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# Hall 3-Chromaticity of the Petersen Graph and the Weak Hall $t$-Chromatic Spectra of Odd Wheels

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**Abstract**

*Hall′s condition* on a finite simple graph equipped with a vertex list assignment and a "color demand" function is a necessary condition for the existence of a proper vertex multicoloring satisfying certain requirements. *Hall′s t-condition* is Hall's condition particularized to the cases when the list assignment is constant. A graph is *Hall t-chromatic* if, for every color demand function on it, the satisfying of Hall's $t$-condition suffices for the existence of a proper multicoloring of the graph, satisfying the demand, from a fixed list of $t$ colors. It is known that all (finite, simple) graphs are Hall $t$-chromatic for each $t \in \{0, 1, 2\}$, and there is a class of graphs, including the odd wheels except for $W_3 = K_4$, each of which is Hall $t$-chromatic only for $t \in \{0, 1, 2\}$.
We show that the Petersen graph is Hall 3-chromatic. Also, by adding a requirement to Hall's $t$-condition, we define *weak* Hall $t$-chromaticity and determine for each odd wheel the values of $t$ for which the wheel is weakly Hall $t$-chromatic.

**Key words and phrases:** List coloring, list multicoloring, list assignment, color demand, Hall's condition, Hall's $t$-condition, vertex independence number, Hall $t$-chromatic spectrum.
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# 1 Introduction

Let $\mathbb{N}$ denote the set of non-negative integers. For $t \in \mathbb{N} \setminus \{0\}$, let $[t] = \{1, 2, \ldots, t\}$ and let $[0] = \emptyset$.

All graphs will be finite and simple. A *color demand* on a graph $G$ is a function $\kappa : V(G) \to \mathbb{N}$. If $\kappa$ is a color demand on $G$ and $t \in \mathbb{N}$, a *proper* $(t, \kappa)$*-coloring* of $G$ is a function $\varphi : V(G) \to 2^{[t]} = \{\text{all subsets of } [t]\}$ satisfying for all $u, v \in V(G)$:

$(i) |\varphi(v)| = \kappa(v)$ and

$(ii)$ if $uv \in E(G)$, then $\varphi(u) \cap \varphi(v) = \emptyset$

Requirement $(ii)$ is equivalent to:

$(ii)'$ for each $j \in [t]$, $U(j, \varphi) = \{v \in V(G) | j \in \varphi(v)\}$ is an independent set of vertices in $G$.

Let $\alpha(H)$ denote the vertex independence number of a graph $H$. Suppose that $\varphi$ is a proper $(t, \kappa)$-coloring of $G$, and $H$ is a subgraph of $G$. Let $\varphi$ and $\kappa$ also denote the restrictions of $\varphi$ and $\kappa$ to $V(H)$. For each $j \in [t]$, the number of appearances of $j$ in the sets $\varphi(v)$, $v \in V(H)$, is $|U(j, \varphi) \cap V(H)| \leq \alpha(H)$. Therefore

$$\sum_{v \in V(H)} \kappa(v) = \sum_{j=1}^{t} |U(j, \varphi) \cap V(H)| \leq t\alpha(H).$$

(The equality above arises by counting the total number of appearances of the colors $1, 2, \ldots, t$ on the sets $\varphi(v)$, $v \in V(H)$, in two ways.)

A graph $G$ and a color demand $\kappa$ on $G$ satisfy *Hall's* $t$*-condition* if and only if for each subgraph $H$ of $G$,

$$t\alpha(H) \geq \sum_{v \in V(H)} \kappa(v) \qquad (*)_H.$$

Since removing an edge from a graph will not decrease its vertex independence number, for $G$ and $\kappa$ to satisfy Hall's $t$-condition it suffices that $(*)_H$ holds for every *induced* subgraph $H$ of $G$. Remarks above show that for any given $t \in \mathbb{N}$, $G$, and $\kappa$, Hall's $t$-condition is necessary for the existence of a proper $(t, \kappa)$-coloring of $G$. We define $G$ to be *Hall* $t$*-chromatic* if this condition is sufficient for the existence of such a coloring. That is, $G$ is Hall $t$-chromatic if and only if for each color demand $\kappa : V(G) \to \mathbb{N}$, if $G$ and $\kappa$ satisfy Hall's $t$-condition, there is a proper $(t, \kappa)$-coloring of $G$.

The *Hall t-chromatic spectrum* of a graph $G$ is $\tau(G) = \{t \in \mathbb{N} \,|\, G$ is Hall $t$-chromatic$\}$. Here is a summary of what is known about Hall $t$-chromatic spectra:

S1. For any graph $G$, $\{0, 1, 2\} \subseteq \tau(G)$ [2]. There are graphs $G$ for which $\tau(G) = \{0, 1, 2\}$ [1, 4]. The only other known value of $\tau$ is $\mathbb{N}$. It is not known that $\tau(G)$ will always be a block of consecutive integers.

S2. $\tau(G) = \mathbb{N}$ if $G$ is: (i) a cycle; (ii) bipartite; (iii) complete multipartite; (iv) the line graph of a tree [2, 3]; or (v) a theta graph [5], which is a graph consisting of three or more internally disjoint paths sharing two end-vertices.

S3. If $G \cap H$ is a clique then $\tau(G) \cap \tau(H) \subseteq \tau(G \cup H)$. In other words, if $G \cap H$ is a complete graph and both $G$ and $H$ are Hall $t$-chromatic, then $G \cup H$ is Hall $t$-chromatic [2].

S4. If $H$ is an induced subgraph of $G$, then $\tau(G) \subseteq \tau(H)$. In other words, if $G$ is Hall $t$-chromatic and $H$ is an induced subgraph of $G$, then $H$ is Hall $t$-chromatic [2].

S5. If $\tau(G)$ is infinite, then the fractional chromatic number of $G$, denoted $\chi_f(G)$, is equal to the Hall ratio of $G$:

$$\chi_f(G) = \rho(G) = max[\frac{|V(H)|}{\alpha(H)}; H \text{ is an induced subgraph of } G] \qquad [4].$$

(See [4] for definitions and proof and [7] for further discussion of $\chi_f$.)

We can easily disprove the converse of S5: take any graph $H$ such that $\tau(H) = \{0, 1, 2\}$, and embed it as an induced subgraph of a graph $G$ such that $\chi_f(G) = \rho(G)$. This can be accomplished by attaching a large clique to $H$. The deeper question, which we expect to be open for a long time, is: If $\rho(H) = \chi_f(H)$ for every induced subgraph $H$ of $G$, does it follow that $\tau(G)$ is infinite?

From S2(iv), it follows that $\tau(K_n) = \mathbb{N}$ for $n \in \mathbb{N}$, $n > 0$, because $K_n$ is the line graph of $K_{1,n}$. Then from S3, it follows that if every block of $G$ is a clique, then $\tau(G) = \mathbb{N}$. This extends part of the main theorem of [6], about arbitrary list assignments with constant color demand $\kappa \equiv 1$ , to the case of constant list assignments with arbitrary color demands.

The following mini-lemmas about Hall's $t$-condition will help simplify the proofs in Sections 4 and 5. Suppose that $\kappa$ is a color demand on a graph $G$ and that $G$ and $\kappa$ satisfy Hall's $t$-condition.

L1. For each $v \in V(G)$, $\kappa(v) \le t$; to see this, take $H = v = K_1$ in the inequality $(*)_H$.

L2. For each $uv \in E(G)$, taking $H = uv = K_2$, we have $\kappa(u) + \kappa(v) \le t$.

L3. If $u \in V(G)$ and $\kappa(u) = t$, then $\kappa(v) = 0$ for each $v \in N_G(u) = \{v \in V(G) | uv \in E(G)\}$. This follows from L2.

Whether or not it is known beforehand that $G$ and $\kappa$ satisfy Hall's condition, if $v \in V(G)$ and $\kappa(v) = 0$, then it is clear that there is a proper $(t, \kappa)$-coloring of $G$ if and only if there is a proper $(t, \kappa)$-coloring of $G - v$.

Finally, a remark about deciding whether or not a given $G$ and $\kappa$ satisfy Hall's $t$-condition. By the definition, it appears to be necessary to verify the inequality $(*)_H$ for every induced subgraph $H$ of $G$. We claim, however, that $G$ and $\kappa$ satisfy Hall's $t$-condition if the single inequality $(*)_G$ holds and if for each graph $G-v$, $v \in V(G)$, there is a proper $(t, \kappa)$-coloring. The claim follows from the observations that each induced subgraph $H$ of $G$ other than $G$ itself is a subgraph of $G - v$ for some $v \in V(G)$ and that Hall's $t$-condition is necessary for the existence of a proper $(t, \kappa)$-coloring.

For instance, consider the 5-wheel, $W_5 = K_1 \vee C_5$, where "$\vee$" denotes the join, with the color demand $\kappa$ as indicated in Figure 1. Suppose also that $t \ge 3$. Clearly $W_5 - v$ is properly $(t, \kappa)$-colorable for each $v \in V(W_5)$. We see that $t\alpha(W_5) = 2t \ge 5 + t - 2 = t + 3$ which holds because $t \ge 3$.

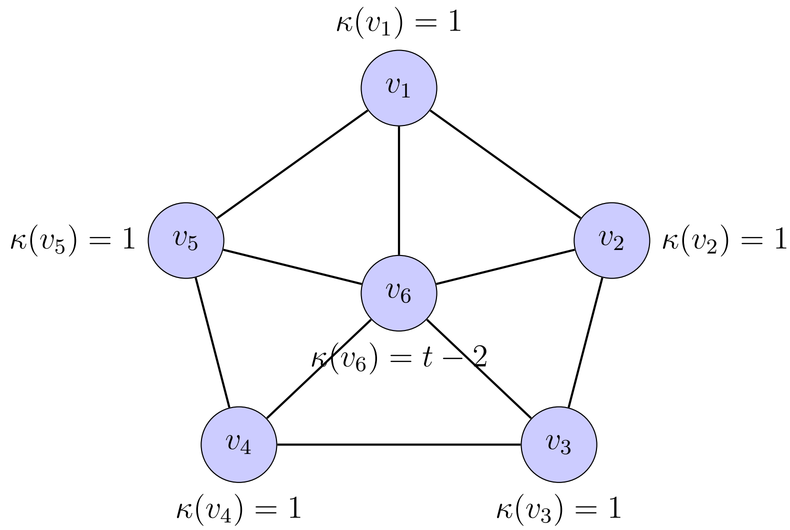


Figure 1: $W_5$ with a color demand satisfying Hall's $t$-condition when $t \geq 3$.

So $W_5$ and $\kappa$ satisfy Hall's $t$-condition. Since there is no proper $(t, \kappa)$-coloring of $W_5$, we conclude that $\tau(W_5) = \{0, 1, 2\}$. This argument, with a similar color demand, was employed in [4] to derive the same conclusion for all wheels $W_{2k+1}$, $k \geq 2$.

## 2 Weak Hall $t$-chromaticity

We will say that a graph $G$ is *weakly Hall $t$-chromatic* if there is a proper $(t, \kappa)$-coloring of $G$ for every color demand $\kappa$ on $G$ such that $G$ and $\kappa$ satisfy Hall's $t$-condition and, in addition,

$$t\alpha(G) = \sum_{v \in V(G)} \kappa(v).$$

That is, $G$ is weakly Hall $t$-chromatic if the satisfaction of the inequalities $(*)_H$ for all $H$, $H$ an induced subgraph of $G$, with equality when $H = G$, is sufficient for the existence of a proper $(t, \kappa)$-coloring of $G$. The weak Hall $t$-chromatic spectrum of $G$ is

$$\tau_w(G) = \{t \in \mathbb{N} \mid G \, is \, weakly \, Hall \, t\text{-}chromatic\}.$$

If $G$ is Hall $t$-chromatic, then $G$ is weakly Hall $t$-chromatic; i.e. $\tau(G) \subseteq \tau_w(G)$. We will soon see that the inclusion can be proper.

There are still quite a few questions about $\tau_w$ that are wide open. For instance:

Is $\tau_w$ always a block of consecutive numbers?

Is it possible for $\tau_w(G)$ to be infinite when $\tau(G)$ is finite?

Does $|\tau_w(G)| = \infty$ imply that $\chi_f(G) = \rho(G)$?

In view of the proof in [4] of the corresponding result for $\tau$ (see S5), this last question may be easily answerable. However, we shall not attempt to answer it here.

# 3 Main Results

Proofs are postponed until Section 4. $G$ will be a graph, $\kappa$ a color demand on $G$, and $t \in \mathbb{N}$.

**Theorem 3.1.** *The Petersen Graph is Hall 3-chromatic.*

**Theorem 3.2.** $\tau_w(W_{2k+1}) = \{0, 1, 2, \ldots, 2k-2\},\ k \geq 2.$

# 4 Proofs and Intermediate Results

**Proof of Theorem 3.1**

In this proof we will refer often to the vertices of $P$ as named in Figure 2, below.

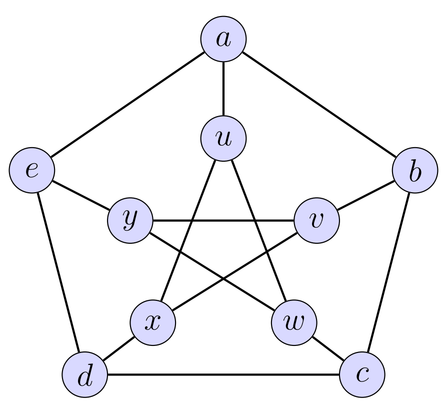


Figure 2: The Petersen graph with vertex labels.

Suppose that $\kappa : V(P) \to \mathbb{N}$ satisfies, with $P$, Hall's 3-condition. We aim to show that there is a proper $(3, \kappa)$-coloring of $P$.

Suppose that $\kappa : V(P) \to \{1,2\}$. That is, suppose that $\kappa$ does not take the values 0,3. If $H \simeq C_5$ is a 5-cycle subgraph of $P$, then $(*)_H$ is $6 \geq \sum_{z \in V(H)} \kappa(z)$. If $\kappa(z) \in \{1,2\}$ for all $z \in V(H)$, then $\kappa(z) = 2$ for at most one $z \in V(H)$, if this inequality holds – which it does, by supposition. Now, $P$ has the interesting property, easily checked, that for any two distinct vertices of $P$ there is a $C_5$ subgraph of $P$ on which both vertices lie. Therefore, there can be at most one vertex $z$ such that $\kappa(z) = 2$. Since $P$ is vertex transitive, it does not matter for which vertex $z$ $\kappa(z) = 2$. Let us suppose that $\kappa(a) = 2$ and $\kappa(z) = 1$ for all $z \in V(P) \backslash \{a\}$. In Figure 3 we indicate a proper $(3, \kappa)$-coloring of $P$.

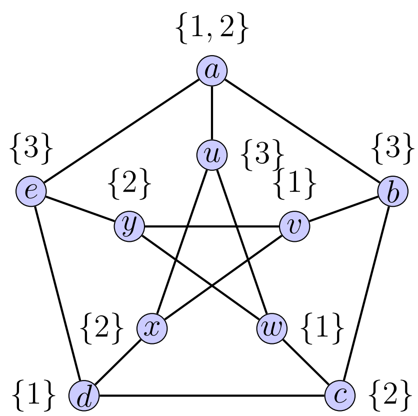


Figure 3: The Petersen graph with a proper $(3, \kappa)$-coloring.

Obviously, if there is a proper $(3, \kappa)$-coloring of $P$ for this $\kappa$, then there is a proper $(3,1)$-coloring of $P$; or, as is well known, $\chi(P) = 3$.

Therefore, we may assume that $\kappa$ takes on the values 0 or 3 somewhere. First suppose that $\kappa$ takes the value 3; without loss of generality, assume that $\kappa(a) = 3$. By mini-lemma L3, we conclude that $\kappa(e) = \kappa(u) = \kappa(b) = 0$. Clearly a proper $(3, \kappa)$-coloring of $H = P - \{a, e, u, b\}$ could be extended to a proper $(3, \kappa)$-coloring of $P$ by setting $\varphi(e) = \varphi(u) = \varphi(b) = \emptyset$ and $\varphi(a) = \{1, 2, 3\}$.

The restriction of $\kappa$ to any subgraph of $P$ satisfies Hall's 3-condition on that subgraph; $H$ is a cycle; therefore $H$ is Hall $t$-chromatic for all $t$, by $S2$; therefore there is a proper $(3, \kappa)$-coloring of $H$.

Now we may assume that $\kappa$ does not take the value 3 on $V(P)$ but does take the value 0. Again, invoking the vertex transitivity of $P$, we may assume that $\kappa(a) = 0$. We will be done with this proof if we can show the existence of a proper $(3, \kappa)$-coloring of $Q = P - a$.

First suppose that $\kappa : V(Q) \to \{1,2\}$; that is, suppose that $\kappa$ does not take the value 0 on $Q$. If $\kappa \equiv 1$, then there is a proper $(3,\kappa)$-coloring of $Q$, because $\chi(Q) = \chi(P) = 3$. If $\kappa(z) = 2$ and $z \in \{c,d,v,w,x,y\}$, then $z$ is the only vertex of $Q$ at which $\kappa$ takes the value 2, because each of the other vertices of $Q$ is on a $C_5$ subgraph of $Q$ with $z$. We have already seen that if a color demand $\kappa'$ on $P$ takes the value 2 at only one vertex and values less than or equal to 1 at all other vertices, then there is a proper $(3,\kappa')$-coloring of $P$, so the same holds for $Q$.

Therefore we may as well assume, under the assumptions that $\kappa$ takes only the values 2 and 1 on $V(Q)$, that $\kappa(z) = 1$ for $z \in \{c,d,v,w,x,y\} = V(Q)\backslash\{e,u,b\}$. We finish off this subcase by showing that if $\kappa(e) = \kappa(u) = \kappa(b) = 2$ and $\kappa(z) = 1$ for all $z \in V(Q)\backslash\{e,u,b\}$, then there is a proper $(3,\kappa)$-coloring of $Q$. This coloring is given in Figure 4.

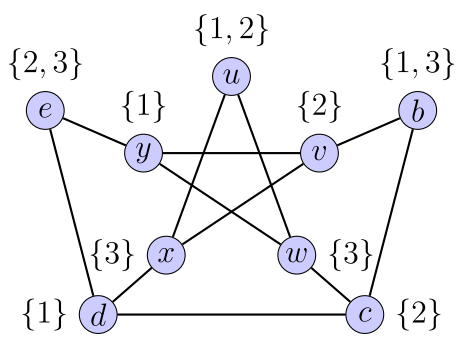


Figure 4: A proper $(3,\kappa)$-coloring of $Q = P - a$ when $\kappa(e) = \kappa(u) = \kappa(b) = 2$ and $\kappa = 1$ otherwise.

We are now down to the case in which $\kappa$ takes only the values 0, 1, 2 on $V(Q)$ and does necessarily take the value 0 at some vertex. Also, we may assume that $\kappa$ takes the value 2 at least twice in general and at least once on $\{v,w,x,y,c,d\}$. As before, if $\kappa(z) = 0$ and there is a proper $(3,\kappa)$-coloring of $Q - z$, then we are done with the subcase in which this occurs.

<u>Subcase 1.</u> $\kappa(z) = 0$ for some $z \in \{c,d\}$. Suppose that $\kappa(d) = 0$.

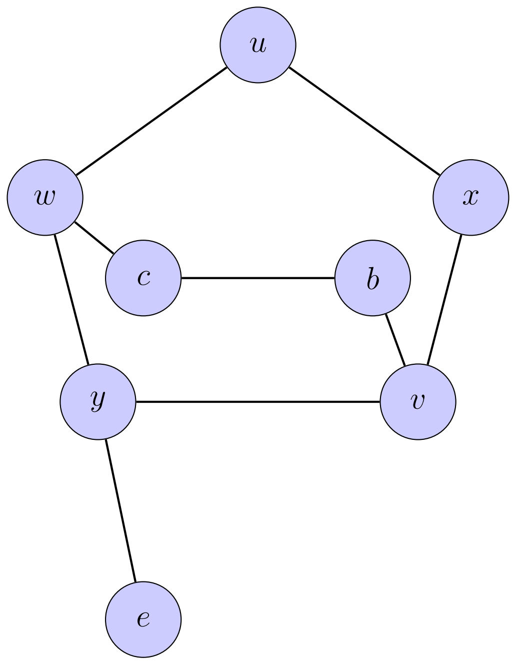


Figure 5: $Q - d$ redrawn.

In Figure 5 we see that $Q-d$ consists of a theta graph with a pendant edge attached. By $S2$ both the theta graph and the attached $K_2$ are Hall $t$-chromatic for all $t$, and they intersect at a single vertex. By $S3$, $Q-d$ is Hall $t$-chromatic for all $t$. Since $Q-d$ and $\kappa$ satisfy Hall's $t$-condition, there is a proper $(3,\kappa)$-coloring of $Q-d$; this finishes Subcase 1.

Subcases 2 and 3. $\kappa(z) = 0$ for some $z \in \{v, w, x, y\}$. These turn out to be the same as Subcase 1.

Subcase 4. $\kappa(z) = 0$ for some $z \in \{e, u, b\}$. It is straightforward to see that the graphs $Q - z$ for $z \in \{e, u, b\}$ are isomorphic. Suppose that $\kappa(u) = 0$. Let $R = Q - u$. If $R$ has a proper $(3, \kappa)$-coloring then the proof is complete. It is straightforward to verify that for every $z \in V(R)$, $R-z$ is Hall $t$-chromatic for all $t \in \mathbb{N}$; if the degree of $z$ in $R$ is 2 then $R-z$ is a theta graph, and if the degree of $z$ in $R$ is 3 then $R-z$ is a cycle with two pendant edges.

Therefore we may assume that $\kappa > 0$ on $V(R)$. Here is a recap of all we may assume about $\kappa$ on $V(R)$, arising from colorings earlier in the proof and the fact that $\kappa$ and $R$ satisfy Hall's 3-condition:

1. $\kappa$ takes only the values 1 and 2.

2. $\kappa$ takes the value 2 at least twice.
3. $\kappa$ takes the value 2 on at least one of $v, w, x, y, c, d$.
4. $\kappa$ cannot take the value 2 at two adjacent vertices.
5. $\kappa$ cannot take the value 2 at two different vertices on a 5-cycle in $R$.

From these - especially 1,2, and 5 - we conclude that $\kappa$ must take the value 2 at vertices $x$ and $w$, and only there. The proper $(3, \kappa)$-coloring of $R$ shown in Figure 6 completes the proof.

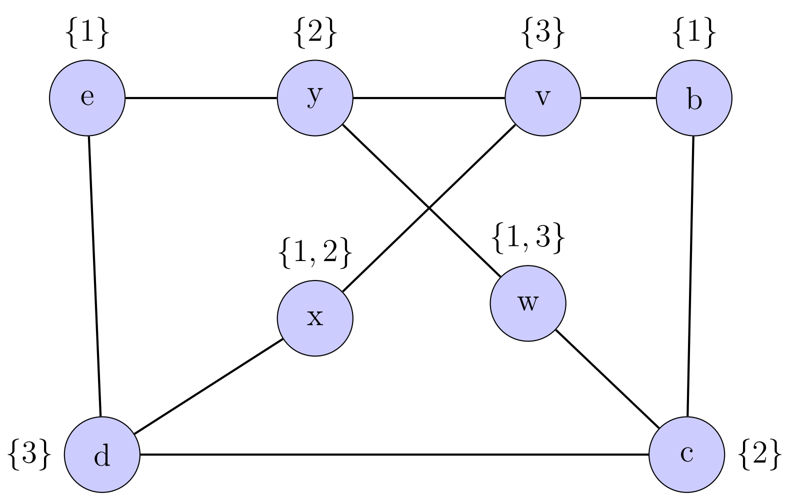


Figure 6: $R = Q - u$ with a proper $(3, \kappa)$-coloring.

$\square$

Next we will prove Theorem 3.2. The lone vertex of $W_{2k+1}$ $(k \geq 2)$ of degree $2k+1$ will be named $v$, and the $2k+1$ vertices of the cycle $C_{2k+1}$ will be $u_1, \ldots, u_{2k+1}$, around the cycle.

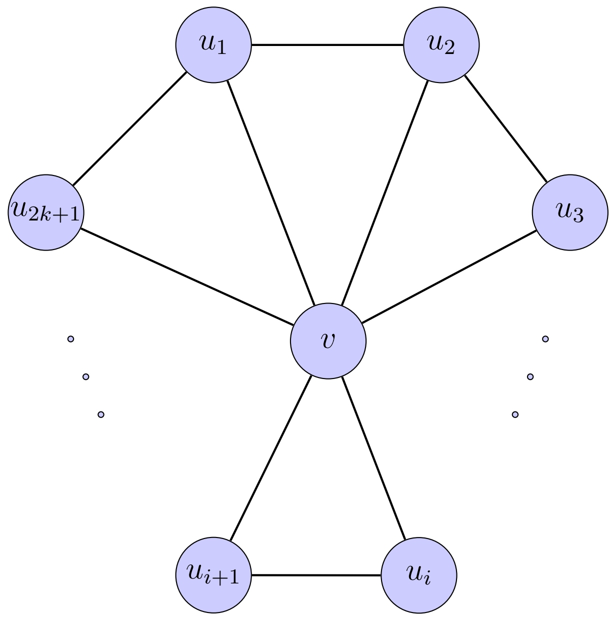


Figure 7: $W_{2k+1}$ with labeled vertices.

**Lemma 4.1.** *For each $w \in V(W_{2k+1})$, $W_{2k+1} - w$ is Hall $t$-chromatic for all $t \in \mathbb{N}$.*

*Proof.* If $w = v$ then $W_{2k+1} - w$ is a cycle; the claim follows from S2(i) in the Introduction.

Without loss of generality we finish the proof by letting $w = u_{2k+1}$. The $K_3$'s $vu_1u_2$ and $vu_2u_3$ each have Hall $t$-chromatic spectrum $\mathbb{N}$; therefore, by S3, their union has Hall $t$-chromatic spectrum $\mathbb{N}$. Therefore, again by S3, the union of $vu_1u_2$, $vu_2u_3$, and $vu_3u_4$ has Hall $t$-chromatic spectrum $\mathbb{N}$. Continuing in this way we find that

$$W_{2k+1} - u_{2k+1} = \cup_{i=1}^{2k-1} vu_iu_{i+1}$$

has Hall $t$-chromatic spectrum $\mathbb{N}$.

□

**Corollary 4.2.** *If $t \in \mathbb{N}$, $\kappa : V(W_{2k+1}) \to [t]$ satisfies Hall's $t$-condition with $W_{2k+1}$, and $\kappa(w) = 0$ for some $w \in V(W_{2k+1})$, then there is a proper $(t, \kappa)$-coloring of $W_{2k+1}$.*

In all that follows, $k \geq 2$ and subscripts are to be read $mod(2k+1)$; e.g., $u_{(2k+1)+1} = u_1$.

**Lemma 4.3.** *Suppose that $t \in \mathbb{N}$, $G = W_{2k+1}$, and $\kappa : V(G) \to [t]$ satisfies the inequality $(*)_H$ for $H = G$ and $H = vu_iu_{i+1} \simeq K_3$, $i = 1, 2, \ldots, 2k+1$. Then $\kappa$ satisfies $(*)_H$ for every subgraph $H$ of $W_{2k+1}$; i.e. $\kappa$ and $W_{2k+1}$ satisfy Hall's $t$-condition.*

*Proof.* It suffices to show that $(*)_H$ holds for every induced subgraph $H$ of $W_{2k+1}$. The inequality for $H = W_{2k+1} - v = C_{2k+1}$ follows from the inequality for $H = W_{2k+1}$. All induced subgraphs of $W_{2k+1}$ other than $W_{2k+1}$, $C_{2k+1}$, and $vu_iu_{i+1}$, $i = 1, 2, \ldots 2k+1$, are subgraphs of one of the graphs $W_{2k+1} - u_i$, $i = 1, 2, \ldots, 2k+1$. Since Hall's $t$-condition is necessary for the existence of a proper $(t, \kappa)$-coloring, it will suffice to show that there is a proper $(t, \kappa)$-coloring of $W_{2k+1} - u_i$ for each $i \in \{1, 2, \ldots, 2k+1\}$. Since the naming of the vertices of $C_{2k+1}$ was arbitrary, it suffices to show that there is a proper $(t, \kappa)$-coloring of $W_{2k+1} - u_{2k+1}$.

Color $v$ with $\emptyset$ if $\kappa(v) = 0$ and with $\{t - \kappa(v) + 1, \ldots, t\}$ otherwise. Then color the vertices of the path $u_1u_2 \ldots u_{2k}$ with subsets of $[t - \kappa(v)]$ by coloring $u_i$ with $\{1, 2, \ldots, \kappa(u_i)\}$ if $i$ is odd and with $\{t - \kappa(v) - \kappa(u_i) + 1, \ldots, t - \kappa(v)\}$ if $i$ is even. [In each case, if $\kappa(u_i) = 0$, the color set is intended to be $\emptyset$.] Since, by $(*)_H$ with $H = vu_iu_{i+1}$, $\kappa(v) + \kappa(u_i) + \kappa(u_{i+1}) \leq t$, it follows that this assignment does indeed result in a proper $(t, \kappa)$-coloring of $W_{2k+1} - u_{2k+1}$. □

**Proof of Theorem 3.2**

Suppose that $2 < t \leq 2k - 2$ where $k \geq 2$ and $\kappa : V(W_{2k+1}) \to [t]$ satisfies Hall's $t$-condition with $W_{2k+1}$. Suppose also that $\sum_{i=1}^{2k+1} \kappa(u_i) = kt - \kappa(v)$ holds. Since $\kappa$ and $W_{2k+1}$ satisfy Hall's $t$-condition, if $\kappa(w) = 0$ for some $w \in V(W_{2k+1})$ then there is a proper $(t, \kappa)$-coloring of $W_{2k+1}$ by Corollary 4.2. Therefore, we may assume that $\kappa(w) \geq 1$, for all $w \in V(W_{2k+1})$.

Since $v$, $u_i$, $u_{i+1}$ (read $i + 1$ $mod(2k+1)$) induce a $K_3$ in $W_{2k+1}$, $\kappa(u_i) + \kappa(u_{i+1}) \leq t - \kappa(v)$, $i = 1, 2, \ldots, 2k+1$. Consequently, $kt - \kappa(v) = \displaystyle\sum_{i=1}^{2k+1} \kappa(u_i) = \frac{1}{2} \sum_{i=1}^{2k+1} [\kappa(u_i) + \kappa(u_{i+1})] \leq \frac{2k+1}{2}(t - \kappa(v))$,
which implies that $kt \leq (k + \frac{1}{2})t - \frac{2k-1}{2}\kappa(v) \leq kt + \frac{t}{2} - \frac{2k-1}{2}$, which implies that $2k - 1 \leq t$.

Therefore, if $t \leq 2k - 2$ then every $\kappa$ satisfying Hall's $t$-condition with $W_{2k+1}$, and also $\sum_{i=1}^{2k+1} \kappa(u_i) + \kappa(v) = kt$, must take the value 0 somewhere, and, therefore, there is a proper $(t, \kappa)$-coloring of $W_{2k+1}$. Thus $\{0, 1, \ldots, 2k-2\} \subseteq \tau_w(W_{2k+1})$.

To show that for $t \geq 2k - 1$, $W_{2k+1}$ is not weakly Hall $t$-chromatic, we will assign $\kappa(v) = 1$ and make $\kappa(u_1), \ldots, \kappa(u_{2k+1})$ as nearly equal as possible, with $\sum_{i=1}^{2k+1} \kappa(u_i) = kt - \kappa(v) = kt - 1$. No proper $(t, \kappa)$-coloring of $W_{2k+1}$ would be possible with such a $\kappa$, because $C_{2k+1}$ would have to be properly $(t-1, \kappa)$ colored, and $\sum_{u \in V(C_{2k+1})} \kappa(u) = kt - 1 > (t-1)k = (t-1)\alpha(C_{2k+1})$.

Now we define $\kappa$. Let $kt-1 = q(2k+1)+r$ where $q, r$ are integers and $0 \leq r < 2k+1$. Observe that $k \geq 2$ and $t \geq 2k-1 \geq 3$ imply $(t-2)k \geq k \geq 2$, which implies that $kt-1 \geq 2k+1$; therefore, $q \geq 1$.

For different values of $\kappa$ and $t$, we will define $\kappa$ by distributing the values $q+1$ and $q$ over the vertices $u_1, \ldots, u_{2k+1}$ such that they appear $r$ and $2k+1-r$ many times, respectively. For any such distribution, $\sum_{i=1}^{2k+1} \kappa(u_i) = kt-1$. We will be finished with the proof of the theorem if, for $\kappa$ defined by some such distribution, $1+\kappa(u_i)+\kappa(u_{i+1}) \leq t$, for $i = 1, 2, \ldots, 2k+1$, because then Lemma 4.3 would imply that $\kappa$ and $W_{2k+1}$ satisfy Hall's $t$-condition.

If $r \leq k$ then we can distribute $q$ and $(q+1)$ so that no two adjacent $u_i$ are assigned $q+1$. If $\kappa$ is defined by such a distribution, we are done if $t \geq 2q+2$. In the very special case where $r = 0$ and all color demand values are $q$, we need only $t \geq 2q+1$. If $k+1 \leq r \leq 2k+1$, then in defining $\kappa$ we cannot avoid $\kappa(u_i) = \kappa(u_{i+1}) = q+1$ for some value of $i$. In these cases we need $t \geq 2q+3$.

We will deal with $t \in \{2k-1, 2k, 2k+1\}$ as special cases and then break $t \geq 2k+2$ into two cases, $r \leq k$ and $r \geq k+1$.

$\mathbf{t = 2k-1}$ : $kt-1 = k(2k-1)-1 = (k-1)(2k+1)$ so $q = k-1$ and $r = 0$. We have $2q+1 = 2k-1 = t$.

$\mathbf{t = 2k}$ : $kt-1 = 2k^2-1 = (k-1)(2k+1)+k$ so $q = k-1$ and $r = k$. We have $2q+2 = 2k = t$.

$\mathbf{t = 2k+1}$ : $kt-1 = k(2k+1)-1 = (k-1)(2k+1)+2k$, so $q = k-1$ and $r = 2k$. We have $2q+3 = 2k+1 = t$.

$\mathbf{t > 2k+1}$ : Then if $q, r$ are integers satisfying $kt-1 = q(2k+1)+r$, and $0 \leq r < 2k+1$, it must be that $q \geq k$. In every case, $kt-1 = q(2k+1)+r \geq q(2k+1)$. Thus $t \geq 2q+\frac{q+1}{k} > 2q+1$ and therefore $t \geq 2q+2$. So we are finished except in the cases where $t > 2k+1$ and $r \geq k+1$. In those cases we have $kt-1 = q(2k+1)+r > q(2k+1)+k$. Thus $t \geq 2q+1+\frac{q+1}{k} > 2q+2$, so $t \geq 2q+3$, as desired.

□

# 5 Other Results

Let $m$ be a positive integer and $H$ a finite simple graph. Let $G = K_m \vee H$. The question we are interested in is: under what conditions on $H$ and integers $m$ and $t$ is $G$ weakly Hall $t$-chromatic?

We omit the proofs of Lemmas 5.1 and 5.2

**Lemma 5.1.** *Suppose that $\kappa$ is a color demand on $G = K_m \vee H$. There is a proper $(t, \kappa)$-coloring of $G$ if and only if there is a proper $(t - \sum_{w \in V(K_m)} \kappa(w),\ \kappa|_{V(H)})$-coloring of $H$.*

**Lemma 5.2.** *Suppose that $\kappa$ is a color demand on $G = K_m \vee H$. Then $\kappa$ and $G$ satisfy Hall's $t$-condition if and only if, for each induced subgraph $X$ of $H$ with $|V(X)| > 0$, $\sum_{x \in V(X)} \kappa(x) \leq t\alpha(X) - \sum_{w \in V(K_m)} \kappa(w)$ and $\sum_{w \in V(K_m)} \kappa(w) \leq t$.*

**Theorem 5.3.** *$K_m \vee H$ is (weakly) Hall $t$-chromatic if and only if $K_1 \vee H$ is (weakly) Hall $t$-chromatic.*

*Proof.* Let $V(K_1) = \{v\}$. If $\kappa$ is a color demand on $K_m \vee H$, define $\hat{\kappa}$ on $V(K_1 \vee H)$ by $\hat{\kappa} = \kappa$ on $V(H)$ and $\hat{\kappa}(v) = \sum_{w \in V(K_m)} \kappa(w)$. Conversely, if $\hat{\kappa}$ is a color demand on $K_1 \vee H$, let $\kappa$ be any color demand on $K_m \vee H$ satisfying $\kappa = \hat{\kappa}$ on $V(H)$ and $\hat{\kappa}(v) = \sum_{w \in V(K_m)} \kappa(w)$.

By Lemmas 5.1 and 5.2, there is a proper $(t, \kappa)$-coloring of $K_m \vee H$ if and only if there is a proper $(t, \hat{\kappa})$-coloring of $K_1 \vee H$, and $\kappa$ and $K_m \vee H$ satisfy Hall's $t$-condition if and only if $\hat{\kappa}$ and $K_1 \vee H$ satisfy Hall's $t$-condition. Also, obviously, $\sum_{w \in V(K_m \vee H)} \kappa(w) = \sum_{x \in V(K_1 \vee H)} \hat{\kappa}(x)$ . The conclusions follow straightforwardly.

□

Henceforward, in this section, $G = K_1 \vee H$ and $V(K_1) = \{v\}$.

The next result makes explicit a path to proving that $G$ is not weakly Hall $t$-chromatic. This path was followed in showing that $W_{2k+1} = K_1 \vee C_{2k+1},\ k \geq 2$, is not weakly Hall $t$-chromatic for $t \geq 2k - 1$.

**Theorem 5.4.** *Suppose that $\kappa$ is a color demand on $G = K_1 \vee H$ which satisfies Hall's $t$-condition with $G$, and suppose that for some induced subgraph $X$ of $H$, $\sum_{x \in V(X)} \kappa(x) >$*

$(t - \kappa(v))\alpha(X)$. *Then there is no proper* $(t, \kappa)$*-coloring of* $G$*; therefore* $G$ *is not Hall* $t$*-chromatic. If, in addition,* $\sum_{w \in V(H)} \kappa(w) = t\alpha(H) - \kappa(v)$*, then the existence of such a subgraph* $X$ *implies that* $G$ *is not weakly Hall* $t$*-chromatic.*

*Proof.* There is a proper $(t, \kappa)$-coloring of $G$ if and only if there is a proper $(t - \kappa(v),\ \kappa|_{V(H)})$-coloring of $H$, by Lemma 5.1, and the satisfaction of Hall's $(t - \kappa(v))$-condition by $\kappa|_{V(H)}$ and $H$ is necessary for such a coloring of $H$ to exist. If such an $X$ exists, then Hall's $(t - \kappa(v))$-condition is not satisfied by $H$ and $\kappa|_{V(H)}$.

□

However, we do have the following:

**Theorem 5.5.** *If* $H$ *is bipartite then* $K_1 \vee H$ *is Hall* $t$*-chromatic for all* $t \in \mathbb{N}$*.*

*Proof.* Let $A, B \subseteq V(H)$ be a bipartition of $H$. Suppose that $\kappa$ is a color demand on $G = K_1 \vee H$ and $\kappa$ and $G$ satisfy Hall's $t$-condition. Color $v$ with $\{t - \kappa(v) + 1, \ldots, t\}$. If $a \in A$, color $a$ with $\{1, \ldots, \kappa(a)\}$ and if $b \in B$ color $b$ with $\{t - \kappa(v) - \kappa(b) + 1, \ldots, t - \kappa(v)\}$. Clearly the color sets on $V(H) = A \cup B$ are disjoint from the color set on $v$. If $a \in A$, $b \in B$ are adjacent in $H$, then $v, a, b$ induce a $K_3$ in $G$ so $\kappa(a) + \kappa(b) \leq t - \kappa(v)$. It follows from this that the color sets on $a$ and $b$ are disjoint. Thus our coloring is a proper $(t, \kappa)$-coloring.

□

**Corollary 5.6.** *If* $H$ *is bipartite then* $K_m \vee H$ *is Hall* $t$*-chromatic for all* $t \in \mathbb{N}$*, for any positive integer* $m$*.*

**Theorem 5.7.** *If* $H$ *is bipartite then* $G = \overline{K_m} \vee H$ *is Hall* $t$*-chromatic for all* $t \in \mathbb{N}$*.*

*Proof.* Suppose that $\kappa$ is a color demand function on $G$ such that $\kappa$ and $H$ satisfy Hall's $t$-condition. Let $v \in V(\overline{K_m})$ be such that $\kappa(v) \geq \kappa(w)$ for all $w \in V(\overline{K_m})$. Let each $w \in V(\overline{K_m})$ have color set $\{t - \kappa(w) + 1, \ldots, t\}$. Let $A, B$ be a bipartition of $H$. Color $a \in A$ with the set $\{1, \ldots, \kappa(a)\}$, and color $b \in B$ with the set $\{t - \kappa(v) - \kappa(b) + 1, \ldots, t - \kappa(v)\}$. The argument that shows this to be a proper $(t, \kappa)$-coloring of $G$ is similar to that in the proof of Theorem 5.5.

□

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