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An extension of the Poisson transform

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Abstract

Let $(G(n))_{n\geq 1}$ be an increasing sequence of complex connected semi-simple Lie groups with finite center. $G(\infty) = \underset{n\geq 1}{\cup} G(n)$ is the inductive limit of $G(n)$ [4]. In this paper, we apply some properties of the set of holomorphic functions defined on $G(\infty)$, differentiable groups, to an extension of Poisson transform.

1 Introduction

Let $(G(n))_{n\geq 1}$ be an increasing sequence of complex connected semi-simple Lie groups with finite center. We denote by $\mathcal{G}(n)$ the Lie algebra of $G(n)$ and by $\mathcal{G}(n) = \mathcal{T}(n) \oplus \mathcal{P}(n)$ the Cartan decomposition of $\mathcal{G}(n)$ obtained by complexifying $\mathcal{T}_{\mathbb{R}}(n)$ and $\mathcal{P}_{\mathbb{R}}(n)$ ($\mathcal{G}_{\mathbb{R}}(n) = \mathcal{T}_{\mathbb{R}}(n) \oplus \mathcal{P}_{\mathbb{R}}(n)$a real form). $K(n)$ is the maximal compact subgroup corresponding to $\mathcal{T}(n)$ and $K_{\mathbb{R}}(n)$ the maximal compact subgroup corresponding to $\mathcal{T}_{\mathbb{R}}(n)$. For each integer $n, P_0(n)$ is $\mathcal{P}(n)$ with its vector space structure and the action of $K(n)$.
Let $G_0(n) = K(n) \ltimes P_0(n)$ (semi-direct product of $K(n)$) and $P_0(n)$) is the Cartan motion group and $\mathcal{G}_0(n) = \mathcal{T}(n) \oplus \mathcal{P}_0(n)$ is its Lie algebra. $\mathcal{A}(n)$ is a maximal abelian subspace of $\mathcal{P}(n)$. Let $\mathcal{Q}(n)$ be the orthogonal of $\mathcal{A}(n)$

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in $\mathcal{P}(n)$ for the Killing form of $\mathcal{G}(n)$ and $M(n)$ be the centralizer of $\mathcal{A}(n)$ in K(n). For each n, we identify the complexified dual $\mathcal{A}^\star(n)$ of $\mathcal{A}(n)$ with the subspace of $\mathcal{P}^\star(n)$ of all nil linear forms on $\mathcal{Q}(n)$. Take λ_n in $\mathcal{A}^\star(n)$. Denote by $K^{\lambda_n}(n)$ the stabilizer of λ_n in $K(n)$ and by

$$G_0(\infty) = \underset{n\geq 1}{\cup} G_0(n), \mathcal{G}_0(\infty) = \underset{n\geq 1}{\cup} \mathcal{G}_0(n), K(\infty) = \underset{n\geq 1}{\cup} K(n), K_{\mathbb{R}}(\infty) = \underset{n\geq 1}{\cup} K_{\mathbb{R}}(n)$$

the respective inductive limits of $G_0(n), \mathcal{G}_0(n), K(n), K_{\mathbb{R}}(n)$.
Let us denote by $\mathcal{H}ol(K(\infty))$ the set of holomorphic functions on $K(\infty)$ and by $\mathcal{M}(K_{\mathbb{R}}(\infty))$ the set of analytical functions on $K_{\mathbb{R}}(\infty)$.
$\mathcal{H}ol(K(\infty))_{K(\infty)}$ (resp. $\mathcal{M}(K_{\mathbb{R}}(\infty))_{K_{\mathbb{R}}(\infty)}$) is the set of $K(\infty)$-finite (resp. $K_{\mathbb{R}}(\infty)$-finite) vectors of $\mathcal{H}ol(K(\infty))$ (resp. $\mathcal{M}(K_{\mathbb{R}}(\infty))$ for the left regular representation of $K(\infty)$ (resp. $K_{\mathbb{R}}(\infty)$).
We obtain the following results:

Theorem A. *(i) Let r be the restriction to $K_{\mathbb{R}}(\infty)$ of holomorphic functions on $K(\infty)$. Then r is a $K_{\mathbb{R}}(\infty)$-isomorphism of $\mathcal{H}ol(K(\infty))_{K(\infty)}$ on $\mathcal{M}(K_{\mathbb{R}}(\infty))_{K_{\mathbb{R}}(\infty)}$.*

(ii) The restriction to $K_{\mathbb{R}}(\infty)$ is an isomorphism of $Hol(\underrightarrow{lim}(K(n)/K^{\lambda}(n)))_{K(\infty)}$ on $\mathcal{M}(\underrightarrow{lim}(K_{\mathbb{R}}(n)/K^{\lambda}_{\mathbb{R}(n)}))_{K_{\mathbb{R}}(\infty)}$.

(iii) $\forall\lambda \in \mathcal{A}^\star = \underleftarrow{\lim}\mathcal{A}^\star(n)$, the restriction of polynomials of $S(\underrightarrow{\lim}\mathcal{P}(n))$ to the orbit of λ, under $K_{\mathbb{R}}(\infty)$, is a surjection of $S(\underrightarrow{\lim}\mathcal{P}(n))$ on $\mathcal{M}(\underrightarrow{\lim}(K_{\mathbb{R}}(n)/K^{\lambda}_{\mathbb{R}(n)}))_{K_{\mathbb{R}}(\infty)}$.

We define the Poisson transform

$$\begin{array}{rccc} \mathbf{P}: & \mathcal{C}^\infty(\underrightarrow{lim}(K(n)/K^{\lambda}(n))) & \longrightarrow & \mathcal{C}^\infty(\underrightarrow{lim}\mathcal{P}(n)) \\ & f & \longmapsto & \mathbf{P}(f) \end{array}$$

with

$$\mathbf{P}(f)(X) = \int_{\underrightarrow{lim}(K_{\mathbb{R}}(n)/K^{\lambda_n}_{\mathbb{R}}(n))} e^{-i\lambda(k^{-1}.X)} f(\dot{k})d\dot{k}, \forall X \in \underrightarrow{lim}\mathcal{P}_{\mathbb{R}}(n)$$

Theorem B. *The Poisson transform so defined is injective.*

Let us denote by $V(\sigma_n, \lambda_n)$ the set of $K(n)$-finite vectors of $V^\infty(\sigma_n, \lambda_n)$ of the representation

$$\Pi_{\sigma_n,\lambda_n} = \underset{K^{\lambda_n}(n)\ltimes P_0(n)\uparrow G_0(n)}{Ind} \sigma_n \otimes \lambda_n$$

of $G_0(n)$, with $V_n^\infty(\sigma_n,\lambda_n)=\{f\in\mathcal{C}^\infty(K(n),E_{\sigma_n}(n))/f(km)=\sigma_n(m^{-1})f(k),\forall k\in K(n),\forall m\in K^{\lambda_n}(n)\}$ and $\forall f\in V_n^\infty(\sigma_n,\lambda_n)$,

$$(\Pi_{\sigma_n,\lambda_n}(e^X.k)f)(k')=e^{i\lambda_n(k^{-1}.X)}f(k^{-1}k').$$

Theorem C. *For any $\lambda\in\mathcal{A}^\star_{\mathbb{C}}(\infty)$, the $(\mathcal{G}_0(\infty),K(\infty))$-module $\varinjlim V(\epsilon,\lambda_n)$ is simple.*

The goal of this work is to study some properties of the set of holomorphic functions on certain differentiable groups and to give an application to an extension of Poison transform.

2 Notations and setup

Let $(G(n))_{n\geqslant 1}$ be the increasing sequences of groups and let $(\pi_n,V(n))$ be a representation of $G(n)$.

Definition 2.1. *We say that a representation (π,V) of the group $G(\infty)$ is the inductive limit of the sequence of representations $(\pi_n,V(n))_{n\geqslant1}$ if for any fixed n_0, the restriction of the representations $\pi_n(n>n_0)$ to the groups $G(n_0)$ converge to the restriction of π to the groups $G(n_0)$ uniformly, when n tends to ∞.*

Let $\theta_{n-1,n}$ be the canonical projection from $G_0(n)$ to $G_0(n)/G_0(n-1)$ and $\sigma_{n-1,n}$ be its continuous section from $G_0(n)/G_0(n-1)$ to $G_0(n)$ such that $\theta_{n-1,n}\circ\sigma_{n-1,n}=id$.
Consider the following projection:

$$\begin{array}{cccc} p_{n-1,n}: & G_0(n) & \longrightarrow & G_0(n-1)\\ & g & \longmapsto & [\sigma_{n-1,n}(\theta_{n-1,n}(g))]^{-1}.g,\end{array}$$

and denote by $\tilde{p}_n$ the map defined by:

$$\begin{array}{cccc} \tilde{p}_n: & G_0(n) & \longrightarrow & G_0(n-1)\times G_0(n)/G_0(n-1)\\ & g & \longmapsto & (p_{n-1,n}(g);\theta_{n-1,n}(g))\end{array}$$

For each n, let λ_n be an element of $\mathcal{A}^\star_{\mathcal{C}}(n)$ and $(\mu_n,E_{\mu_n}(n))$ be an element of $\widehat{M}(n)$ (The unitary dual of $M(n)$).
Consider the representation $\mu_n\otimes\lambda_n$ of

$$H(n)=M(n)\ltimes P_0(n)$$

defined by

$$\mu_n \otimes \lambda_n(e^X, m)x = e^{i\lambda_n(X)}\mu_n(m)x,\ \forall m \in M(n), \forall X \in \mathcal{P}_0(n), \forall x \in E_{\mu_n}.$$

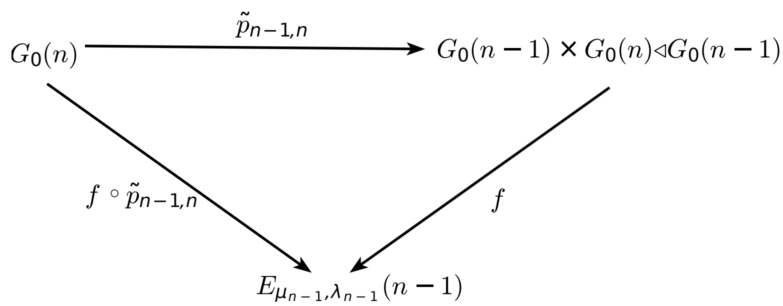
and denote by $E_{\mu_n,\lambda_n}(n)$ the representation space of $\mu_n \otimes \lambda_n$. The principal series of $G_0(n)$ is the family of the representations

$$\Pi_{\mu_n,\lambda_n} = \underset{H(n)\uparrow G_0(n)}{Ind}\ \mu_n \otimes \lambda_n$$

whose action on the space $V(n) = L_n^{\infty}(\mu_n, \lambda_n)$ is defined by

$$(\Pi_{\mu_n,\lambda_n}(g)f)(g') = f(g^{-1}g').$$

We have the following diagram:



For each integer n, let $V^0(n)$ denote the subspace of $V(n)$ defined by

$$V^0(n) = \{f \circ \tilde{p}_n \in V(n) \mid f \in V(n-1) \otimes \mathcal{C}^{\infty}(G_0(n)/G_0(n-1); E_{\mu_{n-1},\lambda_{n-1}}(n-1))\}$$

3 Generalized Poisson transform

Let us denote by $\mathcal{H}ol(K(\infty))$ the set of holomorphic functions on $K(\infty)$ and denote by $\mathcal{M}(K_{\mathbb{R}}(\infty))$ the set of analytical functions on $K_{\mathbb{R}}(\infty)$.
$\mathcal{H}ol(K(\infty))_{K(\infty)}$ $(resp.\mathcal{M}(K_{\mathbb{R}}(\infty))_{K_{\mathbb{R}}(\infty)}$ is the set of $K(\infty)-$finite $(resp.K_{\mathbb{R}}(\infty)-$finite vectors of $\mathcal{H}ol(K(\infty))$ $(resp.\mathcal{M}(K_{\mathbb{R}}(\infty)))$ for the left regular representation of $K(\infty)$ $(resp.K_{\mathbb{R}}(\infty))$.

Theorem 3.1. *(i) Let r be the restriction to $K_{\mathbb{R}}(\infty)$ of holomorphic functions on $K(\infty)$. Then r is a $K_{\mathbb{R}}(\infty)-$isomorphism from $\mathcal{H}ol(K(\infty))_{K(\infty)}$ to $\mathcal{M}(K_{\mathbb{R}}(\infty))_{K_{\mathbb{R}}(\infty)}$.*

(ii) The restriction to $K_{\mathbb{R}}(\infty)$ *is an isomorphism from* $\mathcal{H}ol(\varinjlim(K(n)/K^{\lambda}(n))_{K(\infty)}$ *to* $\mathcal{M}(\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda}(n))_{K_{\mathbb{R}}(\infty)}$.

(iii) $\forall\lambda \in \mathcal{A}^{\star} = \varprojlim\mathcal{A}^{\star}(n)$, *the restriction of polynomials of* $S(\varinjlim\mathcal{P}(n))$ *to the orbit of* λ *under* $K_{\mathbb{R}}(\infty)$ *is a surjection from* $S(\varinjlim\mathcal{P}(n))$ *to* $\mathcal{M}(\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda}(n))_{K_{\mathbb{R}}(\infty)}$.

Proof

(i) For each $n \geq 1$ the map

$$\begin{array}{rcll} r_n: & \mathcal{H}ol(K(n))_{K(n)} & \longrightarrow & \mathcal{M}(K_{\mathbb{R}}(n))_{K_{\mathbb{R}}(n)} \\ & f & \longmapsto & r_n(f) = f_{|K_{\mathbb{R}}(n)} \end{array}$$

is injective. In fact, if the restriction to $K_{\mathbb{R}}(n)$ of any holomorphic function f on $K(n)$ is nil, then

$$\begin{aligned} \forall X \in \mathcal{T}_{\mathbb{R}}(n), (Xf)(e) &= \frac{d}{dt}f(e^{-tX})_{|t=0} \\ &= 0 \end{aligned}$$

Let J be the complex structure of $\mathcal{T}_{\mathbb{R}}(n)$.

$$\begin{aligned} \forall X \in \mathcal{T}_{\mathbb{R}}(n), (Xf)(e) &= \frac{d}{dt}f(e^{-tJ(X)})_{|t=0} \\ &= (-1)^{\frac{1}{2}}\frac{d}{dt}f(e^{-tX})_{|t=0} \\ &= 0 \end{aligned}$$

Since

$$\mathcal{H}ol(K(n))_{K(n)} = \bigoplus_{\delta\in\widehat{K}_{hol}(n)} \dim_{\mathbb{C}}(V_\delta)V_\delta$$

and

$$\mathcal{M}(K_{\mathbb{R}}(n))_{K_{\mathbb{R}}(n)} = \bigoplus_{\delta\in\widehat{K}_{\mathbb{R}}(n)} \dim_{\mathbb{C}}(V_\delta)V_\delta,$$

where $\widehat{K}_{hol}(n)$ is the set of classes of finite dimensional irreducible holomorphic representation of $K(n)$, $\widehat{K}_{\mathbb{R}}$ is the set of classes of irreducible representations of $K_{\mathbb{R}(n)}$, and V_{δ_n} the space of the representation δ_n in $\widehat{K}_{hol}(n)$ (resp. $\widehat{K}_{\mathbb{R}}(n)$).

Then, for each n, $\mathcal{H}ol(K(n))_{K(n)}$ is identified with $\mathcal{M}(K_{\mathbb{R}}(n))_{K_{\mathbb{R}}(n)}$ under this map.
The following map

$$\begin{array}{rccc} d_F: & \mathcal{H}ol(\underrightarrow{\lim}K(n))_{\underrightarrow{\lim}K(n)} & \longrightarrow & \underleftarrow{\lim}\mathcal{H}ol(K(n))_{K(n)} \\ & f & \longmapsto & f \circ i_n \end{array}$$

is an isomorphism, where $i_n : K(n) \longrightarrow \underrightarrow{\lim}K(n)$ is canonical injection [1]. We deduce that the space $\mathcal{H}ol(\underrightarrow{\lim}K(n))_{\underrightarrow{\lim}K(n)}$ is isomorphic to $\mathcal{M}(\underrightarrow{\lim}K_{\mathbb{R}}(n))_{\underrightarrow{\lim}K_{\mathbb{R}}(n)}$.

(ii) For each n, let $K^{\lambda_n}(n)$ denote the stabilizer of λ_n in $K(n)$.
$\mathcal{H}ol(K(n)/K^{\lambda_n}(n))_{K(n)}$ (resp. $\mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}$) is identified with the set of functions of $\mathcal{H}ol(K(n))_{K(n)}$ (resp. $\mathcal{M}(K_{\mathbb{R}}(n))_{K_{\mathbb{R}}(n)}$) invariant by right regular action of $K^{\lambda_n}(n)$ (resp. $K_{\mathbb{R}}^{\lambda_n}(n)$). Therefore, for each n, the space

$$\mathcal{H}ol(K(n)/K^{\lambda_n}(n))$$

is isomorphic to

$$\{f \in \mathcal{H}ol(K(n))_{K(n)}/\mathcal{T}_a f = f, \forall a \in K^{\lambda_n}(n)\}$$

where $\mathcal{T}_a f(k) = f(ka),\ \forall k \in K(n)$ and

$$\mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))$$

is identified with

$$\{f \in \mathcal{M}(K_{\mathbb{R}}(n))_{K_{\mathbb{R}}(n)}/\mathcal{T}_a f = f, \forall a \in K_{\mathbb{R}}^{\lambda_n}(n)\}$$

Accordingly, the map r_n remains injective of $\mathcal{H}ol(K(n)/K^{\lambda_n}(n))_{K(n)}$ in $\mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}$. If $f \in \mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}$, we have r_n^{-1} which commutes with $\mathcal{T}_a$. Since r_n is $K_{\mathbb{R}}(n)$-isomorphism from $\mathcal{H}ol(K(n))_{K(n)}$ to $\mathcal{M}(K_{\mathbb{R}}(n))_{K_{\mathbb{R}}(n)}$ and $\mathcal{T}^{\lambda_n}(n) = (\mathcal{T}_{\mathbb{R}}^{\lambda_n}(n))_{\mathbb{C}}$. We deduce that $r_n^{-1}f$ is invariant by $\mathcal{T}^{\lambda_n}(n)$, so right invariant by $(K^{\lambda_n}(n))^{\circ} = \exp(ad\mathcal{T}^{\lambda_n}(n))$ and by $K_{\mathbb{R}}^{\lambda_n}(n)$. Since $K_{\mathbb{R}}^{\lambda_n}(n)$ meets all the connected components of $K^{\lambda_n}(n)$, it is right invariant by $K^{\lambda_n}(n)$.
In conclusion, we have

$$\mathcal{H}ol(K(n)/K^{\lambda_n}(n))_{K(n)}$$

is identified with

$$\mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}.$$

So we have the following diagram

$$\begin{array}{ccc}
\mathcal{H}ol(\underrightarrow{\lim}(K(n)\lhd K^{\lambda_n}(n)))_{\underrightarrow{\lim}K(n)} & \xrightarrow{\ r\ } & \mathcal{M}(\underrightarrow{\lim}(K_{\mathbb{R}}(n)\lhd K_{\mathbb{R}}^{\lambda_n}(n)))_{\underrightarrow{\lim}K_{\mathbb{R}}(n)} \\
d_F \Big| & & \Big| d'_F \\
\underleftarrow{\lim}\mathcal{H}ol(K(n)\lhd K^{\lambda_n}(n))_{K(n)} & \xrightarrow[r']{} & \underleftarrow{\lim}\mathcal{M}(K_{\mathbb{R}}(n)\lhd K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}
\end{array}$$

where d_F, d'_F, and r' are the bijections. Consequently

$$\mathcal{H}ol(\underrightarrow{\lim}(K(n)/K^{\lambda_n}(n)))_{\underrightarrow{\lim}K(n)}$$

is isomorphic to

$$\mathcal{M}(\underrightarrow{\lim}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)))_{\underrightarrow{\lim}K_{\mathbb{R}}(n)}.$$

(iii) Let $G(n)$ be the adjoint group of $\mathcal{G}(n)$, $G_{\mathbb{R}}(n) = \{a \in G(n)/a.\mathcal{G}_{\mathbb{R}}(n) \subset \mathcal{G}_{\mathbb{R}}(n)\}$, θ_n be the extension of the Cartan involution of $\mathcal{G}_{\mathbb{R}}(n)$ to $\mathcal{G}(n)$, and $K_{\theta_n}(n) = \{a \in G(n)/Ad_{G(n)}a \circ \theta_n = \theta_n \circ Ad_{G(n)}a\}$.
For each n, $K_{\theta_n}(n) = F(n)K(n)$ where $F(n)$ is the finite abelian subgroup of order 2 in $A(n) = \exp ad\mathcal{A}(n)$ and

$$\begin{aligned} K(n) &= \exp ad\mathcal{T}(n) \\ &= (K_{\theta_n}(n))^{\circ} \end{aligned}$$

Let $S(\mathcal{P}(n))$ be the symmetric algebra of $\mathcal{P}(n)$.
$S(\mathcal{P}(n))$ is identified with the algebra of polynomial functions on $\mathcal{P}^{\star}(n)$.
Let λ_n be a fixed element of $\mathcal{A}^{\star}(n)$. Let us denote by $\mathcal{O}_{\lambda_n}(n)$ the orbit of λ_n under $K_{\theta_n}(n)$. We have

$$\mathcal{O}_{\lambda_n}(n)$$

which is identified with

$$K(n)/K^{\lambda_n}(n).$$

Let $\mathcal{R}(\mathcal{O}_{\lambda_n}(n))$ denote the set of rational functions defined everywhere on $\mathcal{O}_{\lambda_n}(n)$. Finally

$$\mathcal{R}(\mathcal{O}_{\lambda_n}(n))$$

is isomorphic to

$$\mathcal{H}ol(K(n)/K^{\lambda_n}(n))_{K(n)}.$$

The restriction of polynomials of $S(\mathcal{P}(n))$ to the orbit $\mathcal{O}_{\lambda_n}(n)$ of λ_n under $K_{\theta_n}(n)$ is a surjection of $S(\mathcal{P}(n))$ on $\mathcal{R}(\mathcal{O}_{\lambda_n}(n))$. As a result,

$\mathcal{R}(\mathcal{O}_{\lambda_n}(n))$ is isomorphic to $\mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}$.
Since

$$\mathcal{R}(\varinjlim \mathcal{O}_{\lambda_n}(n))$$

is identified with

$$\varprojlim \mathcal{H}ol(K(n)/K^{\lambda_n}(n))_{K(n)}$$

and

$$\mathcal{M}(\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)))_{\varinjlim K_{\mathbb{R}}(n)}$$

is identified with

$$\varprojlim \mathcal{M}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))_{K_{\mathbb{R}}(n)}.$$

The map $p: \quad S(\varinjlim \mathcal{P}(n)) \quad \longrightarrow \quad \mathcal{M}(\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)))_{\varinjlim K_{\mathbb{R}}(n)}$ is a surjection.

Remark 3.2. *(i) Let $\lambda_n \in (\varinjlim \mathcal{A}(n))^\star$. We extend λ_n to $\varinjlim \mathcal{P}(n)$ by taking it nil on $\varinjlim(\mathcal{Q}_{\mathbb{R}}(n))_{\mathbb{C}}$, where $\varinjlim \mathcal{Q}_{\mathbb{R}}(n)$ is the orthogonal of $\varinjlim \mathcal{A}_{\mathbb{R}}(n)$ in $\varinjlim \mathcal{P}_{\mathbb{R}}(n)$ for the bilinear form $\mathcal{B}_{\varinjlim \mathcal{P}_{\mathbb{R}}(n)}$.*
The following map is the generalized Poisson transform.

$$\begin{array}{rccc} \mathbf{P}: & \mathcal{C}^\infty(\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))) & \longrightarrow & \mathcal{C}^\infty(\varinjlim \mathcal{P}_{\mathbb{R}}(n)) \\ & f & \longmapsto & \mathbf{P}(f) \end{array}$$

with

$$\begin{aligned} \mathbf{P}(f)(X) &= \int_{\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))} e^{-i\lambda(k^{-1}\cdot X)} f(\dot{k})\, d\dot{k} \\ &= \lim_n \int_{K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)} e^{-i\lambda(k^{-1}\cdot X)} f(\dot{k})\, d\dot{k}, \quad \forall X \in \varinjlim \mathcal{P}_{\mathbb{R}}(n) \end{aligned}$$

where $d\dot{k}$ is the normalised measure $\varinjlim K_{\mathbb{R}}(n)-$invariant on $\varinjlim(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))$.

(ii) For each n, let us consider λ_n an element of $\mathcal{A}_{\mathbb{C}}^\star(n)$ and $(\sigma_n, E_{\sigma_n}(n))$ an element of $\widehat{K}^{\lambda_n}(n)$ (the unitary dual of $K^{\lambda_n}(n)$)
Let $\sigma_n \otimes \lambda_n$ be the representation of $K^{\lambda_n}(n) \ltimes \mathbf{P}_0(n)$ defined by $\sigma_n \otimes \lambda_n(e^{X,m}) = e^{i\lambda_n(X)}\sigma_n(m)x, \ \forall m \in K^{\lambda_n}(n), \ \forall X \in \mathcal{P}_0(n), \ \forall x \in E_{\sigma_n}$.
$V(\sigma_n, \lambda_n)$ denote all the $K(n)-$finite vectors of $V^\infty(\sigma_n, \lambda_n)$ for the representation

$$\Pi_{\sigma_n,\lambda_n} = \underset{K^{\lambda_n}(n) \ltimes \mathbf{P}_0(n) \uparrow G_0(n)}{Ind} \sigma_n \otimes \lambda_n$$

of $G_0(n)$*, with*
$V^\infty(\sigma_n, \lambda_n) = \{f \in \mathcal{C}^\infty(K(n), E_{\sigma_n}(n))/f(km) = \sigma_n(m^{-1})f(k),\ \forall k \in\ K(n),\ \forall m \in\ K^{\lambda_n}(n))\}$
and $\forall f \in V^\infty(\sigma_n, \lambda_n)$*,*

$$(\Pi_{\sigma_n,\lambda_n}(e^X.k)f)(k') = e^{i\lambda_n(k^{-1}.X)}f(k^{-1}k').$$

Let ϵ *be the trivial representation of* $K^{\lambda_n(n)}$*. Then* $\epsilon = \epsilon^\star,\ \ V^\infty(\epsilon, \lambda_n) \backsimeq\ \mathcal{C}^\infty(K(n)/K^{\lambda_n}(n))$ *and* $\underrightarrow{\lim} V^\infty(\epsilon, \lambda_n) \backsimeq \underrightarrow{\lim}\mathcal{C}^\infty(K(n)/K^{\lambda_n}(n))$

Theorem 3.3. *(i) The Poisson transform so defined is injective.*

(ii) For any $\lambda \in \mathcal{A}^\star_{\mathbb{C}}(\infty)$*, the* $(\mathcal{G}_0(\infty), K(\infty))-$*module* $\underrightarrow{\lim} V(\epsilon, \lambda_n)$ *is simple.*

Proof

(i) Let us suppose $\mathbf{P}(f) = 0$. By differentiating $\mathbf{P}(f)$ at the origin, we have

$$\begin{aligned}\int_{\underrightarrow{\lim}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))} p(-ik.\lambda)f(\dot{k})\mathrm{d}\dot{k} &= \lim_n \int_{K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)} p(-ik\cdot\lambda)f(\dot{k})\mathrm{d}\dot{k},\ \ \forall p \in S(\underrightarrow{\lim}\mathcal{P}(n))\\ &= 0.\end{aligned}$$

Since

$$\overline{\mathcal{M}(\underrightarrow{\lim}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)))_{\underrightarrow{\lim}K_{\mathbb{R}}(n)}} = \mathbf{L}^2(\underrightarrow{\lim}(K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n))),\quad f = 0.$$

(ii) For any representation δ, we denote by $\delta^\star$ the contra gradient representation.
Consider the linear map:

$$\begin{aligned}\underrightarrow{\lim} V^\infty(\sigma_n, \lambda_n) \times \underrightarrow{\lim} V^\infty(\sigma_n^\star, \lambda_n) &\longrightarrow \mathbb{C}\\ (f, g) &\longmapsto \langle f, g\rangle = \lim_n \int_{K_{\mathbb{R}}(n)/K_{\mathbb{R}}^{\lambda_n}(n)} \langle f(k), g(k)\rangle \mathrm{d}\dot{k}\end{aligned}$$

Since $\langle\cdot,\cdot\rangle$ is $\underrightarrow{\lim} G_0(n)-$invariant, it is particularly $\underrightarrow{\lim} K(n)-$invariant.
Let $\not\Vdash$ be the constant function equal to 1 on $AdK(n)$.
Then $\not\Vdash\ \in\ V^\infty(\epsilon, \lambda_n)$ and $(\Pi_{\epsilon,\lambda_n}(e^X.k)\not\Vdash)(k') = e^{i\lambda_n(k^{-1}.X)}, \forall k, k' \in AdK(n),\ \forall X \in \mathcal{P}_0(n)$. So, $\not\Vdash$ is $AdK(n)-$invariant. $V(\epsilon, \lambda_n)$, so far as $K(n)-$module, has as isotopic component $\mathbb{C}\cdot\not\Vdash$ of the type $\epsilon_{K(n)}$

and $\nVdash$ generates $V^{\infty}(\epsilon, \lambda_n)$.
$\nVdash$ is cyclic in $V^{\infty}(\epsilon, \lambda_n)$.
Let $f \in V^{\infty}(\epsilon^{\star}, -\lambda_n) = V^{\infty}(\epsilon, -\lambda_n)$ because $\epsilon^{\star} = \epsilon$.
For each n, $V^{\infty}(\epsilon, -\lambda_n) \perp V^{\infty}(\epsilon, \lambda_n)$. Then

$$\int_{AdK(n)/(AdK(n))^{\lambda_n}} e^{i\lambda(k^{-1}\cdot X)} f(\dot{k}) \mathrm{d}\dot{k} = \mathbf{P}_n(f) = 0 \Rightarrow f = 0.$$

Likewise $\nVdash$ is cyclic in $V^{\infty}(\epsilon, -\lambda_n)$.
So, for each n, $V^{\infty}(\epsilon, \lambda_n)$ is topologically irreducible $AdK(n) \ltimes P_0(n)-$module.
Thus $V^{\infty}(\epsilon, \lambda_n)$ is irreducible $(\mathcal{G}_0(n), K(n))-$module.
$\underrightarrow{\lim} V(\epsilon, \lambda_n)$ is irreducible. $\nVdash_{\infty}$ be constant function equal to 1 on $\underrightarrow{\lim} AdK(n)$. We have

$$\begin{aligned} (\lim \Pi_{\epsilon,\lambda_n}(e^X \cdot k) \nVdash_{\infty})(k') &= \lim e^{i\lambda_n(k^{-1}\cdot X)} \\ &= e^{i\lambda(k^{-1}\cdot X)}, \ \forall k, k' \in \underrightarrow{\lim} AdK(n), \ \forall X \in \mathcal{P}_0(n) \end{aligned}$$

So, $\nVdash_{\infty}$ is $\underrightarrow{\lim} AdK(n)-$invariant.
As result

$$\int_{AdK(n)/(AdK(n))^{\lambda_n}} e^{i\lambda(k^{-1}\cdot X)} f(\dot{k}) \mathrm{d}\dot{k} = \mathbf{P}_n(f) = 0$$

implies

$$\int_{\underrightarrow{\lim}(AdK(n)/(AdK(n))^{\lambda_n})} e^{i\lambda(k^{-1}\cdot X)} f(\dot{k}) \mathrm{d}\dot{k} = \mathbf{P}(f) = 0.$$

Since the Poisson transform is injective, $f = 0$. Consequently $\nVdash_{\infty}$ is cyclic in $\underrightarrow{\lim} V(\epsilon, \lambda_n)$. $\nVdash_{\infty}$ is also cyclic in $\underrightarrow{\lim} V(\epsilon, -\lambda_n)$. Then $\underrightarrow{\lim} V(\epsilon, \lambda_n)$ is topologically irreducible $\underrightarrow{\lim} AdK(n) \ltimes \underrightarrow{\lim} P_0(n)-$module.
Finally $\underrightarrow{\lim} V(\epsilon, -\lambda_n)$. is an irreducible $(\underrightarrow{\lim}\mathcal{G}_0(n), \underrightarrow{\lim} K(n))-$module.

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