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On Pure Fuzzy Ideals in Ordered Semigroups

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Abstract

In this paper the concepts of pure fuzzy and weakly pure fuzzy ideals in ordered semigroups are introduced. Characterizations of right weakly regular ordered semigroups by fuzzy ideals are discussed. Also, we obtain that for each pure fuzzy ideal of ordered semigroups is weakly pure.

1 Introduction and Preliminaries

The notions of pure ideal and purely prime ideals in semigroups were introduced by Ahsan and Takahashi [2] and those of pure fuzzy, purely fuzzy maximal and purely fuzzy prime ideals of a semigroups without order were introduced by Ahsan, et. al. [1]. Recently, the notion of pure fuzzy, purely fuzzy maximal and purely fuzzy prime ideals of ternary semigroups were

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introduced by Bashir, Shabir and Rehman [3]. The authors gave some characterizations of pure fuzzy ideals and proved that each fuzzy ideal is weakly pure.

This paper is in line with Bushir et. al. [3]. We introduce the concepts of pure fuzzy and weakly pure fuzzy ideals in ordered semigroups. Besides, we also characterize those ordered semigroups for which fuzzy two-sided ideal is right weakly pure fuzzy.

We recall some basic definitions and results that are relevant to this paper.

A semigroup $(S, \cdot)$ with an order relation $\leq$ is called an *ordered semigroups* ([4], [5], [6]) if $x \leq y$ implies $cx \leq cy$ and $xc \leq yc$ for all $x, y, c \in S$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. The *zero* of an ordered semigroup S is an element of S usually denoted by 0 such that $0 \leq a$ and $0a = a0 = 0$ for all $a \in S$. Let $A, B \subseteq S$. Then

$$AB = \{xy \in S \mid x \in A, y \in B\}$$

and

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

For an element $x \in S$, we write $\{x\}A$ as xA and $A\{x\}$ as Ax. A non-empty subset A of S is a subsemigroup of S if and only if $AA \subseteq A$. It is easy to check that the following hold:

(1) $A \subseteq B$ implies $(A] \subseteq (B]$;

(2) $((A]] = (A]$;

(3) $(A](B] \subseteq (AB]$;

(4) $((A]B] = (A(B]] = ((A](B]] = (AB]$;

(5) $(A] \cup (B] = (A \cup B]$.

For further information we refer the reader to [10].

Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left* (resp. *right*) *ideal* of an ordered semigroup S if satisfies:

(i) $SA \subseteq A$ (resp. $AS \subseteq A$);

(ii) for any $x \in A$ and $y \in S$, if $y \leq x$ then $y \in A$, or equivalently, $A = (A]$.

If A is both a left and a right ideal of S, then A is called a *two-sided ideal* (or an *ideal*) of S. It is well-known that the a nonempty intersection of ideals of S is an ideal of S and the union of ideals of S is an ideal of S.

We denote by f_A the characteristic mapping of a subset A of an ordered semigroup S; that is, the mapping of S into $[0,1]$ is defined by

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $(S, \cdot, \leq)$ be an ordered semigroups. A *fuzzy subset* of S is mapping f of S into the unit interval $[0,1]$ of real numbers. For $a \in S$, define

$$\mathcal{A}_a = \{(y,z) \in S \times S \mid a \leq yz\}.$$

For two fuzzy subsets f and g of S, define

$$f \circ g(a) = \begin{cases} \bigvee_{(y,z) \in \mathcal{A}_a} \min\{f(y), g(z)\} & \text{if } \mathcal{A}_a \neq \emptyset \\ 0 & \text{if } \mathcal{A}_a = \emptyset. \end{cases}$$

We define the operations $f \vee g$ and $f \wedge g$ as the fuzzy subsets of S defined by:

$$(f \vee g)(x) = max\{f(x), g(x)\},\ (f \wedge g)(x) = \min\{f(x), g(x)\}.$$

We denote by $F(S)$ the set of all fuzzy subsets of an ordered semigroup S. It is easy to verify that $(F(S), \circ)$ is a semigroup. Any intersection of fuzzy ideals of S is a fuzzy ideal of S. Moreover, the union of any family of fuzzy ideals is a fuzzy ideal of S. A is a left (resp. right) ideal of an ordered semigroup S if and only if f_A, the characteristic function of , is fuzzy left (resp. right) ideal of S. We define an order relation $\preceq$ on $F(S)$ as follows: for $x \in S$, $f \preceq g$ implies $f(x) \leq g(x)$. It is easy to see that $(F(S), \prec)$ is a poset with the least element 0 and the greatest element 1.

Definition 1.1. ([9]) Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy subsemigroup* of S if

$$f(xy) \geq \min\{f(x), f(y)\}$$

for all $x, y \in S$.

Definition 1.2. ([9]) Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *ideal* of S if

(i) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$);

(ii) $x \leq y$ implies $f(x) \geq f(y)$ for all $x, y \in S$.

If f is both a fuzzy left and right ideal of S, then f is called a *fuzzy two-sided ideal of S (or fuzzy ideal) of S.*

Definition 1.3. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy subset f of S is called a *idempotent* of S if $f \circ f = f$.

2 Pure fuzzy ideals

In this section, the notion of a pure fuzzy ideal in ordered semigroups will be introduced and studied.

Definition 2.1. Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal A of S is called a *left* (resp. *right*) *pure ideal* [7] of S if for each $a \in A$ there exists $x \in S$ such that $a \leq xa$ (resp. $a \leq ax$). Similarly, we define one-sided left and right pure ideals.

Definition 2.2. Let $(S, \cdot, \leq)$ be an ordered semigroup. A fuzzy ideal δ of S is called a *pure fuzzy ideal* of S if $\xi \wedge \delta = \xi \circ \delta$ for all right fuzzy ideals ξ of S.

Lemma 2.3. ([7]) *Let $(S, \cdot, \leq)$ be an ordered semigroup and A an ideal of S. Then A is right pure if and only if $B \cap A = (BA]$ for all right ideals B of S.*

Lemma 2.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup and A a right ideal of S. Then the following conditions are equivalent:*

(i) $B \cap A = (BA]$ for all right ideals B of S;

(ii) $f_B \wedge g_A = f_B \circ g_A$.

Proof. (i)$\Rightarrow$ (ii) Suppose that $B \cap A = (BA]$ for all right ideals B of S. To show that $f_B \wedge g_A = f_B \circ g_A$. Let $a \in S$. If $a \in B \cap A$, then $(f_B \wedge g_A)(a) = \min\{f_B(a), g_A(a)\} = 1$. By assumption, $a \leq uv$ for some $u \in B$ and $v \in A$. That is $\mathcal{A}_a \neq \emptyset$. We obtain

$$(f_B \circ g_A)(a) = \bigvee_{(y,z)\in\mathcal{A}_a} \min\{f_B(y), g_A(z)\} \geq \min\{f_B(u), g_A(v)\} = 1.$$

It follows that $(f_B \circ g_A)(a) = 1$. If $a \notin A$ or $a \notin B$, then $(f_B \wedge g_A)(a) = \min\{f_B(a), g_A(a)\} = 0$ and $\mathcal{A}_a = \emptyset$. This means that $(f_B \circ g_A)(a) = 0$. Hence, $f_B \wedge g_A = f_B \circ g_A$.

(ii)$\Rightarrow$ (i) Assume that $f_B \wedge g_A = f_B \circ g_A$. Now let B be a right ideal of S and $a \in B \cap A$. By hypothesis this implies that $1 = (f_B \wedge g_B)(a) = (f_B \circ g_A)(a)$; that is, $(f_B \circ g_A)(a) = 1$. It follows that $a \leq u'v'$ for some $u' \in B$ and $v' \in A$. We conclude that $a \in (BA]$. On the other hand, we have $(BA] \subseteq B \cap A$. Therefore, $B \cap A = (BA]$.

□

Lemma 2.5. ([8]) *Let $(S, \cdot, \leq)$ be an ordered semigroup, n a natural number, $n \geq 2$ and $\{A_1, A_2, \ldots, A_n\}$ a set of non-empty subset of S. Then we have*

$$f_{A_1} \circ f_{A_2} \circ \cdots \circ f_{A_n} = f_{(A_1 A_2 \cdots A_n]}.$$

Theorem 2.6. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then an ideal A of S is right pure in S if and only if the characteristic function of A, denoted by α_A, is a pure fuzzy ideal of S.*

Proof. Let A be a right pure ideal in S. Let α_A be the characteristic function of A. We will show that α_A is a pure fuzzy ideal. Let $a \in S$. Since A is an ideal, α_A is a fuzzy ideal. Assume that β is a fuzzy right ideal of S. We consider

$$\begin{aligned}(\beta \circ \alpha_A)(a) &= \bigvee_{(y,z) \in \mathcal{A}_a} \min\{\beta(y), \alpha_A(z)\} \\ &\leq \bigvee_{(y,z) \in \mathcal{A}_a} \min\{\beta(yz), \alpha_A(yz)\} \\ &\leq \min\{\beta(a), \alpha_A(a)\} \\ &= (\beta \wedge \alpha_A)(a).\end{aligned}$$

This shows that $\beta \circ \alpha_A \leq \beta \wedge \alpha_A$. On the other hand, we have $(\beta \wedge \alpha_A)(a) = \beta(a) \wedge \alpha_A(a) = 0$ if $a \notin A$, hence $(\beta \wedge \alpha_A)(a) \leq (\beta \circ \alpha_A)(a)$. We consider the case $a \in A$. Since A is right pure, $a \leq ax$ for some $x \in A$. It follows that $\mathcal{A}_a \neq \emptyset$. Also, $\alpha_A(a) = 1 = \alpha_A(x)$. Therefore, we have

$$\begin{aligned}(\beta \circ \alpha_A)(a) &= \bigvee_{(y,z) \in \mathcal{A}_a} \min\{\beta(y), \alpha_A(z)\} \\ &\geq \min\{\beta(a), \alpha_A(x)\} \qquad (2.1) \\ &= \min\{\beta(a), \alpha_A(a)\} \\ &= (\beta \wedge \alpha_A)(a)\end{aligned}$$

This implies that $\beta \wedge \alpha_A \leq \beta \circ \alpha_A$. Consequently, $\beta \wedge \alpha_A = \beta \circ \alpha_A$.

Conversely, assume that α_A is a pure fuzzy ideal in S. To show that A is a right pure of S, let B be a right ideal of S. Since B is right ideal of S, the characteristic function α_B of B is a fuzzy right ideal of S. Since α_A is pure fuzzy, $\alpha_B \wedge \alpha_A = \alpha_B \circ \alpha_A$. By Lemma 2.4, $B \cap A = (BA]$. Hence, A is right pure by Lemma 2.3. □

Theorem 2.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the fuzzy ideals ϕ and f_S of S, defined respectively as*

$$\phi(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and $f_S(x) = 1$ for all $x \in S$, are pure fuzzy ideals of S.

Proof. Let β be a fuzzy right ideal of S. We will prove that $\beta \wedge \phi = \beta \circ \phi$ and $\beta \wedge f_S = \beta \circ f_S$. If $a \neq 0$, then

$$(\beta \wedge \phi)(a) = \min\{\beta(a), \phi(a)\} = \min\{\beta(a), 0\} = 0 \leq (\beta \circ \phi)(a).$$

If $a = 0$, then $\mathcal{A}_a \neq \emptyset$. We have

$$(\beta \wedge \phi)(0) = 0 \leq \bigvee_{(x,y)\in\mathcal{A}_a} \min\{\beta(y), \phi(z)\} = (\beta \circ \phi)(0).$$

This proves that $\beta \wedge \phi \leq \beta \circ \phi$. On the other hand, let $a \in S$. Then

$$\begin{aligned}(\beta \circ \phi)(a) &= \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\beta(y), \phi(z)\} \leq \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\beta(yz), \phi(yz)\} \\ &\leq \min\{\beta(a), \phi(a)\} = (\beta \wedge \phi)(a).\end{aligned}$$

This shows that $\beta \circ \phi \leq \beta \wedge \phi$. Thus $\beta \wedge \phi = \beta \circ \phi$.

Since $f_S(u) = 1$ for all $u \in S$,

$$(\beta \circ f_S)(a) = \bigvee_{(y,z)\in\mathcal{A}_a} (\beta(y) \wedge f_S(z)) \geq \min\{\beta(a), f_S(a)\} = (\beta \wedge f_S)(a).$$

This proves that $\beta \circ f_S \geq \beta \wedge f_S$. For the converse

$$\begin{aligned}(\beta \circ f_S)(a) &= \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\beta(y), f_S(z)\} \leq \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\beta(yz), f_S(yz)\} \\ &\leq \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\beta(a), f_S(a)\} = \min\{\beta(a), f_S(a)\} \\ &= (\beta \wedge f_S)(a).\end{aligned}$$

This implies that $\beta \circ f_S \leq \beta \wedge f_S$. Hence, $\beta \wedge f_S = \beta \circ f_S$. □

Theorem 2.8. *If* $\{\delta_i : i \in \Lambda\}$ *is a family of pure fuzzy ideals of an ordered semigroup* S, *then* $\bigvee_{i\in\Lambda} \delta_i$ *is a pure fuzzy ideal of* S.

Proof. Let $\{\delta_i : i \in \Lambda\}$ be a family of pure fuzzy ideals of an ordered semigroup S. Assume that ξ is a fuzzy right ideal of S. For each $a \in S$, we have

$$\begin{aligned}(\xi \circ (\bigvee_{i\in\Lambda} \delta_i))(a) &= \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\xi(y), (\bigvee_{i\in\Lambda} \delta_i)(z)\} \\ &\leq \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\xi(yz), (\bigvee_{i\in\Lambda} \delta_i)(yz)\} \\ &\leq \min\{\xi(a), (\bigvee_{i\in\Lambda} \delta_i)(a)\} \\ &= (\xi \wedge (\bigvee_{i\in\Lambda} \delta_i))(a).\end{aligned}$$

This implies that $\xi \circ (\bigvee_{i\in\Lambda} \delta_i) \leq \xi \wedge (\bigvee_{i\in\Lambda} \delta_i)$. On the other hand, we consider

$$\begin{aligned}(\xi \circ (\bigvee_{i\in\Lambda} \delta_i))(a) &= \xi(a) \circ (\bigvee_{i\in\Lambda} \delta_i)(a) = \bigvee_{(y,z)\in\mathcal{A}_a} \min\{\xi(y), (\bigvee_{i\in\Lambda} \delta_i)(z)\} \\ &= \bigvee_{i\in\Lambda} \left(\bigvee_{(y,z)\in\mathcal{A}_a} \min\{\xi(y), \delta_i(z)\} \right) \geq \left(\bigvee_{(y,z)\in\mathcal{A}_a} \min\{\xi(y), \delta_i(z)\} \right) \\ &= (\xi \circ \delta_i)(a).\end{aligned}$$

Since δ_i is a pure fuzzy ideal,

$$\begin{aligned}\bigvee_{i\in\Lambda} (\xi \circ \delta_i)(a) &= \bigvee_{i\in\Lambda} (\xi \wedge \delta_i)(a) = \bigvee_{i\in\Lambda} \min\{\xi(a), \delta_i(a)\} \\ &= \min\{\xi(a), (\bigvee_{i\in\Lambda} \delta_i)(a)\} = (\xi \wedge (\bigvee_{i\in\Lambda} \delta_i))(a).\end{aligned}$$

This proves that $(\xi \circ (\bigvee_{i\in\Lambda} \delta_i))(a) \geq \bigvee_{i\in\Lambda} (\xi \circ \delta_i)(a) = (\xi \wedge (\bigvee_{i\in\Lambda} \delta_i))(a)$. We obtain $\xi \wedge (\bigvee_{i\in\Lambda} \delta_i) = \xi \circ (\bigvee_{i\in\Lambda} \delta_i)$. Thus, $\bigvee_{i\in\Lambda} \delta_i$ is a pure fuzzy ideal of S. □

Theorem 2.9. *Let* $(S, \cdot, \leq)$ *be an ordered semigroup. If* δ_1 *and* δ_2 *are pure fuzzy ideals of* S, *then* $\delta_1 \wedge \delta_2$ *is a pure fuzzy ideal of* S.

Proof. Assume that δ_1 and δ_2 are pure fuzzy ideals of S. Let ξ be a right fuzzy ideal of S. By hypothesis, we then have $\xi \wedge \delta_1 = \xi \circ \delta_1$ and $\xi \wedge \delta_2 = \xi \circ \delta_2$. Since δ_2 is a right pure fuzzy ideal of S. It follows that $\delta_1 \wedge \delta_2 = \delta_1 \circ \delta_2$. Consequently, $\xi \circ (\delta_1 \wedge \delta_2) = \xi \circ (\delta_1 \circ \delta_2)$. Since $\delta_1 \circ \delta_2$ is a fuzzy ideal of S, hence

$$\xi \circ (\delta_1 \wedge \delta_2) = \xi \circ (\delta_1 \circ \delta_2) = \xi \wedge (\delta_1 \circ \delta_2) = \xi \wedge (\delta_1 \wedge \delta_2).$$

As a result, $\delta_1 \wedge \delta_2$ is pure fuzzy ideal of S.

□

3 Right Weakly Pure Ideals

Definition 3.1. ([7]) Let $(S, \cdot, \leq)$ be an ordered semigroup. An ideal A of S is called *right weakly regular* if for any $x \in S$ then there exist $y, z \in S$ such that $x \leq xyxz$.

In an ordered semigroup, every left (right) pure two-sided ideal is left (right) weakly pure.

Theorem 3.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following conditions are equivalent:*

(i) *S is a right weakly regular*

(ii) *Each every fuzzy right ideal is idempotent*

(iii) *$\xi \wedge \delta = \xi \circ \delta$ for every fuzzy right ideal ξ and for every fuzzy ideal δ of S.*

Proof. (i)⇒(ii) Assume that S is a right weakly regular. Let δ be a fuzzy right ideal of S. We will show that $\delta^2 = \delta \circ \delta = \delta$. Let $a \in S$. By assumption, we then have $a \leq ayaz = (ay)(az)$ for some $y, z \in S$; that is, $\mathcal{A}_a \neq \emptyset$. Therefore,

$$\begin{aligned} \delta^2(a) &= \delta(a) \circ \delta(a) = \bigvee_{(u,v) \in \mathcal{A}_a} \min\{\delta(u), \delta(v)\} \\ &\geq \min\{\delta(ay), \delta(az)\} \geq \min\{\delta(a), \delta(a)\} \\ &= (\delta \wedge \delta)(a) = \delta(a). \end{aligned}$$

This proves that $\delta^2(a) \geq \delta(a)$. For the reverse inclusion, we have

$$\begin{aligned} \delta(a) &\leq \bigvee_{(ay,az)\in\mathcal{A}_a} \min\{\delta(ay),\delta(az)\} \\ &= \bigvee_{(u,v)\in\mathcal{A}_a} \min\{\delta(u),\delta(v)\} = \delta(a)\circ\delta(a) \end{aligned}$$

That is $\delta^2(a) \leq \delta(a)$. Thus, $\delta^2 = \delta$.

(ii)⇒(i) Assume that every fuzzy right ideal is idempotent. Let $a \in S$ and $A = (a \cup aS]$ be a right ideal generated by a. Let δ_A be the characteristic function of A. By Lemma 2.5, we have $\delta_A = \delta_A \circ \delta_A = \delta_{(A^2]}$. This proves that $A = (A^2]$. Since $a \in A$, so

$$a \in (A^2] = ((a\cup aS](a\cup aS]] = (a^2\cup a^2S\cup aSa\cup aSaS] = (aSaS].$$

This implies that $a \in (aSaS]$; that is, $a \leq axay$ for some $x, y \in S$. Hence, S is right weakly regular.

(i)⇒(iii) Let ξ and δ be a fuzzy right ideal and a fuzzy ideal of S respectively. Let f_S be the characteristic mapping of S. Since $\xi\circ\delta \leq f_S\circ\delta \leq \delta$ and $\xi\circ\delta \leq \xi\circ f_S \leq \xi$, $\xi\circ\delta \leq \xi\wedge\delta$. Now let $a \in S$. Since S is right weakly regular, there exist $x, y \in S$ such that $a \leq axay$; that is, $\mathcal{A}_a \neq \emptyset$. We have

$$\begin{aligned} (\xi\circ\delta)(a) &= \bigvee_{(u,v)\in\mathcal{A}_a} \min\{\xi(u),\delta(v)\} \geq \min\{\xi(ax),\delta(ay)\} \\ &\geq \min\{\xi(a),\delta(a)\} = (\xi\wedge\delta)(a) \end{aligned}$$

This proves that $\xi\circ\delta \geq \xi\wedge\delta$. We conclude that $\xi\circ\delta = \xi\wedge\delta$.

(iii)⇒(i) Suppose $\xi\wedge\delta = \xi\circ\delta$ for every fuzzy right ideal ξ and for every fuzzy ideal δ of S. We will prove that S is right weakly regular. Now let $a \in S$ and $A = (a\cup aS\cup Sa\cup SaS]$. Then A is an ideal of S. Let δ_A be the characteristic function δ_A of A. Then δ_A is a fuzzy ideal of S. By assumption, δ_A is a pure fuzzy ideal of S. By Theorem 2.6, A is right pure in S. Since $a \in A$ and A is right pure ideal, there exists $x \in A$ such that $a \leq ax$; that is, $a \in (aA]$. Since

$$(aA] = (a(a\cup aS\cup Sa\cup SaS]] = (a^2\cup a^2S\cup aSa\cup aSaS] = (aSaS].$$

That is $a \in (aSaS]$. Hence, S is right weakly regular. □

Theorem 3.3. *Let $(S,\cdot,\leq)$ be an ordered semigroup. Then the following conditions are equivalent:*

(i) *S is a right weakly regular;*

(ii) *Every fuzzy ideal δ of S is fuzzy pure.*

Proof. This follows from Theorem 3.2 and Theorem 2.6. □

4 Weakly pure fuzzy ideals

Definition 4.1. A fuzzy ideal δ of an ordered semigroup S is said to be *left* (resp. *right*) *weakly pure* if $\delta \wedge \xi = \delta \circ \xi$ (resp. $\xi \wedge \delta = \xi \circ \delta$) for all fuzzy deals ξ of S.

Theorem 4.2. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then the following conditions are equivalent:*

(i) *Each every fuzzy ideal is left weakly pure fuzzy*

(ii) *Each every fuzzy ideal is idempotent*

(iii) *Each every fuzzy ideal right weakly pure fuzzy.*

Proof. (i)⇒(ii) Assume that every fuzzy ideal is left weakly pure fuzzy. Let δ be a fuzzy ideal of S. Then for any a fuzzy ideal ξ of S, we have $\delta \wedge \xi = \delta \circ \xi$. Then $\delta = \delta \wedge \delta = \delta \circ \delta$; that is, δ is idempotent.

(ii)⇒(i) Assume that every fuzzy ideal is idempotent. Let δ, ξ be fuzzy ideals of S. Then

$$\delta \wedge \xi = (\delta \wedge \xi) \circ (\delta \wedge \xi) \leq \delta \circ \xi.$$

On the other hand, $\delta \circ \xi \leq \delta \wedge \xi$ always. Therefore, $\delta \circ \xi = \delta \wedge \xi$. Consequently, δ is left weakly pure fuzzy.

(ii)⇒(iii) Assume that every fuzzy ideal is idempotent. Let δ be a fuzzy ideal of S. Then for any fuzzy ideal ξ of S, we have

$$\xi \wedge \delta = (\xi \wedge \delta) \circ (\xi \wedge \delta) \leq \xi \circ \delta.$$

On the other hand, $\xi \circ \delta \leq \xi \wedge \delta$. Thus, $\xi \wedge \delta = \xi \circ \delta$. Hence δ is right weakly pure fuzzy.

(iii)⇒(ii) Assume that every fuzzy ideal is right weakly pure fuzzy ideal. Let δ be a fuzzy ideal of S. Then for any fuzzy ideal ξ of S, we have $\xi \wedge \delta = \xi \circ \delta$. Therefore, $\delta = \delta \wedge \delta = \delta \circ \delta$. Consequently, δ is an idempotent of S. □

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