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Characterization of Some Types of Semigroups by using Interval Valued Fuzzy Weakly Interior Ideals

Thiti Gaketem

School of Sciences
University of Phayao
Phayao 56000, Thailand

email: newtonisaac41@yahoo.com

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Abstract

In this article, we give a definition of an interval valued fuzzy weakly interior ideal. We study some interesting properties of interval valued fuzzy weakly interior ideals and the relationship between interval valued fuzzy weakly interior ideals and interval valued fuzzy ideals. We characterize some semigroups by using interval valued fuzzy weakly interior ideals. Moreover, we establish theorems of the homomorphic image and the preimage of an interval valued fuzzy weakly interior ideal in semigroups.

1 Introduction

In 1965, Zadeh introduced the concept of fuzzy sets. Since then fuzzy sets have been applied to many branches in Mathematics and in addition to many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, topology, etc. In 1971, Rosenfeld [11] was the first to give the definition of fuzzy subgroups fuzzy left (right, two-sided) ideals. In 1979, Kuroki [8] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups

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and characterized them. The concept of fuzzy interior ideals in a semigroup was introduced by Hong [7] and he obtained some related properties of such ideals.

In 1975 [15], Zadeh made an extension of the concept of fuzzy sets by an interval valued fuzzy sets, where the values of the membership functions are intervals of numbers instead of numbers.

The notion of interval valued fuzzy sets has many applications in medical science [3], image processing [1] and decision-making [17] to mention a few. In 1994 Biswas [2] defined the interval valued fuzzy subgroups of the same nature which are of the fuzzy subgroups of Rosenfeld. Later, in 2006 [10], Narayanan and Manikantan studied the concept of an interval valued fuzzy left ideal (right ideal, interior ideal, bi-ideal) in a semigroup. In 2013 [13], Singaram and Kandasamy characterized regular and intra-regular semigroups in terms of IVF left (right) ideals.

In this article, we give a definition of an interval valued fuzzy weakly interior ideal. We provide some interesting properties of interval valued fuzzy weakly interior ideals and the relationship between interval valued fuzzy weakly interior ideals and interval valued fuzzy ideals. Our goal is to characterize some semigroups by using interval valued fuzzy weakly interior ideals. Moreover, we establish the theorems of the homomorphic image and the preimage of an interval valued fuzzy weakly interior ideal in semigroups.

2 Preliminaries

Before we begin our study, we give some basic definitions and results that we need to use later.

A non-empty subset K of a semigroup S is called a *subsemigroup* of S if $K^2 \subseteq K$. A non-empty subset K of a semigroup S is called a *left* (right) ideal of S if $SK \subseteq K$ $(KS \subseteq K)$. By an *ideal* K of a semigroup S we mean a left ideal and a right ideal of S A non-empty subset K of S is called a *generalized bi-ideal* of S if $KSK \subseteq K$. A subsemigroup K of a semigroup S is called a *bi-ideal* of S if $KSK \subseteq K$. A subsemigroup K of a semigroup S is called a *interior ideal* of S if $SKS \subseteq K$. A subsemigroup K of a semigroup S is called a *quasi-ideal* of S if $SK \cap KS \subseteq K$. A subsemigroup K of a semigroup S is called a *quasi-ideal* of S if $SK \cap KS \subseteq K$. A subsemigroup K of a semigroup S is called an *interior ideal* of S if $SKS \subseteq K$. A semigroup S is said to be *regular* if for each $u \in S$, there exists $x \in S$ such that $u = uxu$. A semigroup S is called *left* (right) regular if for each $u \in S$, there exists

$a \in S$ such that $u = au^2$ ($u = u^2a$). A semigroup S is said to be *intra-regular* if for each $u \in S$, there exist $a.b \in S$ such that $u = au^2b$. A semigroup S is called *semisimple* if every ideal of S is an idempotent. It is evident that S is semisimple if and only if $u \in (SuS)(SuS)$ for every $u \in S$, that is there exist $w, y, z \in S$ such that $u = wuyuz$. A semigroup S is said to be *weakly regular* if for every $u \in S, u \in (uS)^2$.

For any $p_i \in [0, 1,]$ where $i \in \mathcal{A}$ define

$$\underset{i\in\mathcal{A}}{\vee} p_i := \sup_{i\in I}\{p_i\} \quad \text{and} \quad \underset{i\in\mathcal{A}}{\wedge} p_i := \inf_{i\in\mathcal{A}}\{p_i\}.$$

For any $p, q \in [0, 1]$, we have

$$p \vee q = \max\{p, q\} \quad \text{and} \quad p \wedge q = \min\{p, q\}.$$

Let $\mathcal{C}$ be the set of all closed subintervals $[0, 1]$; i.e.,

$$\mathcal{C} = \{\overline{p} = [p^-, p^+] \mid 0 \leq p^- \leq p^+ \leq 1\}.$$

We note that $[p, p] := \{p\}$ for all $p \in [0, 1]$. For $p = 0$ or 1 we shall denote $\overline{0} = [0, 0] = \{0\}$ and $\overline{1} = [1, 1] = \{1\}$.

Let $\overline{p} := [p^-, p^+]$ and $\overline{q} := [q^-, q^+]$ in $\mathcal{C}$. Define the operations "$\preceq$", "$=$", "$\curlywedge$" "$\curlyvee$" as follows:

(1) $\overline{p} \preceq \overline{q}$ if and only if $p^- \leq q^-$ and $p^+ \leq q^+$

(2) $\overline{p} = \overline{q}$ if and only if $p^- = q^-$ and $p^+ = q^+$

(3) $\overline{p} \curlywedge \overline{q} = [(p^- \curlywedge q^-), (p^+ \curlywedge q^+)]$

(4) $\overline{p} \curlyvee \overline{q} = [(p^- \vee q^-), (p^+ \vee q^+)]$.

If $\overline{p} \succeq \overline{q}$, we mean $\overline{q} \preceq \overline{p}$.

Proposition 2.1. *[5] Let $\overline{p}, \overline{q}, \overline{r} \in \mathcal{C}$. Then the following properties are true:*

(1) $\overline{p} \curlywedge \overline{p} = \overline{p}$ *and* $\overline{p} \curlyvee \overline{p} = \overline{p}$,

(2) $\overline{p} \curlywedge \overline{q} = \overline{q} \curlywedge \overline{p}$ *and* $\overline{p} \curlyvee \overline{q} = \overline{q} \curlyvee \overline{p}$,

(3) $(\overline{p} \curlywedge \overline{q}) \curlywedge \overline{r} = \overline{p} \curlywedge (\overline{q} \curlywedge \overline{r})$ *and* $(\overline{p} \curlyvee \overline{q}) \curlyvee \overline{r} = \overline{p} \curlyvee (\overline{q} \curlyvee \overline{r})$,

(4) $(\overline{p} \curlywedge \overline{q}) \curlyvee \overline{r} = (\overline{a} \curlyvee \overline{r}) \curlywedge (\overline{q} \curlyvee \overline{r})$ *and* $(\overline{p} \curlyvee \overline{q}) \curlywedge \overline{r} = (\overline{p} \curlywedge \overline{r}) \curlyvee (\overline{q} \curlywedge \overline{r})$,

(5) *If* $\overline{p} \preceq \overline{q}$, *then* $\overline{p} \curlywedge \overline{r} \preceq \overline{q} \curlywedge \overline{r}$ *and* $\overline{p} \curlyvee \overline{r} \preceq \overline{q} \curlyvee \overline{r}$.

For each interval $\overline{p}_i = [p_i^-, p_i^+] \in \mathcal{C}$, $i \in \mathcal{A}$ where $\mathcal{A}$ is an index set, we define

$$\underset{i\in\mathcal{A}}{\curlywedge}\overline{p}_i = [\underset{i\in\mathcal{A}}{\wedge} p_i^-, \underset{i\in\mathcal{A}}{\wedge} p_i^+] \quad \text{and} \quad \underset{i\in\mathcal{A}}{\curlyvee}\overline{p}_i = [\underset{i\in\mathcal{A}}{\vee} p_i^-, \underset{i\in\mathcal{A}}{\vee} p_i^+].$$

A fuzzy subset (fuzzy set) of a non-empty set T is a function $f : T \to [0, 1]$.

Definition 2.2. *[15] Let T be a non-empty set. An interval valued fuzzy subset (IVF subset, for short) of T is a function $\overline{f} : T \to \mathcal{C}$ satisfying $\overline{f}(a) = [f^-(a), f^+(a)]$, f^- and f^+ are two fuzzy subsets of T and $f^-(a) \leq f^+(a)$ for all $a \in T$.*

Definition 2.3. *[13] Let $A \subseteq T$. An interval valued characteristic function $\overline{\chi}_A$ of A is a function $\overline{\lambda}_A : T \to \mathcal{C}$ defined as*

$$\overline{\lambda}_A(u) = \begin{cases} \overline{1} & \textit{if} \quad u \in A \\ \overline{0} & \textit{if} \quad u \notin A \end{cases}$$

for all $u \in T$.

For two IVF subsets $\overline{f}$ and $\overline{g}$ in a semigroup S, define

(1) $\overline{f} \sqsubseteq \overline{g} \Leftrightarrow \overline{f}(u) \preceq \overline{g}(u)$ for all $u \in S$,

(2) $\overline{f} = \overline{g} \Leftrightarrow \overline{f} \sqsubseteq \overline{g}$ and $\overline{g} \sqsubseteq \overline{f}$,

(3) $(\overline{f} \sqcap \overline{g})(u) = \overline{f}(u) \curlywedge \overline{g}(u)$ for all $u \in S$.

For two IVF subsets $\overline{f}$ and $\overline{g}$ in a semigroup S, define the product $\overline{f} \circ \overline{g}$ as follows : for all $u \in S$,

$$(\overline{f} \circ \overline{g})(u) = \begin{cases} \underset{(x,y)\in F_u}{\curlyvee} \{\overline{f}(x) \curlywedge \overline{g}(y)\} & \text{if} \quad F_u \neq \emptyset, \\ \overline{0} & \text{if} \quad F_u = \emptyset, \end{cases}$$

where $F_u := \{(x, y) \in S \times S \mid u = xy\}$.

Since the semigroup S is associative, the product is associative [14].

Definition 2.4. *[10, 13] An IVF subset $\overline{f}$ of a semigroup S is said to be*

(1) *an* IVF subsemigroup *of S if $\overline{f}(uv) \succeq \overline{f}(u) \curlywedge \overline{f}(v)$ for all $u, v \in S$,*

(2) *an* IVF left (right) ideal *of S if $\overline{f}(uv) \succeq \overline{f}(v)(\overline{f}(uv) \succeq \overline{f}(u))$ for all $u, v \in S$. An IVF subset $\overline{f}$ of S is called an* IVF ideal *of S if it is both an IVF left ideal and an IVF right ideal of S,*

(3) *an* IVF generalized bi-ideal *of* S *if* $\overline{f}(uvw) \succeq \overline{f}(u) \curlywedge \overline{f}(w)$ *for all* $u, v, w \in S$.

(4) *an* IVF bi-ideal *of* S *if* $\overline{f}$ *is an IVF subsemigroup and is an IVF generalized bi-ideal of* S,

(5) *an* IVF interior ideal *of* S *if* $\overline{f}$ *is an IVF subsemigroup and* $\overline{f}(uav) \succeq \overline{f}(a)$ *for all* $a, u, v \in S$.

3 Interval valued fuzzy weakly interior ideals in semigroups

In this section, we give the concepts of IVF weakly interior ideals and discuss important properties.

Definition 3.1. *An IVF subset* $\overline{f}$ *of a semigroup* S *is said to be a* IVF weakly interior *ideal of* S *if* $\overline{f}(uav) \succeq \overline{f}(a)$ *for all* $a, u, v \in S$.

Example 3.2. *Consider a semigroup* $(S, \cdot)$ *defined by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\overline{f}$ *be an IVF subset of* S *such that* $\overline{f}(a) = [0.8, 0.9]$, $\overline{f}(b) = [0.2, 0.4]$, $\overline{f}(c) = [0.5, 0.7]$, $\overline{f}(d) = [0.3, 0.6]$. *By routine calculation,* $\overline{f}$ *is an IVF weakly interior ideal. But* $\overline{f}$ *is not an IVF interior ideal, since* $\overline{f}(dc) = \overline{f}(b) = [0.2, 0.4] \not\succeq [0.3, 0.6] = \overline{f}(d) \curlywedge \overline{f}(c)$.

The following theorem is an important property for an equivalent of a $(\overline{s}, \overline{t})$-IVF interior ideal of a semigroup.

Theorem 3.3. *An IVF subset* $\overline{f}$ *is an IVF weakly interior ideal of a semigroup* S *if and only if* $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} \sqsubseteq \overline{f}$, *where* $\overline{\mathcal{S}}$ *is an IVF subset of* S *mapping every element of* S *to* 1.

Proof. ($\Rightarrow$) Assume that $\overline{f}$ is an IVF weakly interior ideal of S and let $u \in S$.

If $F_u = \emptyset$, then it is easy to verify that, $(\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}})(u) \preceq \overline{f}(u)$.

If $F_u \neq \emptyset$ then

$$\begin{aligned}
(\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}})(u) &= \curlyvee_{(i,y)\in F_u} \{(\overline{\mathcal{S}} \circ \overline{f})(i) \curlywedge \overline{\mathcal{S}}(y)\} = \curlyvee_{(i,y)\in F_u} \{(\curlyvee_{(x,a)\in F_i} (\{\overline{\mathcal{S}}(x) \curlywedge \overline{f}(a)\})) \curlywedge \overline{\mathcal{S}}(y)\} \\
&= \curlyvee_{(i,y)\in F_u} \{(\curlyvee_{(x,a)\in F_i} (\{1 \curlywedge \overline{f}(a)\})) \curlywedge 1\} = \curlyvee_{(i,y)\in F_u} \{(\curlyvee_{(x,a)\in F_i} \{\overline{f}(a)\})\} \\
&\preceq \curlyvee_{(i,y)\in F_u} \{(\curlyvee_{(x,a)\in F_i} \{\overline{f}(xay)\})\} = \overline{f}(u).
\end{aligned}$$

Thus, $(\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}})(u) \preceq \overline{f}(u)$. Hence, $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} \sqsubseteq \overline{f}$.

($\Leftarrow$) Let $a, u, v \in S$. Since $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} \sqsubseteq \overline{f}$ we have $(\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}})(uav) \preceq \overline{f}(uav)$. Thus

$$\begin{aligned}
\overline{f}(uav) \succeq (\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}})(uav) &= \curlyvee_{(i,j)\in F_{uav}} \{(\overline{\mathcal{S}} \circ \overline{f})(i) \curlywedge \overline{\mathcal{S}}(j)\} \\
&= \curlyvee_{(i,j)\in F_{uav}} \{(\curlyvee_{(p,q)\in F_i} (\{\overline{\mathcal{S}}(p) \curlywedge \overline{f}(q)\}) \curlywedge \overline{\mathcal{S}}(j))\} \\
&= \curlyvee_{(i,j)\in F_{uav}} \{(\curlyvee_{(p,q)\in F_i} (\{1 \curlywedge \overline{f}(q)\}) \curlywedge 1)\} \\
&= \curlyvee_{(i,j)\in F_{uav}} \{(\curlyvee_{(p,q)\in F_i} \overline{f}(q)\})\} \succeq \overline{f}(a).
\end{aligned}$$

Hence, $\overline{f}(uav) \succeq \overline{f}(a)$. Therefore f is an IVF weakly interior ideal of S. $\square$

The following theorem is an important property for an equivalent of an IVF weakly interior ideal of a left (right, intra-) regular semigroup.

Theorem 3.4. *For any left (right, intra-) regular semigroup S, f is an IVF weakly interior ideal if and only if $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} = \overline{f}$.*

Proof. Assume that f is an IVF weakly interior ideal of a left regular semigroup S and let $u \in S$. Then there exists $x \in S$ such that $u = xu^2 = xuu = x(xuu)u$. Thus

$$\begin{aligned}
\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}}(u) &= \curlyvee_{(i,j)\in F_u} \{(\overline{\mathcal{S}} \circ \overline{f})(i) \curlywedge \overline{\mathcal{S}}(j)\} = \curlyvee_{(i,j)\in F_{x(xuu)u}} \{(\overline{\mathcal{S}} \circ \overline{f})(i) \curlywedge \overline{\mathcal{S}}(j)\} \\
&\succeq (\overline{\mathcal{S}} \circ \overline{f})(x(xuu)) \curlywedge \overline{\mathcal{S}}(u) = (\overline{\mathcal{S}} \circ \overline{f})(x(xuu)) \curlywedge 1 = (\overline{\mathcal{S}} \circ \overline{f})(x(xuu)) \\
&= \curlyvee_{(i,j)\in F_{x(xuu)}} \{\overline{\mathcal{S}}(i) \curlywedge \overline{f}(j)\} \succeq (\overline{\mathcal{S}}(x) \curlywedge \overline{f}(xuu)) = (1 \curlywedge \overline{f}(xuu)) = \overline{f}(xuu) \\
&\succeq \overline{f}(u).
\end{aligned}$$

Hence, $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}}(u) \succeq \overline{f}(u)$. Therefore, $\overline{f} \sqsubseteq \overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}}$. By Theorem 3.3 we have $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} \sqsubseteq \overline{f}$. Thus, $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} = \overline{f}$.

The converse follows from Theorem 3.3.

Similarly we can prove the other cases. $\square$

The following theorem is an important property for an equivalent of a IVF weakly interior ideal in case of a regular semigroup.

Theorem 3.5. *Let S be regular semigroups. Then $\overline{f}$ is a IVF weakly interior ideal of S if and only if $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} = \overline{f}$.*

Proof. Assume that f is an IVF weakly interior ideal of S and let $u \in S$. Since S is regular, there exists $x \in S$ such that $u = uxu = (uxu)xu = u(xux)u$. Thus

$$\begin{aligned}
\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}}(u) &= \curlyvee_{(i,j)\in F_u} \{(\overline{\mathcal{S}} \circ \overline{f})(i) \curlywedge \overline{\mathcal{S}}(j)\} = \curlyvee_{(i,j)\in F_{u(xux)u}} \{(\overline{\mathcal{S}} \circ \overline{f})(i) \curlywedge \overline{\mathcal{S}}(j)\} \\
&\succeq (\overline{\mathcal{S}} \circ \overline{f})(u(xux)) \curlywedge \overline{\mathcal{S}}(u) = (\overline{\mathcal{S}} \circ \overline{f})(u(xux)) \curlywedge \overline{1} \\
&= (\overline{\mathcal{S}} \circ \overline{f})(u(xux)) = \curlyvee_{(i,j)\in F_{u(xux)}} \{\overline{\mathcal{S}}(i) \curlywedge \overline{f}(j)\} \\
&\succeq (\overline{\mathcal{S}}(u) \curlywedge \overline{f}(xux)) = (\overline{1} \curlywedge \overline{f}(xux)) = \overline{f}(xux) \succeq \overline{f}(u).
\end{aligned}$$

Hence, $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}}(u) \succeq \overline{f}(u)$. Therefore, $\overline{f} \sqsubseteq \overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}}$. By Theorem 3.3 we have $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} \sqsubseteq \overline{f}$. Thus, $\overline{\mathcal{S}} \circ \overline{f} \circ \overline{\mathcal{S}} = \overline{f}$.

The converse follows from Theorem 3.3. □

In the following easy-to prove theorem shows the relationship between an IVF ideal and an interior ideal.

Theorem 3.6. *Every IVF ideal of a semigroup S is a IVF weakly interior ideal of S.*

Remark 3.7. *Example 3.2 shows that the converse of the above theorem is not true in general, since $\overline{f}(dc) = [0.2, 0.4] \not\succeq [0.5, 0.7] = \overline{f}(c)$.*

The following theorem shows that the IVF weakly interior ideals and IVF ideals coincide for some types of semigroups.

Theorem 3.8. *In regular, left (right) regular, intra-regular, weakly regular and semisimple semigroup, the IVF weakly interior ideals and the IVF ideals coincide.*

Proof. Suppose that $\overline{f}$ is an IVF weakly interior ideal of a regular semigroup and let $u, v \in S$.
Since S is a regular, there exists $x \in S$ such that $u = uxu$. Thus, $\overline{f}(uv) = \overline{f}((uxu)v) = \overline{f}((ux)uv) \succeq \overline{f}(u)$. Hence $\overline{f}$ is an IVF right ideal of S. Similarly, we can prove that $\overline{f}$ is an IVF left ideal of S. Thus $\overline{f}$ is an IVF ideal of S.

Similarly we can prove the other cases. □

The following theorem follows easily.

Theorem 3.9. *If $\overline{f}$ and g are two IVF weakly interior ideals of a semigroup S, then $f \sqcap g$ is also an IVF weakly interior ideals of S.*

In the following theorems, we give a relationship between a weakly interior ideal and the interval valued characteristic function.

Theorem 3.10. *Let K be a non-empty subset of a semigroup S. Then K is a weakly interior ideal of S if and only if $\overline{\lambda}_K$ is an IVF weakly interior ideal of S.*

Proof. Let $a, u, v \in S$ and let K be a weakly interior ideal of S.

If $a \in K$, then $uav \in K$. Thus, $\overline{\lambda}_K(a) = \overline{\lambda}_K(uav) = 1$. Hence, $\overline{\lambda}_K(uav) \succeq \overline{\lambda}_K(a)$ and $\overline{\lambda}_K(uav) \succeq \overline{\lambda}_K(a)$

Suppose that $a \notin K$. Then, $\overline{\lambda}_K(a) = 0$. Thus, $\overline{\lambda}_K(uav) \succeq \overline{\lambda}_K(a)$. Thus, $\overline{\lambda}_K$ is an IVF weakly interior ideal of S.

Conversely let $a, u, v \in S$ and $a \in K$. Since $\overline{\lambda}_K$ is an IVF weakly interior ideal of S, we have $\overline{\lambda}_K(uav) \geq \overline{\lambda}_K(a)$. Thus, $uav \in K$. Hence K is a weakly interior ideal of S. $\square$

Theorem 3.11. *Let S be a semigroup. Then the following statements hold*

(1) $\overline{\lambda}_{(a \cap SaS)}$ *is an IVF weakly interior ideal of S.*

(2) $\overline{\lambda}_{(a \cap aSa)}$ *is an IVF generalized bi-ideal of S.*

Proof. (1) Let $u, a, v \in S$. If $a \in \overline{\lambda}_{(a \cap SaS)}$ then $uav \in (a \cap SaS)$. Thus, $\overline{\lambda}_{(a \cap SaS)}(uav) \succeq \overline{\lambda}_{(a \cap SaS)}(uav)$. Similarly, if $a \notin \overline{\lambda}_{(a \cap SaS)}$ then we can prove that, $\overline{\lambda}_{(a \cap SaS)}(uav) \succeq \overline{\lambda}_{(a \cap SaS)}(uav)$. Hence, $\overline{\lambda}_{(a \cap SaS)}$ is an IVF weakly interior ideal of S.

(2) The proof is similar to (1).

$\square$

4 Characterizing some semigroups in terms of interval valued fuzzy weakly interior ideals and interval valued fuzzy ideals.

In this section, we will characterize some semigroups in terms of IVF weakly interior ideals and IVF ideals.

Theorem 4.1. *Let S be a semigroup. Then the following are equivalent:*

(1) *S is a regular,*

(2) $\overline{f} \sqcap \overline{g} \sqcap \overline{h} \sqsubseteq \overline{f} \circ \overline{g} \circ \overline{h}$, *for every IVF right ideal f, every weakly interior ideal g and for every IVF left ideal h of S,*

(3) $\overline{f} \sqcap \overline{g} \sqcap \overline{h} \sqsubseteq \overline{f} \circ \overline{g} \circ \overline{h}$,*for every IVF right ideal f, every weakly interior ideal g and for every IVF left ideal h of S,*

(4) $\overline{f} \sqcap \overline{g} \sqcap \overline{h} \sqsubseteq \overline{f} \circ \overline{g} \circ \overline{h}$, *for every IVF right ideal f, every weakly interior ideal g and for every IVF left ideal h of S,*

(5) $\overline{\lambda}(k)_r \sqcap \overline{\lambda}(k)_{k\cup SkS} \sqcap \overline{\lambda}(k)_l \sqsubseteq \overline{\lambda}(k)_r \circ \overline{\lambda}(k)_{k\cup SkS} \circ \overline{\lambda}(k)_l$, *for all $K \in S$, where $(k)_r$ is the right ideal generated by k and $(k)_l$ is the left ideal generated by k.*

Proof. (1) $\Rightarrow$ (2) Suppose that f, g and h is an IVF right ideal, an IVF weakly interior ideal and an IVF left ideal of S respectively and let $u \in S$. Since S is regular, there exists $x \in S$ such that $u = uxu = (uxu)xu = (ux)u(xu)$. Thus

$$\begin{aligned}
(\overline{f} \circ \overline{g} \circ \overline{h})(u) &= (\curlyvee_{(i,j)\in F_u} \{\overline{f}(i) \curlywedge (\overline{g} \circ \overline{h})(j)\} = (\curlyvee_{(i,j)\in F_{(ux)u(xu)}} \{\overline{f}(i) \curlywedge (\overline{g} \circ \overline{h})(j)\} \\
&\geq \overline{f}(ux) \curlywedge (\overline{g} \circ \overline{h})u(xu) = \overline{f}(ux) \curlywedge (\curlyvee_{(a,b)\in F_{u(xu)}} \{\overline{g}(a) \curlywedge \overline{h}(b)\}) \\
&\geq \overline{f}(ux) \curlywedge (\overline{g}(u) \curlywedge \overline{h}(xu)) \succeq \overline{f}(u) \curlywedge (\overline{g}(u) \curlywedge \overline{h}(u) \\
&= \overline{f}(u) \curlywedge (\overline{g} \curlywedge \overline{h})(u) = (\overline{f} \curlywedge \overline{g} \curlywedge \overline{h})(u).
\end{aligned}$$

Hence, $(\overline{f} \circ \overline{g} \circ \overline{h})(u) \succeq (\overline{f} \curlywedge \overline{g} \curlywedge \overline{h})(u)$. Therefore, $\overline{f} \sqcap \overline{g} \sqcap \overline{h} \sqsubseteq \overline{f} \circ \overline{g} \circ \overline{h}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) This is obvious because every IVF ideal is an IVF bi-ideal of S.

(4) $\Rightarrow$ (5) By Theorem 3.11 we have $\overline{\lambda}_{u\cap SuS}$ is an IVF weakly interior ideal of S. Then for all $u \in S$,

$$\overline{\lambda}(u)_r \sqcap \overline{\lambda}(u)_{u\cup SuS} \sqcap \overline{\lambda}(u)_l \sqsubseteq \overline{\lambda}(u)_r \circ \overline{\lambda}(u)_{u\cup SuS} \circ \overline{\lambda}(u)_l.$$

(5) $\Rightarrow$ (1) Let $k \in S$ and let $(k)_r$ is the right ideal generated by k and $(k)_l$ is the left ideal generated by k respectively. Then by Theorem 3.11, $\overline{\lambda}(k)_r$ and $\overline{\lambda}(k)_l$ is an IVF right ideal generated by k and IVF left ideal generated by k respectively. By assumption,

$$\overline{1} = \overline{\lambda}(k)_r \sqcap \overline{\lambda}(k)_{k\cup SkS} \sqcap \overline{\lambda}(k)_l \sqsubseteq \overline{\lambda}(k)_r \circ \overline{\lambda}(k)_{k\cup SkS} \circ \overline{\lambda}(k)_l = \overline{1}.$$

Thus, $u \in (u \cap uS \circ (u \cap SuS \circ u \cap Su))$ so, $u \in uSu$. Hence, $u = uxu$. Therefore S is regular. $\square$

Theorem 4.2. *A semigroup S is a semisimple if and only if $\overline{f} \sqcap \overline{g} \sqsubseteq \overline{f} \circ \overline{g}$, for each IVF weakly interior ideal f and g of S.*

Proof. ($\Rightarrow$) Let f and g be IVF weakly interior ideals of S let $u \in S$. Since S is semisimple, there exist $x, y, z \in S$ such that $u = xuyuz = (xuy)(xuyuzz) = (xuy)(xu(yuz^2))$. Thus

$$\begin{aligned}(\overline{f} \circ \overline{g})(u) &= \curlyvee_{(i,j)\in F_u} \{\overline{f}(i) \curlywedge \overline{g}(j)\} = (\curlyvee_{(i,j)\in F_{(xuy)(xu(yuz^2))}} \{\overline{f}(i) \curlywedge \overline{g}(j)\} \\ &\geq \overline{f}(xuy) \curlywedge \overline{g}(xu(yuz^2)) = (\overline{f}(u) \curlywedge \overline{g}(u)) = (\overline{f} \curlywedge \overline{g})(u).\end{aligned}$$

Hence, $(\overline{f} \curlywedge \overline{g})(u) \leq (\overline{f} \circ \overline{g})(u)$. Therefore, $\overline{f} \sqcap \overline{g} \sqsubseteq \overline{f} \circ \overline{g}$.

($\Leftarrow$) Let $k \in S$. Then by Theorem 3.11 $\overline{\lambda}(k)_{k\cup SkS}$ is an IVF weakly interior ideal of S. By assumption, $\overline{1} = \overline{\lambda}_{k\cup SkS} \sqcap \overline{\lambda}_{k\cup SkS} \sqsubseteq \overline{\lambda}_{k\cup SkS} \circ \overline{\lambda}_{k\cup SkS} = \overline{1}$. Thus, $u \in (u \cap SuS \circ (u \cap SuS)$ so, $u \in SuSuS$. Hence, $u = xuyuz$. Therefore S is semisimple. $\square$

Theorem 4.3. *A semigroup S is a semisimple if and only if $\overline{f} \sqcap \overline{g} \sqcap \overline{h} \sqsubseteq \overline{f} \circ \overline{g} \circ \overline{h}$, for every right ideal g and for every IVF weakly interior ideal f and h of S.*

Proof. ($\Rightarrow$) Suppose that f and h are IVF weakly interior ideals and g is an IVF right ideal of S and let $u \in S$. Since S is semisimple, there exist $x, y, z \in S$ such that
$u = xuyuz = (x^2uy)(uz)(yuz)$. Thus

$$\begin{aligned}(\overline{f} \circ \overline{g} \circ \overline{h})(u) &= (\curlyvee_{(i,j)\in F_u} \{\overline{f}(i) \curlywedge (\overline{g} \circ \overline{h})(j)\} = \curlyvee_{(i,j)\in F_{(x^2uy)(uz)(yuz)}} \{\overline{f}(i) \curlywedge (\overline{g} \circ \overline{h})(j)\} \\ &\geq \overline{f}(x^2uy) \curlywedge (\overline{g} \circ \overline{h})(uz)(yuz) = \overline{f}(ux) \curlywedge (\curlyvee_{(a,b)\in F_{(uz)(yuz)}} \{\overline{g}(a) \curlywedge \overline{h}(b)\}) \\ &\geq \overline{f}(ux) \curlywedge (\overline{g}(uz) \curlywedge \overline{h}(yuz)) \succeq \overline{f}(u) \curlywedge (\overline{g}(u) \curlywedge \overline{h}(u)) \\ &= \overline{f}(u) \curlywedge (\overline{g} \curlywedge \overline{h})(u) = (\overline{f} \curlywedge \overline{g} \curlywedge \overline{h})(u).\end{aligned}$$

Hence, $(\overline{f} \circ \overline{g} \circ \overline{h})(u) \succeq \overline{f} \curlywedge (\overline{g} \curlywedge \overline{h})(u)$. Therefore, $\overline{f} \sqcap \overline{g} \sqcap \overline{h} \sqsubseteq \overline{f} \circ \overline{g} \circ \overline{h}$.

($\Leftarrow$) Let $k \in S$ and let $(k)_r$ be a right ideal generated by k. Then by Theorem 3.11, $\overline{\lambda}(k)_r$ is a IVF right ideal generated by k. By assumption,

$$\overline{1} = \overline{\lambda}_{k\cup SkS} \sqcap \overline{\lambda}(k)_r \sqcap \overline{\lambda}(k)_{k\cup SkS} \sqsubseteq \overline{\lambda}(k)_{k\cup SkS} \circ \overline{\lambda}(k)_r \circ \overline{\lambda}(k)_{k\cup SkS} = \overline{1}.$$

Thus, $u \in (u \cap SuS \circ (u \cap uS \circ u \cap SuS))$ so, $u \in SuSuS$. Hence, $u = xuyuz$. Therefore S is semisimple. $\square$

Theorem 4.4. *A semigroup S is weakly regular if and only if $\overline{f} \sqcap \overline{g} \sqsubseteq \overline{f} \circ \overline{g}$, for every IVF generalized bi-ideal f and for every IVF weakly interior ideal g of S.*

Proof. ($\Rightarrow$) Let f and g be an IVF generalized bi-ideal and IVF an weakly interior ideal of S respectively let $u \in S$. Since S is weakly regular, there exist $x, y \in S$ such that $u = uxuy = (uxu)(xuy^2)$. Thus

$$\begin{aligned}(\overline{f} \circ \overline{g})(u) &= \curlyvee_{(i,j)\in F_u} \{\overline{f}(i) \curlywedge \overline{g}(j)\} = (\curlyvee_{(i,j)\in F_{(uxu)(xuy^2)}} \{\overline{f}(i) \curlywedge \overline{g}(j)\} \\ &\succeq \overline{f}(uxu) \curlywedge \overline{g}(xuy^2) \succeq \overline{f}(u) \curlywedge \overline{f}(u) \curlywedge \overline{g}(u) = \overline{f}(u) \curlywedge \overline{g}(u) = (\overline{f} \curlywedge \overline{g})(u).\end{aligned}$$

Hence, $(\overline{f} \curlywedge \overline{g})(u) \succeq (\overline{f} \circ \overline{g})(u)$. Therefore, $\overline{f} \sqcap \overline{g} \sqsubseteq \overline{f} \circ \overline{g}$.

($\Leftarrow$) Let $k \in S$. Then by Theorem 3.11 $\overline{\lambda}(k)_{k\cup SkS}$ and $\overline{\lambda}(k)_{k\cup kSk}$ is an IVF weakly interior ideal and IVF generalized bi-ideal of S respectively. By assumption, $\overline{1} = \overline{\lambda}_{k\cup SkS} \sqcap \overline{\lambda}_{k\cup kSk} \sqsubseteq \overline{\lambda}_{k\cup SkS} \circ \overline{\lambda}_{k\cup kSk} = \overline{1}$. Thus, $u \in (u \cap uSu \circ (u \cap SuS))$ so, $u \in uSuS$. Hence, $u = uxuy$. Therefore S is weakly regular. □

5 The Image and Preimage of Interval Valued Fuzzy Weakly Interior Ideals

In this section, we propose definitions of image and preimage of an IVF subset. And we establish the theorems of the homomorphic image and the preimage of an IVF weakly interior ideal in semigroups.

Definition 5.1. *[12] Let U and V be two non-empty sets and $\phi : U \to V$ be a function.*
Let $\overline{f} \in IVF(U)$ and $\overline{g} \in IVF(V)$. Then image $\phi(\overline{f})$ of $\overline{f}$ under the function ϕ is an IVF subset of V defined by

$$\phi(\overline{f})(v) = \begin{cases} \curlyvee_{z\in\phi^{-1}(v)} \overline{f}(z), & \text{if} \quad \phi^{-1}(v) \neq 0, \\ \overline{0}, & \text{otherwise} \end{cases}$$

for all $v \in V$ and $\phi^{-1}(v) = \{u \in U : \phi(u) = v\}$

Preimage $\phi^{-1}(\overline{g})$ of $\overline{g}$ under the function ϕ is an IVF subsets of U defined by $\phi^{-1}(\overline{g})(u) = (\overline{g})(\phi(u))$ for any $u \in U$.

Theorem 5.2. *Let U and V be two semigroups and let $\phi : U \to V$ be an epimorphism.*
If $\overline{f}$ and $\overline{g}$ are IVF weakly interior ideal of U and V respectively, then

(1) $\phi(\overline{f})$ *is an IVF weakly interior ideal of* V*,*

(2) $\phi^{-1}(\overline{g})$ *is an IVF weakly interior ideal of* U*.*

Proof. Suppose that $\overline{f}$ and $\overline{g}$ are IVF weakly interior ideal of U and V respectively.

1. Let $v_1, v_2, v_3 \in V$. Since ϕ is a surjective we have there exist $u_1, u_2, u_3 \in U$ such that $\phi(u_1) = v_1, \phi(u_2) = v_2$ and $\phi(u_3) = v_3$. Thus

$$\begin{aligned} \phi(\overline{f})(v_1v_2, v_3) &= \curlyvee_{z\in\phi^{-1}(v_1v_2v_3)} \overline{f}(z) = \curlyvee_{\phi(u_1)=v_1,\phi(u_2)=v_2,\phi(u_3)=v_3} \overline{f}(u_1u_2v_3) \\ &\succeq \curlyvee_{\phi(u_1)=v_1,\phi(u_2)=v_2,,\phi(u_3)=v_3} \overline{f}(u_2) = \curlyvee_{z=\phi^{-1}(v_2)} \overline{f}(z) = \phi(\overline{f})(v_2). \end{aligned}$$

   Hence, $\phi(\overline{f})(v_1v_2, v_3) \succeq \phi(\overline{f})(v_2)$. Therefore $\phi(\overline{f})$ is an IVF weakly interior ideal of V.

2. Let $a, b, c \in U$. Then

$$\phi^{-1}(\overline{g})(abc) = \overline{g}(\phi((abc)) = \overline{g}(\phi(a)\phi(b))\phi(c) \succeq (\overline{g})(\phi(b)) = \phi^{-1}(\overline{g})(b).$$

   Thus, $\phi^{-1}(\overline{g})(abc) \succeq \phi^{-1}(\overline{g})(b)$. Hence $\phi^{-1}(\overline{g})$ is an IVF weakly interior ideal of U.

□

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