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Riccati-Bernoulli Sub-ODE approach on the partial differential equations and applications

A. R. Alharbi, M. B. Almatrafi

Department of Mathematics
College of Science
Taibah University
Al-Madinah Al-Munawarah, Saudi Arabia

email: arharbi@taibahu.edu.sa, mmutrafi@taibahu.edu.sa

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Abstract

Essentially, this article aims to implement the Riccati-Bernoulli Sub-ODE approach on four applications played a significant role in mathematical physics. This method is utilized to determine new traveling wave solutions for the thin film equation, the dispersive long wave equation (DLWE), the modified KdV-KP equation and the nonlinear ZK-MEW equation. The exact traveling wave solutions for the considered equations are obtained by utilizing this method and expressed in terms of trigonometric functions, hyperbolic functions and rational functions. We also compare the obtained results with some results obtained by using the first integral method. 2D and 3D figures are illustrated under an appropriate selection of parameters. The applied technique is suitable to be used in gaining new exact solutions for most nonlinear partial differential equations appeared in natural phenomena

1 Introduction

Currently, the investigation of the traveling wave solutions of a numerous number of nonlinear evolution equations plays a considerable role in different

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aspects of mathematical and physical phenomena. Most natural phenomena arising in applied science, such as nuclear physics, chemical reactions, optical fibres, fluid mechanics, plasma, physics and ecology, can be sometimes modelled and described by nonlinear partial differential equations (NPDEs) [1, 2, 3, 4, 5]. Constructing the solitary wave solutions of these equations has become a global interest in recent years. Hence, a massive number of mathematical experts has attempted to invent various approaches by which one can obtain the exact solutions of such equations. Nowadays, some new effective techniques have been developed and established. Among several existing methods, we mention the Riccati-Bernoulli sub-ODE method [6], optimal homotopy asymptotic method [7, 8, 9, 10], exp-function method [12, 13], sine-cosine method [14, 15], tanh-sech method [16, 17], extended tanh-method [18, 19], F-expansion method [20, 21], homogeneous balance method [22, 23], Jacobi elliptic function method [24, 25], $(\frac{G'}{G})-$ expansion method [26, 27] and several others.

A massive number of NLPDEs can be simply solved by the above-mentioned methods. However, there is no a specific approach by which we can deal with all NLPDEs. In addition, some fractional NLPDEs cannot be easily solved by most traditional methods. The proposed method, which allows us to execute tedious and sophisticated algebraic calculations, is utilized to establish solitary wave solutions, peaked wave solutions and exact wave solutions for NLPDEs [6]. Once a traveling wave transformation and the Riccati-Bernoulli equation are applied, the proposed NLPDEs are simply converted into a system of algebraic equations. Applying this technique also leads to a useful mathematical tool for solving some applications in mathematical physics. The performance of this technique is obtained powerful and efficient. This work is established to utilize the Riccati-Bernoulli sub-ODE technique in determining the exact solution of the thin film equation. This method which produces infinite solutions gives reliable and efficient results for the considered equation. To be specific, the most crucial character of this approach is that it introduces infinite solutions for the NPDEs. Bäcklund transformation is also used to show new infinite sequence of the solution. Moreover, this paper aims to apply the considered techniques for solving three more applications, namely dispersive long wave equation, the modified KdV-KP equation and the nonlinear ZK-MEW equation. Verily, the obtained results powerfully contribute in interpreting some practical physical problems and other sophisticated ones.

In brief, this paragraph puts forward the organization of this article. In

Section 2, the explanation of the Riccati-Bernoulli Sub-ODE method is obviously presented. Following this, we introduce the Bäcklund transformation. Section 3 concerns with discussing the wave solutions of the considered four applications. In Subsection 3.1, we thoroughly investigate the exact solution of the thin film equation. The Riccati-Bernoulli Sub-ODE technique is applied to solve the DLWE, the modified KdV-KP equation and the nonlinear ZK-MEW equation in Subsections 3.2, 3.3 and 3.4, respectively. Finally, Section 5 highlights the most important results shown in this work.

2 The Riccati-Bernoulli Sub-ODE method

The Riccati-Bernoulli Sub-ODE technique [6] is concisely explained in this section to construct traveling wave solutions of nonlinear partial differential equations (NPDEs). We first consider a given NPDEs in two independent variables x and t on the form

$$R(\Omega, \Omega_t, \Omega_x, \Omega_{tt}, \Omega_{xx}, ...) = 0, \tag{2.1}$$

where $\Omega(x,t)$ is assumed to be a solution of (2.1). The authors in [6] have listed various steps of this method described as follows:

- Introduce a new transformation on the form

$$\Omega(x,t) = q(\eta), \qquad \eta = \frac{x - w\,t}{\sqrt[3]{Ca}}. \tag{2.2}$$

  This transformation is useful in converting (2.1) into the following ordinary differential equation (ODE):

$$Q(q, q', q'', q''', ...) = 0, \tag{2.3}$$

  where the appeared derivatives of q are with respect of η. Now, Q is a polynomial in $q(\eta)$.

- Let

$$q' = aq^{2-n} + bq + cq^n, \tag{2.4}$$

  be a solution of (2.3), where the parameters a, b, c and n are obtained later.

- Respectively, Take the first and second derivatives of (2.4) to have

$$q'' = ab(3-n)q^{2-n} + a^2(2-n)q^{3-2n} + nc^2q^{2n-1} + bc(n+1)q^n + (2ac+b^2)q, \tag{2.5}$$

$$q''' = (ab(3-n)(2-n)q^{1-n} + a^2(2-n)(3-2n)q^{2-2n} + n(2n-1)c^2q^{2n-2} + bcn(n+1)q^{n-1} + (2ac+b^2))q'. \tag{2.6}$$

- The exact solution of (2.4) is now given by
  - When $n = 1$, the solution is shown by

$$q(\eta) = \mu e^{(a+b+c)\eta}. \tag{2.7}$$

  - If $n \neq 1$, $b = 0$ and $c = 0$, then the solution is given by

$$q(\eta) = (a(n-1)(\eta+\mu))^{\frac{1}{n-1}}. \tag{2.8}$$

  - If $n \neq 1$, $b \neq 0$ and $c = 0$, then the solution is given by

$$q(\eta) = \left(\frac{-a}{b} + \mu e^{b(n-1)\eta}\right)^{\frac{1}{n-1}}. \tag{2.9}$$

  - If $n \neq 1$, $a \neq 0$ and $b^2 - 4ac < 0$, then the solution is illustrated as follows:

$$q(\eta) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\eta+\mu)\right)\right)^{\frac{1}{1-n}}, \tag{2.10}$$

and

$$q(\eta) = \left(\frac{-b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}\cot\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\eta+\mu)\right)\right)^{\frac{1}{1-n}}. \tag{2.11}$$

  - For $n \neq 1$, $a \neq 0$ and $b^2 - 4ac > 0$, we have

$$q(\eta) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\coth\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\eta+\mu)\right)\right)^{\frac{1}{1-n}}, \tag{2.12}$$

and

$$q(\eta) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\tanh\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\eta+\mu)\right)\right)^{\frac{1}{1-n}}. \tag{2.13}$$

– For $n \neq 1$, $a \neq 0$ and $b^2 - 4ac = 0$, we have

$$q(\eta) = \left(\frac{1}{a(n-1)(\eta+\mu)} - \frac{b}{2a} \right)^{\frac{1}{1-n}}. \tag{2.14}$$

where μ is supposed to be an arbitrary constant.

2.0.1 Bäcklund transformation

Assume that $q_{k-1}(\eta)$ and $q_k(\eta)(q_k(\eta) = q_k(q_{k-1}(\eta)))$ are the solutions of (2.4). This leads to

$$\begin{aligned} \frac{dq_k(\eta)}{d\eta} &= \frac{dq_k(\eta)}{dq_{k-1}(\eta)} \frac{dq_{k-1}(\eta)}{d\eta} \\ &= \frac{dq_k(\eta)}{dq_{k-1}(\eta)} (aq_{k-1}^{2-n} + bq_{k-1} + cq_{k-1}^n), \end{aligned}$$

namely

$$\frac{dq_k(\eta)}{aq_k^{2-n} + bq_k + cq_k^n} = \frac{dq_{k-1}(\eta)}{aq_{k-1}^{2-n} + bq_{k-1} + cq_{k-1}^n}. \tag{2.15}$$

Next, we integrate both sides of (2.15) with respect to η to attain the Bäcklund transformation of (2.4) which is given by

$$q(\eta) = \left(\frac{-cK_1 + aK_2 \left(q_{k-1}(\eta)\right)^{1-n}}{bK_1 + aK_2 + aK_1 \left(q_{k-1}(\eta)\right)^{1-n}} \right)^{\frac{1}{1-n}}. \tag{2.16}$$

The constants K_1 and K_2 are assumed to be arbitrary. Whenever a solution for our equations gained, (2.16) can be utilized to generate infinite solutions of (2.4), as well of (2.1).

3 Results and Discussion

3.1 First application

In this subsection, we study the following thin film equation [28, 29, 30, 31, 32, 33]:

$$h_t + \left(Ca\frac{h^3}{3}\, h_{xxx} - D(\theta)\frac{h^3}{3} h_x + \frac{h^3}{3} \right)_x = 0, \tag{3.1}$$

with the following boundary conditions:

$$\begin{aligned} &h_{xxx}(a,t) = h_x(a,t) = 0, \\ &h(b,t) = p,\ h_{xxx}(b,t) = h_x(b,t) = 0. \end{aligned} \tag{3.2}$$

Here, a, b are supposed to be constants, $a \leq x \leq b$ represents the spatial variable, t is time and $p \in (0,1)$ indicates the precursor film thickness. The analysis of the solution starts by introducing a new transformation on the form

$$h(x,t) = \rho(\eta), \qquad \eta = \frac{x - w\,t}{\sqrt[3]{Ca}}. \tag{3.3}$$

The constant w is called a wave speed. (3.3) converts (3.1) into the following ODE:

$$-w\,\rho_\eta + \left(\frac{\rho^3}{3}\,\rho_{\eta\eta\eta} - \beta\frac{\rho^3}{3}\rho_\eta + \frac{\rho^3}{3}\right)_\eta = 0, \tag{3.4}$$

where $\beta = D(\theta)/\sqrt[3]{Ca}$, and the new boundary conditions are shown as follows:

$$\rho_{\eta\eta\eta} \to 0 \text{ as } \eta \to \pm\infty,\ \text{ and } \rho \to p,\ \text{ as } \eta \to +\infty. \tag{3.5}$$

Taking the integration for both sides of (3.4) and applying the boundary conditions given in (3.5) leads to

$$-3\,w\,\rho + \rho^3\,\rho_{\eta\eta\eta} - \beta\rho^3\rho_\eta + \rho^3 = C_1. \tag{3.6}$$

Applying the boundary conditions, leads $w = (p^2 + p + 1)/3$ and $C_1 = -(p^2 + p)$. Now, the derivatives appeared in (3.6) are replaced by the ones appeared in (2.5) to have

$$\begin{aligned} &u^3(a^2(2-n)(3-2n)u^{2-2n} + \left(2ac + b^2\right)\left(au^{2-n} + bu + cu^n\right) + ab(2-n)(3-n)u^{1-n} \\ &+b\,c\,(n+1)\,nu^{n-1} + c^2(2n-1)n\,u^{2n-2}) - \beta\,u^3\,(au^{2-n} + b\,u + c\,u^n) \\ &-C_1 + u^3 + u\,w = 0. \end{aligned} \tag{3.7}$$

Setting $n = 0$, (3.7) becomes

$$\begin{aligned} &2\,a^2\,c\,u^5 + 6\,a^2u^5 + a\,b^2u^5 + 2\,a\,b\,c\,u^4 + 6\,a\,b\,u^4 \\ &+2\,a\,c^2u^3 - a\,\beta\,u^5 + b^3u^4 + b^2c\,u^3 - b\,\beta\,u^4 - c\,\beta u^3 \\ &-C_1 + u^3 + u\,w = 0. \end{aligned} \tag{3.8}$$

Next, we equate the coefficients of $\hat{\rho}^i$ $(i = 0,1,2,3)$ to zero to attain

$$\begin{aligned} &2ac^2 + b^2c - c\beta + 1 = 0, \\ &2abc + 6ab + b^3 - b\beta = 0, \\ &2a^2c + 6a^2 + ab^2 - a\beta = 0. \end{aligned} \tag{3.9}$$

Hence, the solutions of the above sytem (3.9) can be now expressed as follows:

$$a \to \frac{1}{6\,c}, \qquad b \to \pm\frac{\sqrt{3\,c\,\beta - c - 3}}{\sqrt{3\,c}}. \tag{3.10}$$

Consequently, several cases for the solutions of (3.4) emerge for various parameters. The solution is given by

1. If $b^2 - 4ac < 0$, then the solution is illustrated as follows:

$$q(\eta) = \left(\pm\sqrt{3\,c\,-(3+c)+3\,c\,\beta} + \gamma_1 \tan\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}\Big(\frac{x - w\,t}{\sqrt[3]{Ca}} + \mu\Big)\right)\right)^{\frac{1}{1-n}}, \tag{3.11}$$

   and

$$q(\eta) = \left(\pm\sqrt{3\,c\,-(3+c)+3\,c\,\beta} - \gamma_1 \cot\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}\Big(\frac{x - w\,t}{\sqrt[3]{Ca}} + \mu\Big)\right)\right)^{\frac{1}{1-n}}. \tag{3.12}$$

2. For $b^2 - 4ac > 0$, we have

$$q(\eta) = \left(\pm\sqrt{3\,c\,-(3+c)+3\,c\,\beta} - \gamma_2 \coth\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}\Big(\frac{x - w\,t}{\sqrt[3]{Ca}} + \mu\Big)\right)\right)^{\frac{1}{1-n}} \tag{3.13}$$

   and

$$q(\eta) = \left(\pm\sqrt{3\,c\,-(3+c)+3\,c\,\beta} - \gamma_2 \tanh\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}\Big(\frac{x - w\,t}{\sqrt[3]{Ca}} + \mu\Big)\right)\right)^{\frac{1}{1-n}}, \tag{3.14}$$

where

$$\gamma_1 = 3\,c\,\sqrt{2/3 - (3\,c\,\beta - (3+c))/(3\,c)},$$

$$\gamma_2 = 3\,c\,\sqrt{(3\,c\,\beta - (3+c))/(3\,c) - 2/3}.$$

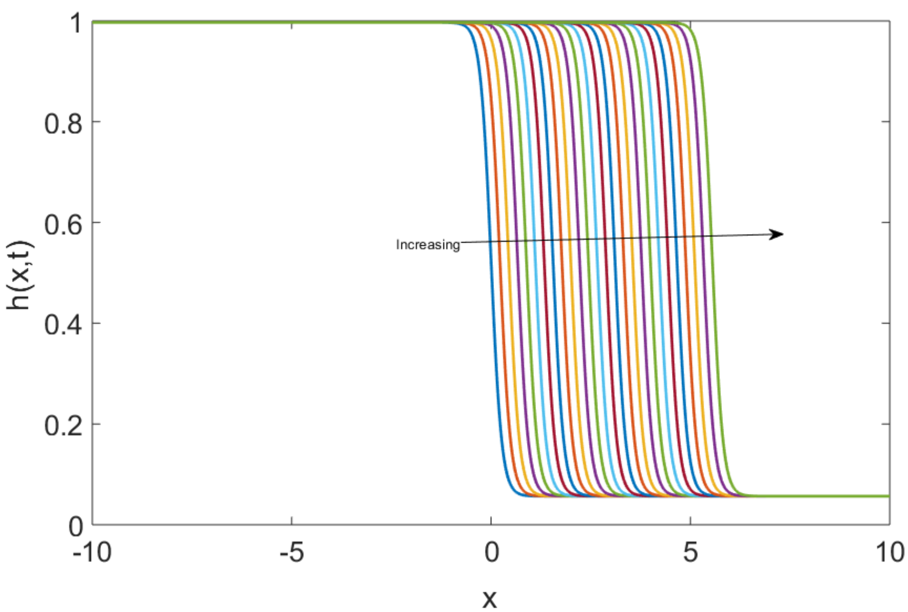

Figure 1: 2D plot for the solution of (3.14) when $Ca = 10^{-2}$, $p = 0.1$, $D(\theta) = 1$ and $c = 0.97$.

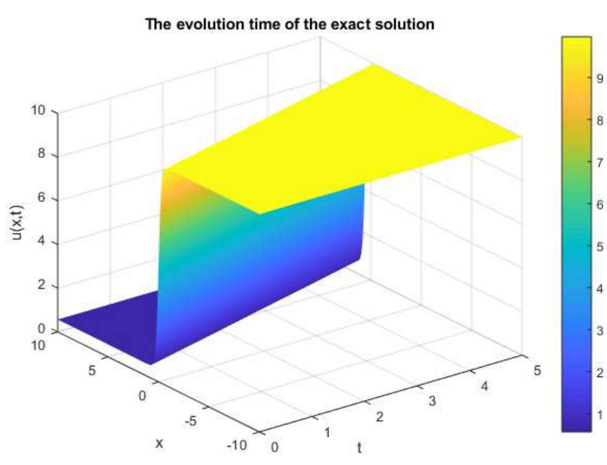

Figure 2: 3D graph for the solution of (3.14) when $Ca = 10^{-2}, p = 0.1$, $D(\theta) = 1$ and $c = 0.97$.

3.2 Second application

Through this subsection, we present the exact solution of DLWE [34, 35, 36] using the considered technique in this paper. The DLWE is given by

$$\begin{aligned} &U_t + U\,U_x + G_x = 0, \\ &G_t + (G\,U)_x + \frac{1}{3}U_{xxx} = 0. \end{aligned} \tag{3.15}$$

We first introduce the transformation

$$U(x,t) = \rho(\eta), \qquad G(x,t) = v(\eta), \qquad \eta = k(x + \alpha t), \tag{3.16}$$

where α is a constant. Equations (3.15) can be now reduced to the ODEs

$$k\alpha\rho_\eta + \frac{k}{2}(\rho^2)_\eta + kv_\eta = 0, \tag{3.17}$$

$$k\alpha v_\eta + k(v\rho)_\eta + \frac{k^3}{3}\rho_{\eta\eta\eta} = 0. \tag{3.18}$$

Taking the first integral for both sides of (3.17) and (3.18) with respect to η and take the integral constant by zero to have

$$v = -\rho(\alpha + 0.5\rho) \tag{3.19}$$

$$\alpha v + v\rho + \frac{k^2}{3}\rho_{\eta\eta} = 0. \tag{3.20}$$

Substituting (3.19) into (3.20) leads to

$$-\rho(\alpha + \rho)(\alpha + 0.5\rho) + \frac{k^2}{3}\rho_{\eta\eta} = 0. \tag{3.21}$$

We now turn to use the derivatives in (2.5) to be replaced with the one in (3.21) to have

$$\begin{aligned} &\frac{1}{6}u^{-2n-1}\left(-2a^2k^2(n-2)u^4 + 2ak^2u^{n+2}(2cu^n\right. \\ &-b(n-3)u) + u^{2n}\left(-6\alpha^2u^2 - 9\alpha u^3 + 2b^2k^2u^2\right. \\ &\left.\left.+2bck^2(n+1)u^{n+1} + 2c^2k^2nu^{2n} - 3u^4\right)\right) = 0. \end{aligned} \tag{3.22}$$

Put $n=0$, (3.22) becomes

$$\frac{2}{3}a^2k^2u^3+abk^2u^2+\frac{2}{3}ack^2u-\alpha^2u-\frac{3\alpha u^2}{2}+\frac{1}{3}b^2k^2u+\frac{1}{3}bck^2-\frac{u^3}{2}=0. \tag{3.23}$$

Next, we equate the coefficients of $\hat{\rho}^j$ $(j=0,1,2,3)$ to zero to obtain the following algebraic system:

$$\begin{aligned}&\frac{1}{3}bck^2=0,\\&\frac{2}{3}ack^2-\alpha^2+\frac{b^2k^2}{3}=0,\\&abk^2-\frac{3\alpha}{2}=0,\\&\frac{2a^2k^2}{3}-\frac{1}{2}=0.\end{aligned} \tag{3.24}$$

Thus, the solutions of the system (3.24) are given by

$$\begin{aligned}&a\to\pm\frac{\sqrt{3}}{2k},\\&b\to\pm\tfrac{\sqrt{3}\alpha}{k},\\&c\to 0.\end{aligned} \tag{3.25}$$

Various cases are now listed here.

- If $b\neq 0$ and $c=0$, the solution is written as

$$U(x,t)=\left(\pm\frac{1}{2\,\alpha}+\mu e^{\pm\sqrt{3}\,\alpha\,(x+\alpha t)}\right)^{-1}. \tag{3.26}$$

- If $\alpha^2>0$, we have

$$U(x,t)=\pm\alpha\left(2\;+\coth\left(\frac{\sqrt{3}\,\alpha}{2\,k}(k(x+\alpha t)+\mu)\right)\right), \tag{3.27}$$

and

$$U(x,t)=\pm\alpha\left(2\;+\tanh\left(\frac{\sqrt{3}\,\alpha}{2\,k}(k(x+\alpha t)+\mu)\right)\right). \tag{3.28}$$

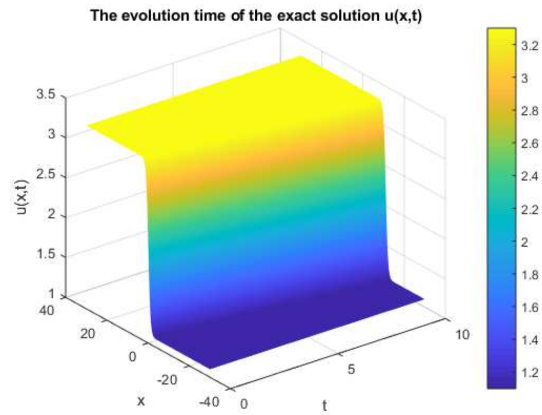

Figure 3: Graph of the solutions of (3.28) and (3.19) for $\alpha = 1.1$, $k = 0.1$ and $\mu = 0.1$.

- If $a \neq 0$ and $\alpha = 0$, we conclude with the solution

$$U(x,t) = \left(\frac{2\,k}{\sqrt{3}(k(x+\alpha t)+\mu)} \right). \tag{3.29}$$

where μ is supposed to be an arbitrary constant.

3.3 Third application

We devote this part to examine new exact solutions for the modified KdV-KP equation [37, 38] which is given by

$$u_{tx} - \frac{3}{2} u_{xx} + 2\,u u_x^2 + u^2\,u_{xx} + u_{xxxx} + u_{yy} = 0. \tag{3.30}$$

The variable

$$u(x,y,t) = \rho(\eta),\ \ \eta = k\,(x + \alpha\,y + w\,t), \tag{3.31}$$

where α, k, and w are constants, is utilized to change (3.30) into the ODE

$$w\,\rho_{\eta\eta} - \frac{3}{2}\rho_{\eta\eta} + 2\,\rho\rho_\eta^2 + \rho^2\,\rho_{\eta\eta} + k^2 \rho_{\eta\eta\eta\eta} + \alpha^2\,\rho_{\eta\eta} = 0. \tag{3.32}$$

Taking the second integral for both sides of (3.32) with respect to η ends with

$$\delta\,\rho + \frac{1}{3}\rho^3 + k^2 \rho_{\eta\eta} = 0, \tag{3.33}$$

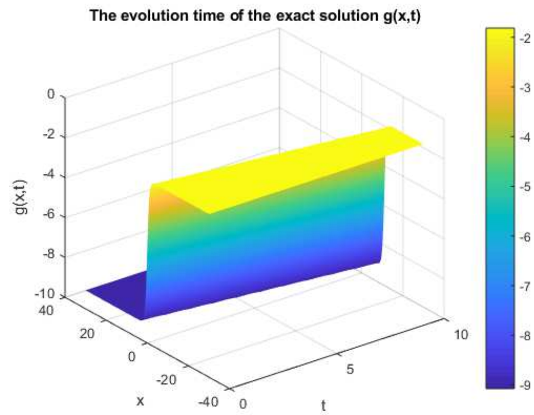

Figure 4: Graph of the solutions of (3.28) and (3.19) for $\alpha = 1.1$, $k = 0.1$ and $\mu = 0.1$.

where $\delta = \alpha^2 + w - 3/2$. The derivative appeared in (3.33) can be eliminated by employing (2.5). This leads to

$$\begin{aligned} &\delta\,\rho + \frac{1}{3}\rho^3 + k^2\left(ab(3-n)\rho^{2-n} + a^2(2-n)\rho^{3-2n}\right. \\ &\left.+nc^2\rho^{2n-1} + bc(n+1)\rho^n + (2ac+b^2)\rho\right) = 0. \end{aligned} \tag{3.34}$$

Assume that $n = 0$, then we have

$$bck^2 + \delta\rho + b^2k^2\rho + 2ack^2\rho + 3abk^2\rho^2 + \frac{1}{3}\rho^3 + 2a^2k^2\rho^3 = 0. \tag{3.35}$$

Equating each coefficients of ρ^j ($j = 0,\ 1,\ 2,\ 3$), to zero yields

$$\begin{aligned} &b\,c\,k^2 = 0, \\ &\delta + b^2\,k^2 + 2\,a\,c\,k^2 = 0, \\ &a\,b\,k^2 = 0, 2\,a^2\,k^2 + \frac{1}{3} = 0. \end{aligned} \tag{3.36}$$

Solving the algebraic system (3.36), using any Mathematical software (for example Mathematica or Maple), gives us

$$\begin{aligned} &b \to 0, \\ &a \to \pm\frac{i}{\sqrt{6}k}, \\ &c \to \pm\frac{i\sqrt{3\delta}}{\sqrt{2}k}, \\ &ac \to -\frac{\delta}{2k^2}, \end{aligned} \tag{3.37}$$

where $\delta = \alpha^2 + w - 3/2$. Under different assumptions, we have some cases given as

- For negative δ, the solution becomes

  $$u_1(x,y,t) = \pm\sqrt{-3\delta}\,\tan\left(\frac{\sqrt{-2\delta}}{2k}(k\,(x+\alpha\,y+w\,t)+\mu)\right), \tag{3.38}$$

  and

  $$u_2(x,y,t) = \pm\sqrt{-3\delta}\,\cot\left(\frac{\sqrt{-2\delta}}{2k}(k\,(x+\alpha\,y+w\,t)+\mu)\right). \tag{3.39}$$

  Here, $\alpha,\ \mu,\ w, k$ are arbitrary constants and $\delta = \alpha^2 + w - 3/2$.

- For positive δ, we have

  $$u_3(x,y,t) = \pm\sqrt{3\delta}\,\coth\left(\frac{\sqrt{2\delta}}{2k}(k\,(x+\alpha\,y+w\,t)+\mu)\right), \tag{3.40}$$

  and

  $$u_4(x,y,t) = \pm i\sqrt{3\delta}\,\tanh\left(\frac{\sqrt{2\delta}}{2}(x+\alpha\,y+w\,t+\mu)\right). \tag{3.41}$$

- If $\delta = 0$, then the solution is assumed to be

  $$u_5(x,y,t) = \frac{i\sqrt{6}}{x+\alpha\,y+w\,t+\mu}. \tag{3.42}$$

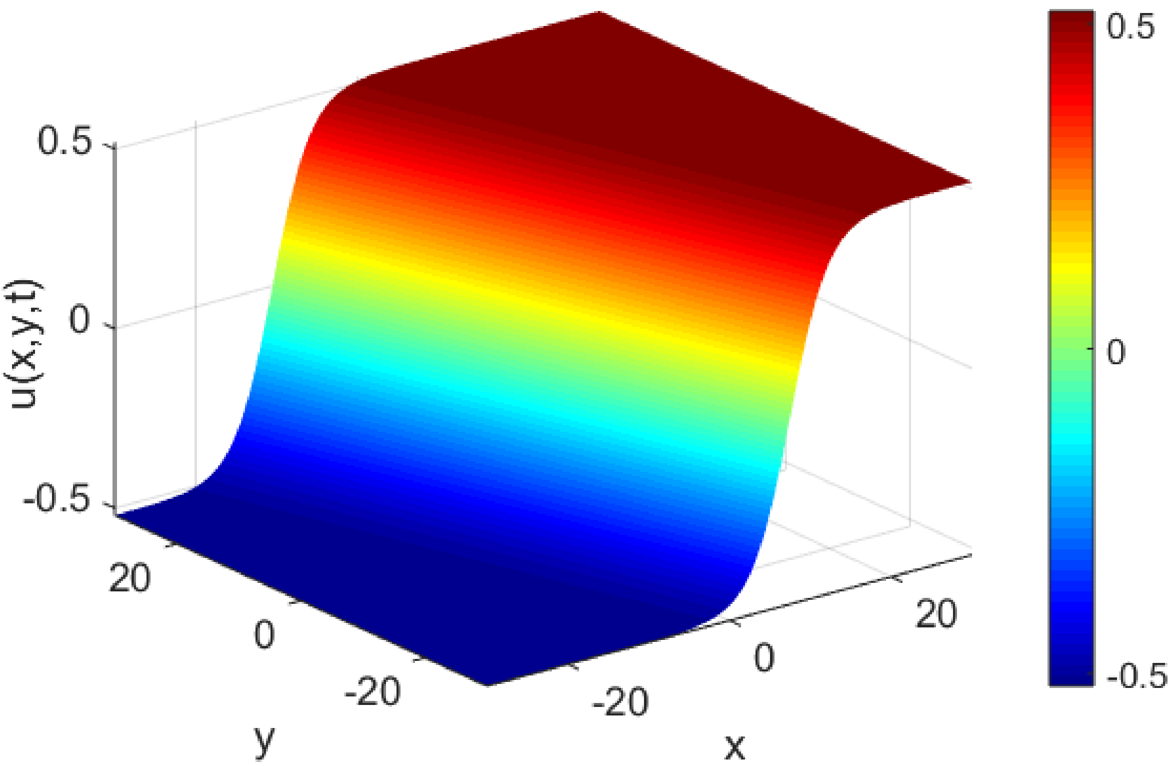

Figure 5: Graph of the solutions of (3.41) for $k = 1$, $\alpha = 0.3$, $w = 1.5$ and x, $y \in [-30, 30]$.

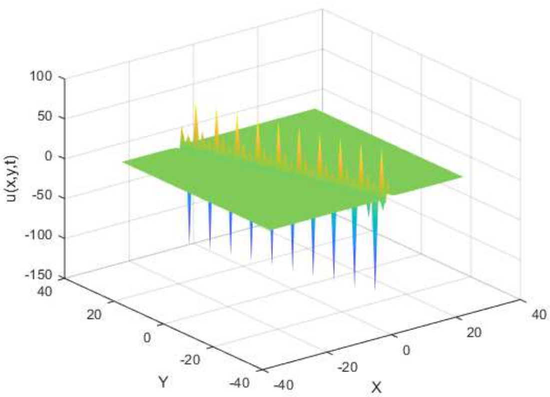

Figure 6: Graph of the solutions of (3.40) for $k = 1$, $\alpha = 0.3$, $w = 1.5$ and $x,\, y \in [-30, 30]$.

3.4 Fourth application

This subsection is allocated to solve a (1+1)-dimensional nonlinear ZK-MEW equation [39, 40] given by

$$\Gamma_t + 3\alpha\Gamma^2\Gamma_x + \beta\Gamma_{xxt} = 0, \tag{3.43}$$

where α, β are assumed to be constants. Use the transformation

$$\Gamma(x,t) = \rho(\eta), \qquad \eta = k(x + wt), \tag{3.44}$$

where w is the wave speed constant, to convert the considered equation into the ODE

$$w\,k\,\rho_\eta + k\,\alpha(\rho^3)_\eta + w\beta\,k^3\rho_{\eta\eta\eta} = 0. \tag{3.45}$$

Applying the integral on both sides of (3.45) and utilizing the boundary conditions lead to

$$w\,\rho + \alpha(\rho^3) + w\beta\,k^2\rho_{\eta\eta} = 0. \tag{3.46}$$

The second derivative emerged in (3.46) can be replaced by the second derivative in (2.5), i.e,

$$\begin{aligned}&2a^2\beta k^2wu^{3-2n} - a^2\beta k^2nwu^{3-2n} + 3ab\beta k^2wu^{2-n} - ab\beta k^2nwu^{2-n} + 2a\beta ck^2uw\\&+\alpha u^3 + b^2\beta k^2uw + b\beta ck^2wu^n + b\beta ck^2nwu^n + \beta c^2k^2nwu^{2n-1} + uw = 0.\end{aligned} \tag{3.47}$$

Suppose that $n = 0$, then (3.47) is reduced as

$$2a^2\beta k^2u^3w + 3ab\beta k^2u^2w + 2a\beta ck^2uw + \alpha u^3 + b^2\beta k^2uw + b\beta ck^2w + uw = 0. \tag{3.48}$$

The coefficients of $\hat{\rho}^j$ $(j = 0, 1, 2, 3)$ are now equated to zero to attain the following system:

$$\begin{aligned} &b\beta ck^2w = 0,\\ &2a\beta ck^2w + b^2\beta k^2w + w = 0,\\ &3ab\beta k^2w = 0,\\ &2a^2\beta k^2w + \alpha = 0. \end{aligned} \tag{3.49}$$

System (3.49) can be simply solved by using Mathematica. The solution is given by

$$a \to \pm\sqrt{\frac{-\alpha}{2\,k^2\,\beta w}}, \tag{3.50}$$

$$c \to \sqrt{\frac{-w}{2\,k^2\,\beta\alpha}}, \tag{3.51}$$

$$b \to 0, \tag{3.52}$$

$$ac \to -\frac{1}{2\,k^2\,\beta}. \tag{3.53}$$

We now consider some different cases.

- If $\beta < 0$, then the solution is shown as follows:

$$\Gamma(x,t) = \left(\pm\frac{\sqrt{4ac - b^2}}{2a}\tan\left(\frac{\sqrt{4ac - b^2}}{2}(k(x + wt) + \mu)\right)\right), \tag{3.54}$$

and

$$\Gamma(x,t) = \left(\pm\frac{\sqrt{4ac - b^2}}{2a}\cot\left(\frac{\sqrt{4ac - b^2}}{2}(k(x + wt) + \mu)\right)\right). \tag{3.55}$$

- If $\beta > 0$, we end up with

$$\Gamma(x,t) = \left(\pm\frac{\sqrt{b^2 - 4ac}}{2a}\coth\left(\frac{\sqrt{b^2 - 4ac}}{2}(k(x + wt) + \mu)\right)\right), \tag{3.56}$$

and

$$\Gamma(x,t) = \left(\pm\frac{\sqrt{b^2 - 4ac}}{2a}\tanh\left(\frac{\sqrt{b^2 - 4ac}}{2}(k(x + wt) + \mu)\right)\right). \tag{3.57}$$

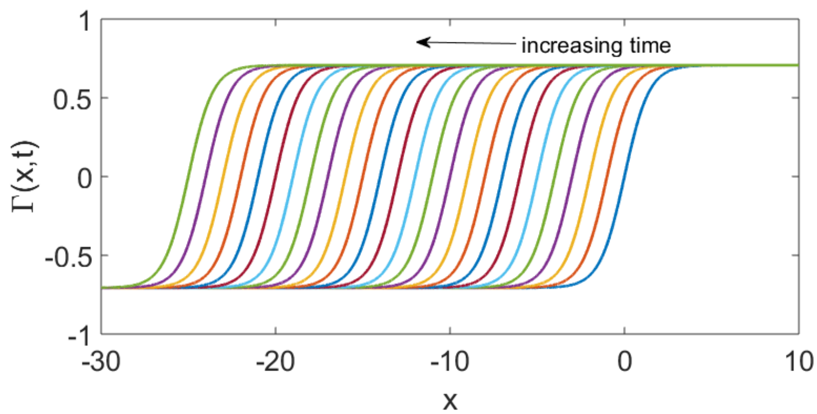

Figure 7: 2D plot for the solution of (3.57) when $w = 1/2$, $k = 0.1$, $\beta = 1$, $\alpha = -1$ and $\mu = 0$.

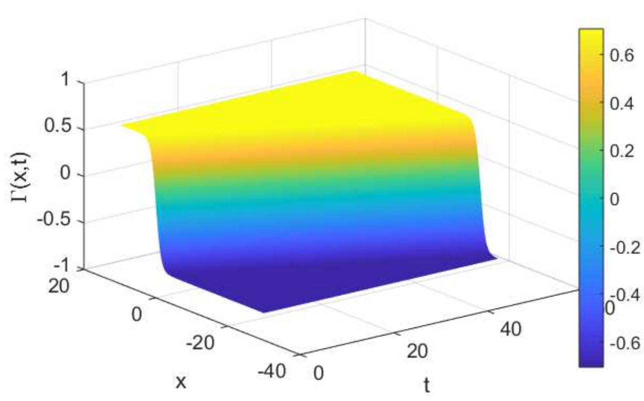

Figure 8: 3D graph for the solution of (3.57) when $w = 1/2$, $k = 0.1$, $\beta = 1$, $\alpha = -1$ and $\mu = 0$.

4 Comparison

In order to show that the Riccati-Bernoulli sub-ODE method is adequate and effective, we compare some results presented in this article with other results. For instance, Taghizadeh et al. [37] have obtained two real solutions and two complex solutions for the modified KdV-KP equation, using the first integral method. Comparing these results with our results, we deduce that the proposed approach gives several new exact traveling wave solutions (including hyperbolic function, trigonometric function and rational function solutions) along with additional free parameters. Furthermore, it can be obtained an infinite sequence of solutions for the applications presented in this article when a Bäcklund transformation is used. These solutions cannot be found by the first integral method or the $(\frac{G'}{G})$-expansion method. Thus, the used method is more effective in obtaining new exact solutions than the first integral method.

5 Conclusion

The Riccati-Bernoulli Sub-ODE approach has been successfully employed to explore the exact traveling wave solutions for the thin film equation, DLWE, the modified KdV-KP equation and the nonlinear ZK-MEW equation. Consequently, we found several new exact traveling wave solutions containing trigonometric functions, hyperbolic functions and rational functions. We also compared some achieved results with other results and discovered that the proposed method is more powerful than the first integral method and $(\frac{G'}{G})$-expansion method. The Bäcklund transformation which introduces an infinite sequence of solutions is given. Finally, the performance of the considered technique is found straightforward and efficient to be used in solving some sophisticated NPDEs.

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