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On JU-algebras and p-Closure Ideals

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Abstract

In this paper, we define JU-algebras and discuss the concept of p-closure ideals of JU-algebras which we denote as J_{pc} for a non-empty subset J of a JU-algebra X. Moreover, we investigate related properties of JU-algebras and p-closures of subsets of JU-algebras. Through p-closure of subsets of X we also define a closure operator. We establish a relation between $f(J_{pc})$ and $(f(J))_{pc}$ for a JU-homomorphism f and prove that J_{pc} is the least closed p-ideal containing J for any ideal J of X.

1 Introduction

Y. Imai and K. Iseki [6] introduced BCK-algebra as a generalization of the concept of set-theoretic difference and proportional calculi which is an important class of algebras. Moreover, at the same time, Iseki [7] introduced BCI-algebras which can be considered to be another class of algebras. BCI/BCK algebras are the two important classes of logical algebras. Afterwards, the notion of BCI/BCK algebras number in different types of logical algebras has been defined by numerous authors and their distinct properties have been discussed in different articles. KU-algebra is one of those logical algebras which

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we will be discussing.

The notion of KU-algebras was introduced by Prabpayak and Leerawat [1]. Homomorphisms of KU-algebras and some of its related properties was given in ([1], [2]). Many authors widely studied KU-algebras in different contexts; e.g., fuzzy, neutrosophic and intuitionistic, soft and rough sense etc. Mostafa et al. [9] defined fuzzy ideals of KU-algebras whereas Mostafa et al. [10] further studied interval valued fuzzy KU-ideals in KU-algebras. Naveed et al. [8] introduced the concept of cubic KU-ideals of KU-algebras. Recently, Moin and Ali introduced roughness in KU-algebras [11]. Ali et al. [12] introduced pseudo-metric on KU-algebras. Senapati and Shum [13] defined Atanassov's intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra.

In this article we define a JU-algebra which is a generalization of KU-algebras. A JU-algebra can also be seen as a Pseudo KU-algebra (see [3] and [4]). We study the p-closure of a non-empty subset J of a JU-algebra X and investigate related properties of p-closure of subsets and ideals of JU-algebras.

2 Preliminaries

In this section, we shall introduce JU-algebras, JU-subalgebras, JU-ideals and other important terminologies with examples and some related results.

Definition 2.1. *An algebra $(X, \diamond, 1)$ of type $(2, 0)$ with a single binary operation $\diamond$ is said to be JU-algebras satisfying the following identities for any $x, y, z \in X$,*
(JU_1) $(y \diamond z) \diamond [(z \diamond x) \diamond (y \diamond x)] = 1$,
(JU_2) $1 \diamond x = x$,
(JU_3) $x \diamond y = y \diamond x = 1$ *implies* $x = y$.

We call the constant 1 of X the fixed element of X. For convenience, we write X instead of $(X, \diamond, 1)$ to represent a JU-algebra. We define a relation " $\leq$ " in X by $y \leq x$ if and only if $x \diamond y = 1$. If we add the condition $x \diamond 1 = 1$ for all $x \in X$ in the definition of JU-algebras, then we get that X is a KU-algebra. Therefore, a JU-algebra is a generalization of KU-algebras.

Lemma 2.2. *If X is a JU-algebra, then $(X, \leq)$ is a partially ordered set; i.e.,*

(J_4) $x \leq x$,
(J_5) $x \leq y, y \leq x$ *imply* $x = y$,
(J_6) $x \leq z, z \leq y$ *imply* $x \leq y$.

Proof. Putting $y = z = 1$ in (JU_1) we get $x \diamond x = 1$; i.e., $x \leq x$ which proves (J_4). (J_5) follows directly from (JU_3). For (J_6), taking $x \leq z$ and $z \leq y$ imply that $z \diamond x = 1$ and $y \diamond z = 1$. Using (JU_1) we have $y \diamond x = 1$ which implies that $x \leq y$. □

Moreover, we have the following lemma for a JU-algebra X.

Lemma 2.3. *If X is a JU-algebra, then the following inequalities hold for any $x, y, z \in X$:*
(J_7) $x \leq y$ *implies* $y \diamond z \leq x \diamond z$,
(J_8) $x \leq y$ *implies* $z \diamond x \leq z \diamond y$,
(J_9) $(z \diamond x) \diamond (y \diamond x) \leq y \diamond z$,
(J_{10}) $(y \diamond x) \diamond x \leq y$,

Proof. (J_7), (J_8) and (J_9) follow from (JU_1) by adequate replacement of elements. (J_{10}) follows from (JU_1) and (JU_2). □

Now we have the following lemmas.

Lemma 2.4. *Any JU-algebra X satisfies the following conditions for any $x, y, z \in X$,*
(J_{11}) $x \diamond x = 1$,
(J_{12}) $z \diamond (y \diamond x) = y \diamond (z \diamond x)$,
(J_{13}) *If* $(x \diamond y) \diamond y = 1$, *then X is a KU-algebra,*
(J_{14}) $(y \diamond x) \diamond 1 = (y \diamond 1) \diamond (x \diamond 1)$.

Proof. Putting $y = z = 1$ in JU_1, we get $x \diamond x = 1$ which proves (J_{11}). For (J_{12}), we have $(z \diamond x) \diamond x \leq z$ by putting $y = 1$ in (JU_1) and now using (J_7) we get

$$z \diamond (y \diamond x) \leq ((z \diamond x) \diamond x) \diamond (y \diamond x). \tag{2.1}$$

Replacing z with $z \diamond x$ in (JU_1) we get $[y \diamond (z \diamond x)] \diamond [((z \diamond x) \diamond x) \diamond (y \diamond x)] = 1$ which implies

$$((z \diamond x) \diamond x) \diamond (y \diamond x) \leq y \diamond (z \diamond x). \tag{2.2}$$

From (2.1), (2.2) and Lemma 2.2 (J_6) we get

$$z \diamond (y \diamond x) \leq y \diamond (z \diamond x). \tag{2.3}$$

In addition, by replacing y with z and z with y in (2.3), we get

$$y \diamond (z \diamond x) \leq z \diamond (y \diamond x). \tag{2.4}$$

Now (2.3), (2.4) and (J_5) yield $z \diamond (y \diamond x) = y \diamond (z \diamond x)$.
In order to prove (J_{13}) we just need to show that $x \diamond 1 = 1, \quad \forall \quad x \in X$. Substituting $y = 1, x = 1, z = x$ in (JU_1), we obtain, $(1 \diamond x) \diamond [(x \diamond 1) \diamond (1 \diamond 1)] = 1 \Rightarrow x \diamond [(x \diamond 1) \diamond 1] = 1 \Rightarrow x \diamond 1 = 1$ (by using $y = 1$ in the given condition of (J_{13})).
Using (J_{12}) for any $x, y \in X$ we see that
$(y \diamond 1) \diamond (x \diamond 1) = (y \diamond 1) \diamond [x \diamond [(y \diamond x) \diamond (y \diamond x)] = (y \diamond 1) \diamond [(y \diamond x) \diamond (x \diamond (y \diamond x))]$
$= (y \diamond x) \diamond [(y \diamond 1) \diamond (y \diamond (x \diamond x))] = (y \diamond x) \diamond [(y \diamond 1) \diamond (y \diamond 1)] = (y \diamond x) \diamond 1$
which shows that (J_{14}) holds. □

Example 1. *[9] Let $X = \{1, 2, 3, 4, 5\}$ in which $\diamond$ is defined by the following table*

$\diamond$	1	2	3	4	5
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

It is easy to see that X is a JU-algebra.

The following example shows that a JU algebra may not be a KU-algebra

Example 2. *[4] Let $X = \{1, 2, 3, 4\}$ in which $\diamond$ is defined by the following table*

$\diamond$	1	2	3	4
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

It is easy to see that X is a JU-algebra but not a KU-algebra whereas the table given below is both a KU-algebra and a JU-algebra with another operation $\diamond'$:

Example 3. *[4] Let $X = \{1, 2, 3, 4\}$ in which $\diamond'$ is defined by the following table*

$\diamond'$	1	2	3	4
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

Definition 2.5. *A non-empty subset J of X is called a JU-subalgebra of X if $y \diamond x \in J$ for all $x, y \in J$. The set $P_X := \{x \in X | (x \diamond 1) \diamond 1 = x\}$ is called the p-semisimple part of X. A JU-algebra X is called a p-semisimple JU-algebra if $(x \diamond 1) \diamond 1 = x$ for all $x \in X$. The element j of X is called a minimal element if $x \leq j$ implies $x = j$ for all $x \in X$. For any minimal element $j \in X$, the branch of j is defined by $K(j) := \{x \in X | x \geq j\}$. The set $B_X = \{x \in X | x \diamond 1 = 1\}$ is called the JU-part of X.*

Definition 2.6. *A non-empty subset J of a JU-algebra X is called a JU-ideal of X if*

1. $1 \in J$,
2. *for all $x, y \in X$, $x, x \diamond y$ imply $y \in J$.*

Definition 2.7. *A subset J of a JU-Algebra X is called a p-ideal of X if $1 \in J$ and $y, (z \diamond y) \diamond (z \diamond x) \in J$ imply $x \in J$ for any $x, y, z \in X$.*

Definition 2.8. *An ideal J of X is called strong if $x \in J$ and $y \notin J$ imply $y \diamond x \notin J$ for any $x, y \in X$.*

Example 4. *[11] Let $X = \{1, 2, 3, 4, 5, 6\}$ in which $\diamond$ is defined by the fol-*

lowing table:

$\diamond$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	3	5	6
3	1	1	1	2	5	6
4	1	1	1	1	5	6
5	5	5	5	5	1	1
6	1	1	2	1	1	1

Clearly $(X, \diamond, 1)$ *is a JU-algebra. It is easy to show that* $A = \{1, 2\}$ *and* $B = \{1, 2, 3, 4, 5\}$ *are JU-ideals of* X.

Every ideal J of X determines a congruence $\sim$ on X in the sense that $x \sim y$ if and only if $x \diamond y$ and $y \diamond x \in J$ for any $x, y \in X$. The symbol X/J will be used instead of the quotient algebra $X/\sim$, which is still a JU-algebra.

In a JU-algebra X, an ideal, is not necessarily subalgebra of X. If J is both a subalgebra and an ideal of X, then it is called closed ideal of X. Let J be a subset of X. Then the least ideal of X containing J is called the generated ideal of X by J and it is denoted by $\langle J \rangle$.
A mapping $f : X \to X'$ of a JU-algebras $(X, \diamond, 1)$ into a JU-algebra $(X', \diamond', 1')$ is called a JU-homomorphism if $f(x \diamond y) = f(x) \diamond' f(y)$ for all $x, y \in X$. Clearly, $f(1) = 1'$. Every ideal A of X determines a congruence $\sim$ on X in the sense that $x \sim y$ if and only if $x \diamond y$ and $y \diamond x \in J$ for any $x, y \in X$. X/A stands for quotient algebra of X in stead of $X/\sim$, which is a JU-algebra.

A mapping $f : E \to E$ is said to be a closure operator on an ordered set $(E, \diamond)$ if it satisfies the following properties
i. $x \leq f(x)$ (extensivity),
ii. $x \diamond y \Rightarrow f(x) \diamond f(y)$ (isotony),
iii. $f(f(x)) = f(x)$ (idempotence).

3 P-Closure Ideals

We start by identifying p-closure for a non-empty subset J of a KU-algebra X.

Definition 3.1. *Let J be a non-empty subset of X. The p-closure of J is defined by $J_{pc} := \{x \in X | x \diamond j \in J \text{ for some } j \in J\}$.*

Lemma 3.2. *If J is a subset of X containing 1, then $J \subseteq J_{pc}$.*

Proof. For any $j \in J$, it follows from $j \diamond j = 1 \in J$ that $j \in J_{pc}$ and consequently $J \subseteq J_{pc}$. □

In the following example, we show that the condition 1 belongs to J is necessary in Lemma 3.2 and cannot be dropped.

Example 5. *Let $(X = \{0, 1, j\}, \diamond, 1)$ be a JU-algebra in which the operation $\diamond$ is defined by the following table*

$\diamond$	1	0	j
1	1	0	j
0	1	1	j
j	j	j	1

For $J = \{j\}$, clearly $J_{pc} = \{0, 1\}$.

Lemma 3.3. *For any non-empty subsets J_{11} and J_{12} of X, if $J_{11} \subseteq J_{12}$, then $(J_{11})_{pc} \subseteq (J_{12})_{pc}$.*

Proposition 3.4. *Let $x, y \in X$. Then*
i. $((y \diamond x) \diamond x)^n \diamond x = y^n \diamond x$ *for any* $n \in N$;
ii. $(x^n \diamond 1) \diamond 1 = (x \diamond 1)^n \diamond 1$ *for any* $n \in N$.

Proof. i. We use induction to prove the two assertions. It is true for $n = 0$. Assume that the equality is valid for $n = k$; that is, $((y \diamond x) \diamond x)^k \diamond x = y^k \diamond x$ for any $k \in N$. Then $((y \diamond x) \diamond x)^{k+1} \diamond x = ((y \diamond x) \diamond x) \diamond (((y \diamond x) \diamond x)^k \diamond x) = ((y \diamond x) \diamond x) \diamond (y^k \diamond x) = y^k \diamond (((y \diamond x) \diamond x) \diamond x) = y^{k+1} \diamond x$.
So, the equality holds for $n = k + 1$ also. Therefore $((y \diamond x) \diamond x)^n \diamond x = y^n \diamond x$ for any $n \in N$;
ii. Without loss of generality, consider $n \geq 1$. By applying the left distributive law we get
$(x^n \diamond 1) \diamond 1 = (x \diamond (x^{n-1} \diamond 1)) \diamond 1 = (x \diamond 1) \diamond ((x^{n-1} \diamond 1) \diamond 1 = (x \diamond 1)^2 \diamond ((x^{n-2} \diamond 1) \diamond 1 = \ldots = (x \diamond 1)^n \diamond 1$. □

Definition 3.5. *Let $x \in X$. If $x \diamond 1 = 1$ (that is, $x \geq 1$), then the element x is called a positive element of X. Therefore the fixed element defined earlier is a positive element. The positive element in Example 5 is* $0, 1$.

Proposition 3.6. *Let X be a JU-algebra. Then $(((x \diamond 1) \diamond 1) \diamond x)$ is a positive element of X for every $x \in X$.*

Proof. Using the left distributive law, we have $(((x \diamond 1) \diamond 1) \diamond x) = (((x \diamond 1) \diamond 1) \diamond 1) \diamond (x \diamond 1) = (x \diamond 1) \diamond (x \diamond 1) = 1$. Then $((x \diamond 1) \diamond 1) \diamond x$ is positive. □

Definition 3.7. *Minimal elements are an important part of all types of algebras including classical and logical. So we define them here. An element j in a JU-algebra X is called minimal if $j \diamond x = 1$ (i.e., $j \leq x$) implies $x = j$ for any $x \in X$. j is called least element of X if $x \diamond j = 1$ (i.e., $x \leq j$) for all $x \in X$. Similarly, it is called a maximal element if $x \diamond j = 1$ (i.e., $j \leq x$) implies that $j = x$ for any $x \in X$.*

We have the following proposition.

Proposition 3.8. *Consider $j \in X$. Then the following conditions are equivalent*

1. *j is minimal;*
2. $(j \diamond 1) \diamond 1 = j$;
3. *there is $x \in X$ such that $j = x \diamond 1$.*

Proof. (1) ⇒ (2). By Definition 2.1 we have $j \diamond ((j \diamond 1) \diamond 1) = 1$. Since j is minimal, $(j \diamond 1) \diamond 1 = j$.
(2) ⇒ (3). By hypothesis, $j = (j \diamond 1) \diamond 1 = x \diamond 1$ where $x = j \diamond 1$.
(3) ⇒ (1). Suppose that $j = x \diamond 1$ for some $x \in X$. For every $y \in X$, if $j \diamond y = 0$, then $(x \diamond 1) \diamond y = 0$. We have $y \diamond j = y \diamond (x \diamond 1) = y \diamond (((x \diamond 1) \diamond 1) \diamond 1) = ((x \diamond 1) \diamond 1) \diamond (y \diamond 1)$. Using the left distributive law, we also have $((x \diamond 1) \diamond 1) \diamond (y \diamond 1) = ((x \diamond 1) \diamond y) \diamond 1 = 1 \diamond 1 = 1$. Hence $y \diamond j = 1$. In addition, $j \diamond y = 1$. It follows that $y = j$ by JU-3. Thus j is a minimal element of X. □

In the following theorem, we give a characterization of the minimal elements of X.

Theorem 3.9. *An element j of X is minimal if and only if $\{j\}_{pc} = B_X$.*

Proof. Let j be a minimal element of X. Assume that $x \in \{j\}_{pc}$. Then $x \diamond j = j$ and so we have $1 = j \diamond (x \diamond j) = x \diamond (j \diamond j) = x \diamond 1$. It follows that $x \in B_X$. Hence $\{j\}_{pc} \subseteq B_X$.
Now, let $x \in B_X$. Then, $x \diamond 1 = 1$ and so by the minimality of j, using Proposition 3.8 we have $j = (j \diamond 1) \diamond 1 = (j \diamond 1) \diamond (x \diamond 1) = (x \diamond (j \diamond 1)) \diamond 1 = x \diamond j$; that is, $x \diamond j = j$ which implies that $x \in j_{pc}$.
Therefore, $B_X \subseteq \{j\}_{pc}$ and so $\{j\}_{pc} = B_X$.
Conversely, assume that $\{j\}_{pc} = B_X$. Let $b \in X$ with $b \leq j$. Then $1 \leq b \diamond j$ and so $b \diamond j \in B_X$. Thus $b \diamond j \in \{j\}_{pc}$ and so $(b \diamond j) \diamond j = j$. It follows that $b \diamond j = (b \diamond [(b \diamond j) \diamond j] = (b \diamond j) \diamond (b \diamond j) = 1$; that is, $j \leq b$. Hence $b = j$ and therefore j is a minimal element of X. □

Proposition 3.10. *If J is an ideal of X and $1 \in J$, then $B_X \subseteq J_{pc}$.*

Proof. Since $1 \in J$ is a minimal element, using Theorem 3.9 and Lemma 3.3, we have $B_X \subseteq J_{pc}$. □

In the following theorem, we give a necessary and sufficient condition for a JU-algebra to be a KU-algebra.

Theorem 3.11. *X is a KU-algebra if and only if $\{1\}_{pc} = X$.*

Proof. Assume that X is a KU-algebra. Then, for any $x \in X, x \diamond 1 = 1$. It follows that $x \in 1_{pc}$ for any $x \in X$. Therefore, $\{1\}_{pc} = X$.
Conversely, let $\{1\}_{pc} = X$. Then, by Theorem 3.9, $X = B_X$. Hence, X is a KU-algebra. □

As a consequence of Lemma 3.3 and Theorem 3.11, we obtain the following result.

Corollary 3.12. *X is a KU-algebra if and only if $J_{pc} = X$ for any subset J of X containing* 1.

Theorem 3.13. *If c is an arbitrary element of X, then $\{J(c)\}_{pc} = B_X$, where $J(c) = \{x \in X | \ x \leq c\}$.*

Proof. Let $x \in B_X$. Then, $x \diamond 1 = 1$ and so $c \diamond (x \diamond c) = x \diamond (c \diamond c) = x \diamond 1 = 1$. This implies that $x \diamond c \leq c$ and so $x \diamond c \in J(c)$. From $c \in J(c)$, we conclude that $x \in \{J(c)\}_{pc}$ and so $B_X \subseteq \{J(c)\}_{pc}$. Now, let $x \in \{J(c)\}_{pc}$. Then there exists $t \in J(c)$ such that $x \diamond t \leq c$; that is, $x \diamond (c \diamond t) = 1$. Thus $c \diamond t = 1$ implies $x \diamond 1 = 1$ and so $x \in B_X$. Hence $\{J(c)\}_{pc} = B_X$. □

Theorem 3.14. *Let X be a JU-algebra. Then*
i. $(P_X)_{pc} = X$,
ii. $(B_X)_{pc} = B_X$.

Proof. i. Let $x \in X$. Since $X = \bigcup_{j \in P_x} K(j)$, we have $x \in K(j)$ for some $j \in P_X$ and so $j \leq x$. Consequently, $x \diamond j = 1 \in P_X$. This implies that $x \in (P_X)_{pc}$. Therefore, $X = (P_X)_{pc}$.
ii. Let $x \in B_X$. Then $x \diamond 1 = 1 \in B_X$ and so $x \in (B_X)_{pc}$. For the reverse inclusion, let $x \in (B_X)_{pc}$. Then, $x \diamond j \in B_X$ for some $j \in B_X$. It follows that $(x \diamond j) \diamond 1 = 1$ and $j \diamond 1 = 1$. Thus by (J_{14}), we obtain $(x \diamond 1) \diamond 1 = (x \diamond 1) \diamond (j \diamond 1) = (x \diamond j) \diamond 1 = 1$. Hence, $1 \leq x \diamond 1$ and so by the minimality of $x \diamond 1$, we get $x \diamond 1 = 1$; that is, $x \in B_X$. Therefore $(B_X)_{pc} = B_X$. □

Theorem 3.15. *Let J be a subset of B_X containing* 1. *Then,* $J_{pc} = B_X$.

Proof. Since $\{1\} \subseteq J \subseteq B_X$, using Lemma 3.3, $\{1\}_{pc} \subseteq J_{pc} \subseteq (B_X)_{pc}$. Thus, by Theorems 3.9 and Theorem 3.14, we obtain $B_X \subseteq J_{pc} \subseteq B_X$, which implies that $J_{pc} = B_X$. □

Note that the condition 1 belongs to J in preceding theorem is necessary:

Example 6. *Let $X = \{1, 0, a\}$ be a JU-algebra as in Example 5.*
Put $J := \{0\}$. Clearly, $J \subseteq B_X$ and $J_{pc} = \{1\}$. Therefore $J_{pc} \neq B_X$.

Lemma 3.16. *Let J be a subalgebra of X. Then,*
i. $x \in J_{pc}$ *implies* $x \diamond 1 \in J$,
ii. $x \in J_{pc}$ *if and only if* $x \diamond 1 \in J_{pc}$.

Proof. i. Let $x \in J_{pc}$. Then $x \diamond j \in J$ for some $j \in J$ and so by the closeness of J, we have $j \diamond (x \diamond j) \in J$. Therefore, by J_{12}, we conclude that $x \diamond 1 \in J$.

ii. Let $x \in J_{pc}$. Then by part i, we have $x \diamond 1 \in J$ and so, by Lemma 3.2, we get $x \diamond 1 \in J_{pc}$. Conversely, let $x \diamond 1 \in J_{pc}$. Then by part i, we have $(x \diamond 1) \diamond 1 \in J$. Since $(x \diamond (x \diamond 1)) \diamond 1 = 1 \in J$, it follows that $x \in J_{pc}$. □

Theorem 3.17. *If J is a subalgebra of X, then J_{pc} is a subalgebra of X containing J.*

Proof. Obviously, $1 \in J$. Then, by Lemma 3.2, $J \subseteq J_{pc}$ and so it remains to show that J_{pc} is a subalgebra of X. Let $x, y \in J_{pc}$. Then, by part i of Lemma 3.16, we have $x \diamond 1$ and $y \diamond 1 \in J$ and so $(y \diamond 1) \diamond (x \diamond 1) \in J$. By (J_{14}), $(y \diamond x) \diamond 1 = (y \diamond 1) \diamond (x \diamond 1) \in J$. Hence, $y \diamond x \in J_{pc}$ and consequently J_{pc} is a subalgebra of X. □

The following theorem gives a characterization of J_{pc} for any subalgebra J of X :

Theorem 3.18. *If J is a subalgebra of X, then*
$J_{pc} = \bigcup_{x \in X, x \diamond 1 \in J} K((x \diamond 1) \diamond 1)$.

Proof. Let $x \in J_{pc}$. Then, by part i of Lemma 3.16, $x \diamond 1 \in J$. Note that $x \in K((x \diamond 1) \diamond 1)$. Therefore $x \in \bigcup_{x \in X, x \diamond 1 \in J} K((x \diamond 1) \diamond 1)$ and so
$J_{pc} \subseteq \bigcup_{x \in X, x \diamond 1 \in J} K((x \diamond 1) \diamond 1)$.
Now, let $y \in \bigcup_{x \in X, 1 \diamond x \in J} K((x \diamond 1) \diamond 1)$. Then $y \in K((x \diamond 1) \diamond 1)$ for some $x \in X$ and $x \diamond 1 \in J$. Thus $((x \diamond 1) \diamond 1) \leq y$ and so, by J_{12} and J_{14}, $y \diamond 1 = ((x \diamond 1) \diamond 1) = x \diamond 1$. Hence, $y \diamond 1 \in J$ and so $y \in J_{pc}$. Therefore, $\bigcup_{x \in X, x \diamond 1 \in J} K((x \diamond 1) \diamond 1) \subseteq J_{pc}$. □

We now establish some important properties of p-closure.

Theorem 3.19. *For any ideals J of X, J_{pc} is a closed ideal of X.*

Proof. Clearly, $1 \in J_{pc}$. Let $x, y \in X$ be such that $x, x \diamond y \in J_{pc}$. Then, $x \diamond j \in J$ and $(x \diamond y) \diamond b \in J$ for some $j, b \in J$. First we show that $(j \diamond 1) \diamond b \in J$. For this, we have $b \diamond ((j \diamond 1) \diamond b) = (j \diamond 1) \diamond (b \diamond b) = (j \diamond 1) \diamond 1 \leq j$.
Thus, since $j, b \in J$, we conclude that $(j \diamond 1) \diamond b \in J$.

Now, we show that $y \in J_{pc}$. For this, we have
$((x \diamond y) \diamond b) \diamond (y \diamond (x \diamond ((j \diamond 1) \diamond b)$
$= y \diamond (((x \diamond y) \diamond b) \diamond ((j \diamond 1 \diamond b)) \quad (by(J_{12}))$
$\leq y \diamond ((j \diamond 1) \diamond x \diamond y)) \quad (by(KU_1))$
$= (j \diamond 1) \diamond (y \diamond (y \diamond x)) \quad (by(J_{12}))$
$= (j \diamond 1) \diamond (x \diamond 1) \quad (by(J_{12}))$
$\leq x \diamond j$ (by (JU-1)).
Thus, since $(x \diamond y) \diamond b \in J$, we get $((j \diamond 1) \diamond b) \diamond y \in J$. Therefore, by the property that $(j \diamond 1) \diamond b \in J$, we get $y \in J_{pc}$ and so J_{pc} is an ideal of X. To prove the closeness of J_{pc}, let $x \in J_{pc}$. Then $x \diamond a \in J$ for some $j \in J$. We get $(x \diamond j) \diamond ((j \diamond x \diamond j)) \leq (j \diamond x) \diamond x \leq j$. Since $j, x \diamond j \in J$, we obtain $(j \diamond x) \diamond j \in J$. Moreover, we have $j \diamond ((j \diamond 1) \diamond (j \diamond x) \diamond j) = (j \diamond 1) \diamond (j \diamond x) \diamond 1 \leq (j \diamond x) \diamond x \leq j$. Thus, by $j \in J$, we get $(x \diamond 1)((j \diamond x) \diamond j) \in J$. Therefore by the property $(j \diamond x) \diamond j \in J$, we conclude that $x \diamond 1 \in J_{pc}$. Hence, by definition of closed ideal, J_{pc} is a closed ideal of X. □

Combining Theorem 3.19, Proposition 3.10, and the definition of p-ideal, we have the following result.

Corollary 3.20. *For any ideal J of X, J_{pc} is a p-ideal.*

The converse of Theorem 3.19 may not be true in general as can be seen in the following example.

Example 7. *Let a be any arbitrary nonzero minimal element of X. Then, by Theorem 3.9, $\{j\}_{pc} = B_X$. It is known that B_X is a closed ideal, but $\{j\}$ is not an ideal of X.*

Theorem 3.21. *For any ideal J of X, $J_{pc} = (J_{pc})_{pc}$.*

Proof. By Lemma 3.2, $J_{pc} \subseteq (J_{pc})_{pc}$. To show the reverse inclusion, let $x \in (J_{pc})_{pc}$. By Theorem 3.19, J_{pc} is a subalgebra of X and so by Lemma 3.16, we get $x \diamond 1 \in J_{pc}$. Applying the closeness of J_{pc}, we obtain $(x \diamond 1) \diamond 1 \in J_{pc}$. Since $(x \diamond 1) \diamond 1) \diamond x \in B_X$, it follows from Proposition 3.10 that $((x \diamond 1) \diamond 1) \diamond x \in J_{pc}$. Now, since $(x \diamond 1) \diamond 1 \in J_{pc}$, we have $x \in J_{pc}$. Therefore, $(J_{pc})_{pc} \subseteq J_{pc}$. □

In the following theorem, we show that the notion of p-closure ideals introduces a closure operator on $(I(X), \subseteq)$, where $I(X)$ denotes the set of all ideals of X.

Theorem 3.22. *For any X, the mapping $pc : I(X) \to I(X)$ defined by $pc(J) = J_{pc}$ for any $A \in I(X)$ is a closure operator on $(I(X), \subseteq)$.*

Proof. Follows immediately from Lemmas 3.2, 3.3 and Theorem 3.21. □

Theorem 3.23. *For every family $\{J_\alpha\}_{\alpha\in I}$ of closed ideals of X, and $(\bigcap_{\alpha\in I} J_\alpha)_{pc} = \bigcap_{\alpha\in I}(J_\alpha)_{pc}$.*

Proof. Obviously, $(\bigcap_{\alpha\in I} J_\alpha)_{pc} \subseteq (J_\alpha)_{pc}$ for every $\alpha \in I$. Thus $(\bigcap_{\alpha\in I} J_\alpha)_{pc} \subseteq \bigcap_{\alpha\in I}(J_\alpha)_{pc}$. Now, let $x \in \bigcap_{\alpha} \in I(J_\alpha)_{pc}$. Then for every $\alpha \in I$, there exists $t_\alpha \in J_\alpha$ such that $x \diamond t_\alpha \in J_\alpha$. Hence, from $(x \diamond t_\alpha) \diamond (x \diamond 1) \leq t_\alpha \diamond 1 \in J_\alpha$, we get $x \diamond 1 \in J_\alpha$ and so $x \diamond 1 \in \bigcap_{\alpha\in I} J_\alpha$. Thus, $x \in (\bigcap_{\alpha\in I} J_\alpha)_{pc}$ and consequently $\bigcap_{\alpha\in I}(J_\alpha)_{pc} \subseteq (\bigcap_{\alpha\in I} J_\alpha)_{pc}$. Therefore $(\bigcap_{\alpha\in I} J_\alpha)_{pc} = \bigcap_{\alpha\in I}(J_\alpha)_{pc}$. □

The closeness condition of ideals in Theorem 3.23 is necessary and cannot be removed.

Example 8. *Suppose that $(\mathbb{Z}, \diamond, 0)$ is the adjoint p-semisimple JU-algebra of integers $(\mathbb{Z}, +)$. Then, $m \diamond n = n - m$ for any $m, n \in \mathbb{Z}$. It is easy to check that $A := \{0, 1, 2, \ldots\}$ and $B := \{0, 1, 2, \ldots\}$ are two ideals of $\mathbb{Z}$, but not closed. Moreover, we can show that $J_{pc} = B_{pc} = \mathbb{Z}$ and $(J \cap B)_{pc} = \{1\}$. Therefore, $(J \cap B)_{pc} \neq J_{pc} \cap B_{pc}$.*

Theorem 3.24. *Let $\{J_\alpha\}_\alpha \in I$ be a family of ideals of X. Then $\langle \bigcup_{\alpha\in I}(J_\alpha)_{pc}\rangle_{pc} = \langle \bigcup_{\alpha\in I} J_\alpha\rangle_{pc}$.*

Proof. Since $J_\alpha \subseteq (J_\alpha)_{pc}$ for every $\alpha \in I$, we have $\bigcup_{\alpha\in I} J_\alpha \subseteq \bigcup_{\alpha\in I}(J_\alpha)_{pc}$. Thus, by Lemma 3.3, we get $\langle \bigcup_{\alpha\in I} J_\alpha\rangle_{pc} \subseteq \langle \bigcup_{\alpha\in I}(J_\alpha)_{pc}\rangle_{pc}$. On the other hand, from $J_\alpha \subseteq \langle \bigcup_{\alpha\in I} J_\alpha\rangle$, we obtain $(J_\alpha)_{pc} \subseteq \langle \bigcup_{\alpha\in I} J_\alpha\rangle_{pc}$ for any $\alpha \in I$. Therefore $\langle \bigcup_{\alpha\in I}(J_\alpha)_{pc}\rangle \subseteq \langle \bigcup_{\alpha\in I}(J_\alpha)\rangle_{pc}$. and so by Lemma 3.3 and Theorem 3.21, we conclude that

$\langle \bigcup_{\alpha\in I}(J_\alpha)_{pc}\rangle_{pc} = (\langle \bigcup_{\alpha\in I} J_\alpha\rangle_{pc})_{pc} = \langle \bigcup_{\alpha\in I} J_\alpha\rangle_{pc}$.
This completes the proof. □

Definition 3.25. *For any non-empty subset J of X, we denote $J^1 := \{x \diamond 1 | x \in J\}$.*

Theorem 3.26. *For any nonempty subset J of X, the ideal $\langle J \cup J^1\rangle$ is the least closed ideal containing J.*

Lemma 3.27. *For any ideals J of X, the following hold:*
i. $J^1 \subseteq J_{pc}$,
ii. $\langle J \cup J^1\rangle_{pc} = J_{pc}$.

Proof. i. For any $x \in J$, since $(x \diamond 1) \diamond 1 \leq x$, we have $(x \diamond 1) \diamond 1 \in J$. This implies that $x \diamond 1 \in J_{pc}$ and so $J^1 \in J_{pc}$.
ii. By part i and Lemma 3.2, we have $J, J^1 \subseteq J_{pc}$. Since J_{pc} is an ideal, we obtain $J \subseteq \langle J \cup J^1\rangle \subseteq J_{pc}$ and so by Lemma 3.3, we get $J_{pc} \subseteq \langle J \cup J^1\rangle_{pc} \subseteq (J_{pc})_{pc}$. Thus, by Theorem 3.21, we conclude that $\langle J \cup J^1\rangle_{pc} = J_{pc}$. □

Theorem 3.28. *For any ideal J of X, J_{pc} is the least closed p-ideal containing J.*

Proof. Combining Lemma 3.2, Theorem 3.19 and Corollary 3.20, we conclude that J_{pc} is a closed p-ideal of X containing J. Let C be another closed p-ideal of X containing J and let $x \in J_{pc}$. Since J_{pc} is a closed ideal of X, we have $x\diamond 1 \in J_{pc}$. Thus $x\diamond 1 \in C_{pc}$ and so by Lemma 3.16, we get $(x\diamond 1)\diamond 1 \in C$. Note that $(x\diamond 1)\diamond 1)\diamond x \in B_X$ and so from $B_X \subseteq C$, we obtain $((x\diamond 1)\diamond 1)\diamond x \in C$. Since $(x \diamond 1) \diamond 1 \in C$, we conclude $x \in C$. Therefore, $J_{pc} \in C$. Hence, J_{pc} is the least closed p-ideal containing J. □

Theorem 3.29. *An ideal J of X is closed if and only if $(J^1)_{pc} = J_{pc}$.*

Proof. Suppose that an ideal J of X is closed. Then, $x \diamond 1 \in J$ for any $x \in J$ and so $J \in J^1$. Clearly, $J^1 \in J$. Thus, $J^1 = J$ and so $(J^1)_{pc} = J_{pc}$.
Conversely, assume that $(J^1)_{pc} = J_{pc}$ and $x \in J$. Then by the closeness of $J_{pc}, x \diamond 1 \in J_{pc}$ and so by assumption, $x \diamond 1 \in (J^1)_{pc}$. Thus, there exists $j \in J^1$

such that $(x \diamond 1) \diamond j \in J^1$. It follows that $((x \diamond 1) \diamond j) \diamond 1 \in J$. Now, we have
$j \diamond (x \diamond 1) = j \diamond (((x \diamond 1) \diamond 1) \diamond 1) \quad (by(J_{14}))$
$= ((x \diamond 1) \diamond 1) \diamond (j \diamond 1) \quad (by(J_{12}))$
$= ((x \diamond 1) \diamond j) \diamond 1 \in J \quad (by(J_{14}))$
Therefore, $j \diamond (x \diamond 1) \in J$ and so by $j \in J^1 \subseteq J$, we conclude $x \diamond 1 \in J$. Hence, J is closed. □

4 JU-homomorphisms under P-closure

In this section, we consider the relation between $f(J_{pc})$ and $(f(J))_{pc}$ in which $f : X \to Y$ is a JU-homomorphism and J is a non-empty subset of X. It is well known that if $f : X \to Y$ is a JU-epimorphism and J is a closed ideal of X, then $f(J)$ is a closed ideal of Y. From this and Theorem 3.19, we have the following result.

Corollary 4.1. *If $f : X \to Y$ is a JU-epimorphism, then $f(J_{pc})$ is a closed ideal of Y for any ideal J of X.*

Theorem 4.2. *Suppose that $f : X \to Y$ is a JU-homomorphism and Y is a p-semisimple JU-algebra. If J is an ideal of X, then $f(J_{pc})$ is a closed ideal of Y.*

Proof. Clearly, $1 \in f(J_{pc})$. Let $x, y \in Y$ be such that $x, x \diamond y \in f(J_{pc})$. Then, $x = f(s)$ and $x \diamond y = f(t)$ for some $s, t \in J_{pc}$. Since J_{pc} is a closed ideal of X, we obtain $(s \diamond 1) \diamond t \in J_{pc}$. Moreover, we have $f((s \diamond 1) \diamond t) = (f(s) \diamond (f(1))) \diamond f(t) = (x \diamond 1) \diamond (x \diamond y) \leq 1 \diamond y = y$. Thus, since Y is a p-semisimple JU-algebra, we conclude that $f((s \diamond 1) \diamond t) = y$. Therefore, $y \in f(J_{pc})$ and consequently $f(J_{pc})$ is an ideal of Y. To prove the closeness of $f(J_{pc})$, let $y \in f(J_{pc})$. Then, there exists $b \in J_{pc}$ such that $y = f(b)$. Thus, $f(b \diamond 1) = f(b) \diamond f(1) = y \diamond 1$ and so, since $b \diamond 1 \in J_{pc}$, we get $y \diamond 1 \in f(J_{pc})$. Hence, $f(J_{pc})$ is a closed ideal of Y. □

The converse of Theorem 4.2 may not be true in general as the following example shows.

Example 9. *Let* $(X = \{1, 0, j\}, \diamond, 1)$ *be a JU-algebra and* $(X' = \{1', b\}, \diamond' 1')$ *be a p-semisimple JU-algebra in which the operations* $\diamond$ *and* $\diamond'$ *are given by the following tables:*

$\diamond$	1	0	j
1	1	0	j
0	1	j	j
j	j	j	1

$\diamond'$	$1'$	b
$1'$	$1'$	b
b	b	$1'$

Define the mapping $f : X \to X'$ by $f(1) = f(0) = 1'$ and $f(j) = b$. Clearly, f is a JU-epimorphism. Putting $J := \{1, j\}$, we get $f(J_{pc}) = X$. But J is not an ideal of X since $j \diamond 0 = j \in J$ and $0 \notin J$.

Lemma 4.3. *If* $f : X \to Y$ *is a JU-homomorphism, then* $f(J_{pc}) \subseteq (f(J))_{pc}$ *for any non-empty subset* A *of* X.

Proof. Let $f(x) \subseteq f(J_{pc})$. Then $x \in J_{pc}$ and so $x \diamond j \in J$ for some $j \in J$. Thus, $f(x) \diamond f(j) \in f(J)$, which implies that $f(x) \in (f(J))_{pc}$. Therefore, $f(J_{pc}) \subseteq (f(J))_{pc}$. □

Theorem 4.4. *Let* $f : X \to Y$ *be a JU-epimorphism and* J *be an ideal of* X. *If* $ker f \subseteq J$, *then* $f(J_{pc}) = (f(J))_{pc}$.

Proof. By Lemma 4.3, it suffices to show that $(f(J))_{pc} \subseteq f((J_{pc}))$. Let $y \in (f(J))_{pc}$. Then, $y \diamond f(j) \in f(J)$ for some $j \in J$. Since f is epimorphic, $y = f(x)$ for some $x \in X$. Thus, there exists $b \in J$ such that $f(x \diamond j) = f(b)$ and so $f(b \diamond (x \diamond j)) = f(b) \diamond f(x \diamond j) = 1$. It follows that $b \diamond (x \diamond j) \in ker f \subseteq J$. Since $b \in J$, we have $x \diamond j \in J$. Hence, $x \in J_{pc}$ and so $y = f(x) \in f(J_{pc})$. Therefore, $(f(J))_{pc} \subseteq f(J_{pc})$. This completes the proof. □

Theorem 4.5. *Suppose that* $f : X \to Y$ *is a JU-homomorphism and* Y *is a p-semisimple JU-algebra. If* J *is a closed ideal of* X, *then* $f(J_{pc}) = (f(J))_{pc}$.

Proof. By Lemma 4.3 it suffices to show that $(f(J))_{pc} \subseteq f(J_{pc})$. Let $y \in (f(J))_{pc}$. Then, $y \diamond f(j) \in f(J)$ for some $j \in J$ and so there exists $b \in J$ such that $y \diamond f(j) = f(b)$. Hence, $f(j) \diamond f(b) = f(j) \diamond [y \diamond f(j)] = y \diamond [f(j) \diamond f(j)] = y \diamond 1$. Now, since Y is a p-semisimple JU-algebra, we get $y = (y \diamond 1) \diamond 1 = (f(j) \diamond f(b)) \diamond 1$, that is, $y = f((j \diamond b) \diamond 1)$. By the closeness of J, we have $(j \diamond b) \diamond 1 \subseteq J \subseteq J_{pc}$, and consequently $y \in f(J_{pc})$. Therefore, $(f(J))_{pc} \subseteq f(J_{pc})$, which completes the proof. □

The converse of Theorem 4.5 may not be true in general.

Example 10. *Let $f : X \to X$ be a JU-homomorphism and $J := \{1, j\}$ as in Example 9. It is easy to see that $f(J_{pc}) = (f(J))_{pc} = X$. But A is not an ideal of X.*

Proposition 4.6. *Let $f : X \to Y$ be a JU-homomorphism and C be an (subalgebra) ideal of Y. Then $f^{-1}(C_{pc})$ is an ideal of X containing $(f^{-1}(C))_{pc}$.*

Proof. We only show that $f^{-1}(C_{pc})$ is an ideal of X containing $(f^{-1}(C))_{pc}$. That $f^{-1}(C)$ is a subalgebra can be proved in similar manner. Since C is an ideal of Y, it follows that $f \diamond 1(C_{pc})$ is an ideal of X. Now let $x \in (f \diamond 1(C))_{pc}$. Then $x \diamond j \in f \diamond 1(C)$ for some $j \in f \diamond 1(C)$. This implies that $f(j) \in C$. Also, $f(x) \diamond f(j) = f(x \diamond j) \in C$ and consequently $f(x) \in C_{pc}$. Therefore $x \in f \diamond 1(C_{pc})$ and so $(f \in 1(C))_{pc} \in f \diamond 1(C_{pc})$. This completes the proof. □

Theorem 4.7. *Let $f : X \to Y$ be an isomorphism and J be an ideal of X. Then $J \in B_X$ if and only if $f(J_{pc}) = B_Y$.*

Proof. Let $A \in B_X$. By Theorem 3.15, we have $J_{pc} = B_X$. Thus, $f(J_{pc}) = \{f(x)|x \in B_X\} = \{f(x)|1 \leq x\} \subseteq \{f(x)|1' \leq f(x)\} \subseteq B_Y$. On the other hand, let $y \in B_Y$. Therefore, $1' \leq y$ and there exists $x \in X$ such that $f(x) = y$ and hence $1' \leq f(x)$. Thus, $f(x \diamond 1) = f(x) \diamond f(1) = f(1)$. Now, since f is monomorphic, we have $x \diamond 1 = 1 \in J$. It follows that $x \in J_{pc}$ and consequently $y = f(x) \in f(J_{pc})$. Therefore, $f(J_{pc}) = B_Y$.

Conversely, let $f(J_{pc}) = B_Y$ and $j \in J$. By Lemma 3.2, $j \in J_{pc}$ and so $f(j) \in B_Y$. Thus $f(j \diamond 1) = f(j) \diamond f(1) = 1' \diamond f(j) = 1' = f(1)$. Hence, $j \diamond 1 = 1$ and so $j \in B_X$. Therefore, $J \subseteq B_X$. □

Note that the condition f is bijective in the preceding theorem is necessary:

Example 11. *Let $X = \{1, 0, j\}$ and $X' = \{1', b\}$ be two JU-algebras as in Example 9. Define the mapping $f : X' \to X$ by $f(1') = 1$ and $f(b) = j$. It is routine to check that f is a JU-monomorphism which is not surjective. Taking $J := \{1'\}$, it is easy to see that $J \subseteq B'_X$ but $f(J_{pc}) = \{1\} \neq \{0, 1\} = B_X$.*

Example 12. *Let $X = \{0, 1, j\}$ be a JU-algebra as in Example 9. Consider the identity mapping $f : X \to X$. Taking $J := \{1\}$, it can be checked that $f(J_{pc}) = B_X$ and $B_X \nsubseteq J$.*

Proposition 4.8. *Let J and I be two ideals of X with $J \subseteq I$. Then, $(I/J)_{pc} = I_{pc}/J$.*

Proof. Note that $J \subseteq I \subseteq I_{pc}$, so I_{pc}/J is well-defined. Now, we have $(I/J)_{pc} = \{J_y \in X/J | J_y \diamond J_x \in I/J \text{ for some } J_x \in I/J\} = \{J_y \in X/J | J_y \diamond x \in I/J \text{ for some } x \in I\} = \{J_y \in X/J | y \diamond x \in I \text{ for some } x \in I\} = \{J_y \in X/J \mid y \in I_{pc}\} = I_{pc}/J$. □

Theorem 4.9. *Let J and B be two ideals of X and Y, respectively. Then*
i. $J_{pc} \times B_{pc} = (J \times B)_{pc}$,
ii. $(X/J_{pc}) \times (Y/B_{pc}) \simeq (X \times Y)/(J_{pc} \times B_{pc})$.

Proof. i. Clearly, $(J \times B)_{pc} = \{(x, y) \in X \times Y | (y, x) \diamond (v, u) \in J \times B \text{ for some } u \in J \text{ and } v \in B\} = \{(x, y) \in X \times Y | (x \diamond u, y \diamond v) \in J \times B \text{ for some } u \in J \text{ and } v \in B\} = \{(x, y) \in X \times Y | x \diamond u \in J, y \diamond v \in B \text{ for some } u \in J \text{ and } v \in B\} = \{x \in X | x \diamond u \in J \text{ for some } u \in J\} \times \{y \in Y | y \diamond v \in B \text{ for some } v \in B\} = J_{pc} \times B_{pc}$.
ii. We know that $J_{pc} \times B_{pc}$ is a closed ideal of $X \times Y$. Now, consider the natural homomorphisms $\pi_X : X \to X/J_{pc}$ and $\pi_Y : Y \to Y/B_{pc}$ with $\pi_X(x) = (J_{pc})_x$ and $\pi_Y(y) = (B_{pc})_y$. Define the mapping $f : X \times Y \to X/J_{pc} \times Y/B_{pc}$ by $f(x, y) = (\pi_X(x), \pi_Y(y)) = ((J_{pc})_x, (B_{pc})_y)$.
Clearly, f is a JU-epimorphism. Moreover, $ker f = \{(x, y) \in X \times Y | f(x, y) = (J_{pc}, B_{pc})\} = \{(x, y) \in X \times Y | (J_{pc})_x = J_{pc} \text{ and } (B_{pc})_y = B_{pc}\} = \{(x, y) \in X \times Y | x \in J_{pc}\} \text{ and } y \in B_{pc}\} = J_{pc} \times B_{pc}$. Therefore, by the first isomorphism theorem, we get $(X \times Y)/(J_{pc} \times B_{pc}) \simeq (X/J_{pc}) \times (Y/B_{pc})$. □

5 Conclusion and future work

In this paper, we have introduced the concept of p-closure for any non-empty subset J of a JU-algebra X and have investigated some related properties showing that J_{pc} is the least closed p-ideal containing J for any ideal J of X. Our future work is to study the concept of p-closure on other logical algebras and investigate fuzzification, roughness, soft sets and other related work based on JU-algebras.

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