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A note on almost hyperideals in semihypergroups

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Abstract

In this paper, we define almost hyperideals in semihypergroups and give some interesting properties.

1 Introduction

In 1934, Marty [10] pioneered the study of hyperstructures which has been further studied by various mathematicians. Marty published some properties of hypergroups. Among algebraic hyperstructures, semihypergroup is an interesting research area. Many authors studied various aspects of semihypergroups, as we can see from [3], [4], [5] and so on. In this work, the authors focus on ideals in semihypergroup. Ideals in semihypergroups were

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studied by many authors; for instance, Hasankhani [6] studied hyperideals and Green's relations in semihypergroups, Hila, Davvaz and Naka [7] studied quasi-hyperideals in semihypergroups, and Changphas and Davvaz [2] studied hyperideals in ordered semihypergroups.

An introductory definition of left, right, two-sided almost ideals of semigroups was launched in 1980 by Grosek and Satko [8]. They characterized these ideals when a semigroup S contains no proper left, right, two-sided almost ideals in [8]. Afterwards, they discovered minimal almost ideals and smallest almost ideals of semigroups in [11] and [9], respectively. In 1981, Bogdanovic [1] introduced the concept of almost bi-ideals in semigroups by using the notions of almost ideals and bi-ideals in semigroups. Likewise, Wattanatripop, Chinram and Changphas [12] examined quasi-almost-ideals and fuzzy almost ideals in semigroups. Furthermore, they provided the properties of quasi-almost-ideals in semigroups and the relationship between almost ideals and fuzzy almost ideals in semigroups. In addition, these authors defined fuzzy almost bi-ideals in semigroups in [13]. In this paper, we study almost hyperideals in semihypergroups and give some interesting properties.

2 Basic Definitions

In this section, we recall the basic definitions of hyperideals in semihypergroups.

Definition 2.1. A *hyperstructure* is a nonempty set H together with a map $\cdot : H \times H \to P^*(H)$ where $P^*(H)$ is the set of all nonempty subsets of H. The operation $\cdot$ is called a *hyperoperation* on H.

If $x \in H$ and A and B are subsets of H, then we let

$$A \cdot B = \bigcup_{a\in A, b\in B} a \cdot b, \quad x \cdot A = \{x\} \cdot A, \quad \text{and} \quad B \cdot x = B \cdot \{x\}.$$

Definition 2.2. A hyperstructure $(H, \cdot)$ is called a *semihypergroup* if for all $x, y, z \in H$,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Definition 2.3. A nonempty subset A of a semihypergroup $(H, \cdot)$ is called a *subsemihypergroup* of H if for all $x, y \in A$, $x \cdot y \subseteq A$.

Definition 2.4. Let $(H, \cdot)$ be a semihypergroup.

(1) A nonempty subset L of H is called a *left hyperideal* of H if $x \cdot L \subseteq L$ for all $x \in H$.

(2) A nonempty subset R of H is called a *right hyperideal* of H if $R \cdot x \subseteq R$ for all $x \in H$.

(3) A nonempty subset I of H is called a *hyperideal* of H if I is a left hyperideal and a right hyperideal of H.

3 Main results

At first we define almost hyperideals in semihypergroups.

Definition 3.1. Let $(H, \cdot)$ be a semihypergroup.

(1) A nonempty subset L of H is called a *left almost hyperideal* of H if

$$x \cdot L \cap L \neq \emptyset \text{ for all } x \in H.$$

(2) A nonempty subset R of H is called a *right almost hyperideal* of H if

$$R \cdot x \cap R \neq \emptyset \text{ for all } x \in H.$$

(3) A nonempty subset I of H is called an *almost hyperideal* of H if I is a left almost hyperideal and a right almost hyperideal.

Theorem 3.1. *Let $(H, \cdot)$ be a semihypergroup.*

(1) *Every left hyperideal of H is a left almost hyperideal of H.*

(2) *Every right hyperideal of H is a right almost hyperideal of H.*

(3) *Every hyperideal of H is an almost hyperideal of H.*

Proof. (1) Assume that L is a left hyperideal of H. Let $x \in H$. Then $x \cdot L \subseteq L$. Therefore, $x \cdot L \cap L \neq \emptyset$.

(2) Similar to (1).

(3) Follows from (1) and (2). □

However, the converse of Theorem 3.1 does not hold true. A counterexample follows.

Example 3.1. Consider $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$ and define $x \cdot y = \{x + y\}$ for all $x, y \in \mathbb{Z}_5$. It is easy to see that $(\mathbb{Z}_5, \cdot)$ is a semihypergroup. Obviously, $L = \{\overline{0}, \overline{1}, \overline{4}\}$ is a left almost hyperideal of $\mathbb{Z}_5$. However, L is not a left hyperideal of $\mathbb{Z}_6$ because $\overline{1} \cdot L = \{\overline{1}, \overline{2}, \overline{0}\} \nsubseteq L$.

Theorem 3.2. *Let $(H, \cdot)$ be a semihypergroup.*

(1) *Let L be a left almost hyperideal of H. If T is a subset of H containing L, then T is a left almost hyperideal of H.*

(2) *Let R be a right almost hyperideal of H. If T is a subset of H containing R, then T is a right almost hyperideal of H.*

(3) *Let I be an almost hyperideal of H. If T is a subset of H containing I, then T is an almost hyperideal of H.*

Proof. (1) Assume that $L \subseteq T$. Then $T \neq \emptyset$. Let $x \in H$. By assumption, $x \cdot L \subseteq x \cdot T$. It follows that $x \cdot L \cap L \subseteq x \cdot T \cap T$. Since $x \cdot L \cap L \neq \emptyset$, T is a left almost hyperideal of H.

(2) Similar to (1).

(3) Follows from (1) and (2). □

Theorem 3.3. *Let $(H, \cdot)$ be a semihypergroup.*

(1) *If L_1 and L_2 are left almost hyperideals of H, then $L_1 \cup L_2$ is also.*

(2) *If R_1 and R_2 are right almost hyperideals of H, then $R_1 \cup R_2$ is also.*

(3) *If I_1 and I_2 are almost hyperideals of H, then $I_1 \cup I_2$ is also.*

Proof. Use Theorem 3.2. □

Example 3.2. Consider the semihypergroup $\mathbb{Z}_5$ under the hyperoperation $\cdot$ defined by $x \cdot y = \{x + y\}$ for all $x, y \in \mathbb{Z}_5$. We have $I_1 = \{\overline{1}, \overline{3}, \overline{4}\}$ and $I_2 = \{\overline{1}, \overline{2}, \overline{4}\}$ are almost hyperideals of $\mathbb{Z}_5$. However, $I_1 \cap I_2 = \{\overline{1}, \overline{4}\}$ is not an almost hyperideal of $\mathbb{Z}_5$ because $\overline{1} \cdot (I_1 \cap I_2) = \{\overline{2}, \overline{0}\}$. Then $[\overline{1} \cdot (I_1 \cap I_2)] \cap (I_1 \cap I_2) = \emptyset$. Therefore, the intersection of two almost hyperideals of a semihypergroup H need not be an almost hyperideal of H.

Theorem 3.4. *Let $(H, \cdot)$ be a semihypergroup and $|H| > 1$.*

(1) *H has no proper left almost hyperideal if and only if for any $a \in H$, there exists an element $h_a \in H$ such that $h_a \cdot (H \setminus \{a\}) = \{a\}$.*

(2) *H has no proper right almost hyperideal if and only if for any $a \in H$, there exists an element $h_a \in H$ such that $(H \setminus \{a\}) \cdot h_a = \{a\}$.*

(3) *H has no proper almost hyperideal if and only if for any $a \in H$, there exists an element $h_a, k_a \in H$ such that $h_a \cdot (H \setminus \{a\}) = \{a\}$ and $(H \setminus \{a\}) \cdot k_a = \{a\}$.*

Proof. (1) Assume that H has no proper left almost hyperideal. Then $H \setminus \{a\}$ is not a left almost hyperideal. Then there exists $h_a \in H$ such that $h_a \cdot (H \setminus \{a\}) \cap (H \setminus \{a\}) = \emptyset$. Therefore $h_a \cdot (H \setminus \{a\}) = \{a\}$.

Conversely, assume that for any $a \in H$, there exists $h_a \in H$ such that $h_a \cdot (H \setminus \{a\}) = \{a\}$. Let $a \in H$. Then $h_a \cdot (H \setminus \{a\}) \cap (H \setminus \{a\}) = \emptyset$. Hence $H \setminus \{a\}$ is not a left almost hyperideal of H. Let A be a proper left almost hyperideal of H. Then $A \subseteq H \setminus \{a\}$ for some $a \in H$, a contradiction. Therefore H has no proper left almost hyperideal.

(2) Similar to (1).

(3)Follows from (1) and (2). □

Theorem 3.5. *Let $(H, \cdot)$ be a semihypergroup and $|H| > 1$.*

(1) *If $H \setminus \{a\}$ is not a left almost hyperideal of S, then $a \in a \cdot a$ or $a \cdot a \cdot a = \{a\}$.*

(2) *If $H \setminus \{a\}$ is not a right almost hyperideal of S, then $a \in a \cdot a$ or $a \cdot a \cdot a = \{a\}$.*

(3) *If $H \setminus \{a\}$ is not an almost hyperideal of S, then $a \in a \cdot a$ or $a \cdot a \cdot a = \{a\}$.*

Proof. (1) Assume that $H \setminus \{a\}$ is not a left almost hyperideal. By Theorem 3.4, there exists $h_a \in H$ such that $h_a \cdot (H \setminus \{a\}) = \{a\}$.

Case 1 : $h_a \neq a$. Then $h_a \in H \setminus \{a\}$. So $h_a \cdot h_a = \{a\}$. Suppose that $a \notin a \cdot a$. This implies $a \cdot a \subseteq H \setminus \{a\}$. We have $h_a \cdot a \cdot a \subseteq h_a \cdot (H \setminus \{a\})$. Hence $h_a \cdot a \cdot a = \{a\}$.

Case 1.1 : If $a \in h_a \cdot a$, then $a \cdot a \subseteq h_a \cdot a \cdot a = \{a\}$, which is a contradiction.

Case 1.2 : If $a \notin h_a \cdot a$, then $h_a \cdot a \subseteq H \setminus \{a\}$. That is $h_a \cdot h_a \cdot a = \{a\}$. Therefore $a \cdot a = \{a\}$, this is a contradiction.
Case 2 : $h_a = a$. Suppose that $a \notin a \cdot a$. Then $a \cdot a \subseteq H \setminus \{a\}$. So $h_a \cdot a \cdot a = \{a\}$. Hence $a \cdot a \cdot a = \{a\}$.

(2) Similar to (1).

(3) Follows from (1) and (2). □

In general, the converse of Theorem 3.5 does not holds true.

Example 3.3. Consider the semihypergroup $H = \{0, 1, -1\}$ under the hyperoperation $\cdot$ defined by $x \cdot y = \{x \times y\}$ for all $x, y \in H$ where $\times$ is the usual multiplication on $\mathbb{Z}$.

1. We have $1 \in 1 \cdot 1$ but $H \setminus \{1\} = \{0, -1\}$ is a left almost hyperideal.

2. We have $-1 \cdot -1 \cdot -1 = \{-1\}$ but $H \setminus \{-1\} = \{0, 1\}$ is a left almost hyperideal.

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