G(\zeta_n, T\zeta_n, Tv) + d^p(2pG(\zeta_n, T\zeta_n, T\zeta_n) + G(\zeta_n, q, q))$$
$$(1-d^p)G(\zeta_n, q, q) \leq (\frac{a(1-d^p)}{1-d} + 2pd^p)G(\zeta_n, T\zeta_n, T\zeta_n)$$
$$G(\zeta_n, q, q) \leq (\frac{a}{1-d} + \frac{2pd^p}{1-d^p})G(\zeta_n, T\zeta_n, T\zeta_n), \text{ for all } n \geq \gamma \tag{2.14}$$

Similarly,

$$G(T\zeta_n, q, q) \leq (\frac{a}{1-d} + \frac{2(p-1)d^{p-1}}{1-d^{p-1}})G(T\zeta_n, T^2\zeta_n, T^2\zeta_n)$$
$$\leq (\frac{a}{1-d} + \frac{2(p-1)d^{p-1}}{1-d^{p-1}})G(\zeta_n, T\zeta_n, T\zeta_n). \tag{2.15}$$

Since $G(\zeta_n, T\zeta_n, T\zeta_n) \leq G(\zeta_n, q, q)+G(q, T\zeta_n, T\zeta_n) \leq G(\zeta_n, q, q)+2G(q, q, Tv)$, by using inequalities (2.14) and (2.15),

$$G(\zeta_n, T\zeta_n, T\zeta_n) \leq (\frac{a}{1-d} + \frac{2pd^p}{1-d^p} + \frac{2a}{1-d} + \frac{4(p-1)d^{p-1}}{1-d^{p-1}})G(\zeta_n, T\zeta_n, T\zeta_n)$$

$$\leq (\frac{3a}{1-d} + \frac{2pd^p}{1-d^p} + \frac{4(p-1)d^{p-1}}{1-d^{p-1}})G(\zeta_n, T\zeta_n, T\zeta_n)), \text{ for all } n \geq \gamma$$

Since $\lim\limits_{p\to\infty} \frac{2pd^p}{1-d^p} = 0$, and $\frac{3a}{1-d} < 1$, we can choose p satisfying $\frac{3a}{1-d} + \frac{2pd^p}{1-d^p} + \frac{4(p-1)d^{p-1}}{1-d^{p-1}} < 1$. Then

$$G(\zeta_n, T\zeta_n, T\zeta_n) < G(\zeta_n, T\zeta_n, T\zeta_n)$$

which is not the case. So there exists $z \in X$ such that $Tz = z$. Fix $\nu \in X$ with $\nu \neq \zeta$. Then $aG(\zeta, T\zeta, T\zeta) + (b+c)G(\nu, T\zeta, T\zeta) = (b+c)G(\nu, \zeta, v) \leq 2(b+c)G(\zeta, \nu, \nu) < G(\zeta, \nu, \nu)$.
$G(T\zeta, T\nu, T\nu) < M(\zeta, \nu, \nu) = \max[G(\zeta, \nu, \nu), G(\zeta, T\zeta, T\zeta), G(v, T\nu, T\nu)] = G(\zeta, \nu, v)$. Hence ν is not a fixed point of T. Consequently, T has a unique fixed point. □

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