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Some fixed point theorems for rational-type contractive maps in ordered metric spaces

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Abstract

In this paper, we prove some fixed point theorems for certain rational-type contractive maps in ordered metric spaces. An example is given to support our results which extend some works in the literature

1 Introduction and Preliminaries

In mathematical analysis, a versatile result that is widely applied to solve functional equations in several branches of mathematics and especially in metric fixed point theory was initiated by Banach [1] in 1922. There are many extensions and generalizations of Banach contraction principle in the literature (see: [2- 8]).
Frechet in 1906, introduced the notions of metric space. This space is generalized by endowing it with a partial ordering. Turinici [10] is the pioneer researcher who established some results in ordered metrizable uniform space. Later, Ran and Reurings [11] established fixed point results in partially ordered metric spaces with application to solving matrix equations. Nieto and

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Rodriguez-Lopez [12] proved fixed point results in partially ordered metric spaces and applied the result to obtain solutions of certain partial differential equations with periodic boundary conditions. Many works have been carried out in this regard (see:[13-15]).
Recently, Dass and Gupta [16] extended the Banach contraction mapping by introducing the concept of rational expressions in contraction mappings. The rational-type contractive map

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y))} + \beta d(x,y)$$

$\forall x,y \in X$, for $\alpha,\beta \in [0,1)$ such that $\alpha+\beta<1$ and T continuous is employed to prove fixed point theorem in a complete metric space.
Jaggi and Dass [17] established the unique fixed point for any continuous mapping $T: X \to X$ on a complete metric space (X,d) that satisfies the rational-type contractive condition

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)d(x,Tx)}{d(x,y)+d(x,Ty)+d(y,Tx)} + \beta d(x,y)$$

$\forall x,y \in X$.

In 2018, Olatinwo and Ishola [18] proved the unique fixed point for a rational type contractive mapping in a complete metric space that satisfies

$$d(Tx,Ty) \leq \alpha \frac{[p+d(x,Tx)][d(y,Ty)]^r[d(y,Tx)]^q}{d(x,y)+\nu d(x,Ty)+\mu d(y,Tx)} + \beta d(x,y)$$
$\forall x,y \in X$

Cabrera et al. [19] established the result of Dass and Gupta [16] in the context of partially ordered metric spaces. Chandok et al. [20] proved the unique fixed point of (ϕ,ψ)-rational type contractive mappings in a complete metric spaces endowed with partial ordered by employing an altering distance function.
In this paper, some fixed point theorems satisfying rational type contractive mapping of Olatinwo and Ishola [18] are proved in the setting of ordered metric spaces.
The following definitions are needed for our proofs.
Definition 1.1. **[11]** Suppose $(X,\leq)$ is a partially ordered set and $T:X \to X$. T is said to be monotone nondecreasing if for all $x,y \in X$,
$x \leq y \Longrightarrow Tx \leq Ty$.

2 Main Results

Theorem 2.1. Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that for all $x, y \in X$ and $x \leq y$ there exist $\alpha, p, q, r, \nu, \mu \in R^+$ and $\beta \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq \alpha \frac{[p+d(x,Tx)][d(y,Ty)]^r[d(y,Tx)]^q}{d(x,y)+\nu d(x,Ty)+\mu d(y,Tx)} + \beta d(x, y) \tag{1}$$

with $d(x, y) + \nu d(x, Ty) + \mu d(y, Tx) > 0$.

For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty} \subset X$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$ be the Picard iteration associated to T. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Proof. If $Tx_0 = x_0$, then x_0 is the fixed point of T. Suppose that $x_0 < Tx_0$. Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for every $n \geq 0$. Since T is a nondecreasing mapping, by induction we obtain
$x_0 \leq Tx_0 = x_1 \leq Tx_1 = x_2 \leq \cdots Tx_{n-1} = x_n \leq Tx_n = x_{n+1} \leq \cdots$
If there is $n \geq 1$ such that $x_{n+1} = x_n$, then x_n is the fixed point of T. Let $x_{n+1} \neq x_n$. Using (1) we have
$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$\leq \alpha \frac{[(p + d(x_{n-1}, Tx_{n-1})][d(x_n, Tx_n)]^r[d(x_n, Tx_{n-1})]^q}{d(x_{n-1}, x_n) + \nu d(x_{n-1}, Tx_n) + \mu d(x_n, Tx_{n-1})} + \beta d(x_{n-1}, x_n)$$

$$= \alpha \frac{[p + d(x_{n-1}, x_n)][d(x_n, x_{n+1})]^r[d(x_n, x_n)]^q}{d(x_{n-1}, x_n) + \nu d(x_{n-1}, x_{n+1}) + \mu d(x_n, x_n)} + \beta d(x_{n-1}, x_n) \leq \beta d(x_{n-1}, x_n)$$

Consequently, we have $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1)$.
For $n > m$ and using the triangle inequality we obtain,

$$\begin{aligned} d(x_m, x_n) \;\leq\; & d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ & \leq \beta^m d(x_0, x_1) + \beta^{m+1} d(x_0, x_1) + \beta^{m+2} d(x_0, x_1) \\ & + \cdots + \beta^{m+n-1} d(x_0, x_1) \\ & \leq (\beta^m + \beta^{m+1} + \beta^{m+2} + \cdots + \beta^{m+n-1}) d(x_0, x_1) \\ & \leq \frac{\beta^m}{1 - \beta} d(x_0, x_1). \end{aligned}$$

Taking the limit as $m \to \infty$ shows that the sequence $\{x_n\}$ is Cauchy. Since (X, d) is a complete metric space, then $\{x_n\}$ converges to some point $u \in X$.

The continuity of T implies that $Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u$. Thus, u is a fixed point of T.

We prove the next theorem by relaxing the continuity assumption of the mapping T in Theorem 2.1 and imposing the following ordered conditions of the metric spaces X.
If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in N$.

Theorem 2.2. Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in N$. Let $T : X \to X$ be a nondecreasing mapping. Suppose (1) holds as in Theorem 2.1. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.
Proof. We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. Then we have $x_0 \leq x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \leq \cdots$ that is , $\{x_n\}$ is a nondecreasing sequence. Also, this sequence converges to u. Then $x_n \leq u$ for all $n \in N$. Suppose that $u \neq Tu$, that is $d(u, Tu) > 0$. Using(1) we have

$$\begin{aligned} d(Tx_n, Tu) \;\; &\leq \;\; \alpha\frac{[p + d(x_n, Tx_n)][d(u, Tu)]^r[d(u, Tx_n)]^q}{d(x_{n-1}, u) + \nu d(x_{n--1}, Tu) + \mu d(u, Tx_n)} + \beta d(x_{n-1}, u) \\ &= \alpha\frac{[p + d(x_n, x_{n+1})][d(u, Tu)]^r[d(u, x_{n+1})]^q}{d(x_{n-1}, u) + \nu d(x_{n--1}, Tu) + \mu d(u, x_{n+1})} + \beta d(x_{n-1}, u) \end{aligned}$$

$d(u, Tu) \leq 0$ as $n \to \infty$.

Since $d(u, Tu)$ is non-negative then $d(u, Tu) = 0$. This implies that $u = Tu$.

Theorem 2.3. In addition to the hypotheses of Theorem 2.1 (or Theorem 2.2), suppose that for every $x, y \in X$, there is $u \in X$ such that $u \leq x$ and $u \leq y$. Then T has a unique fixed point.
Proof. It has been established in Theorem 2.1 (or Theorem 2.2) that the set of fixed points for T is non-empty. We shall prove that if $x*$ and $y*$ are two fixed point of T, that is $x* = Tx*$ and $y* = Ty*$, then $x* = y*$.
By our assumption, there exists $u_0 \in X$ such that $u_0 \leq x*$ and $u_0 \leq y*$. Similarly as in the proof of Theorem 2.1, we define the sequence $\{u_n\}$ such that $u_{n+1} = Tu_n = T^{n+1}u_0$, $n = 0, 1, 2, \cdots$. Monotonicity of T implies that $T_0^u = u_n \leq x* = T^n x*$ and $T_0^u = u_n \leq y* = T^n y*$.
If there exists a positive integer m such that $x* = u_n$, then $x* = Tx* =$

$Tu_n = u_{n+1}$, for all $n \geq m$. Then $u_n \to x*$ as $n \to \infty$. Now we suppose that $x* \neq u_n$, for all $n \geq 0$. Then $d(u_n, x*) \neq 0$ for all $n \geq 0$.
Since $u_n < x*$, for all $n \geq 0$, applying (1) yields

$$d(u_{n+1}, x*) = d(Tu_n, Tx*) \leq \alpha \frac{[p+d(u_n,Tu_n)][d(x*,Tx*)]^r[d(x*,Tu_n)]^q}{d(u_n,x*)+\mu d(x*,Tu_n)+\nu d(u_u,Tx*)} + \beta d(u_n, x*)$$

Taking the limit as $n \to \infty$ yields $d(u_{n+1}, x*) \to 0$ that $u_n \to x*$ as $n \to \infty$. Using similar argument, we can prove that $u_n \to y*$ as $n \to \infty$. By the uniqueness of limit we have $x* = y*$. Thus T has a unique fixed point.

Remark 2.4. Theorem 2.1 and Theorem 2.2 are generalizations of the result of Olatinwo and Ishola [18] (Theorem 6) in the setting of ordered metric spaces. Theorems 2.1, 2.2 and 2.3 are generalizations of Theorems 2, 3 and 4 of Cabrera et al. [19] with respect to the maps.

For $\alpha = 0$ in (1) we obtain the following corollaries.
Corollary 2.4. Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that for all $x, y \in X$ and $x \leq y$ there exists $\beta \in [0, 1)$ satisfying

$$d(Tx, Ty) \leq \beta d(x, y) \tag{2}$$

For $x_0 \in X$, let $\{x_n\}_{n=0}^{\infty} \subset X$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \cdots$ be the Picard iteration associated to T. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Corollary 2.5. Let $(X, \leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \leq x$ for all $n \in N$. Let $T : X \to X$ be a nondecreasing mapping. Suppose (2) holds as in Corollary 2.4. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Corollary 2.6. In addition to the hypotheses of Corollary 2.4 (or Corollary 2.5), suppose that for every $x, y \in X$, there is $u \in X$ such that $u \leq x$ and $u \leq y$. Then T has a unique fixed point.

Example 2.4. Let $X = [0, 1]$ with partial ordered " $\leq$ " and usual metric "d" be a partially ordered metric space. Let $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{1}{2} & if\ x \in [0, \frac{1}{4}) \\ x - \frac{1}{4} & if\ x \in [\frac{1}{4}, 1] \end{cases}$$

To show that T satisfies the contractive conditions (1) of Theorem 2.2 for $x = \frac{1}{2}$, $y = \frac{1}{6}$, $p = 0$, $r = q = \nu = \mu = 1$ and $\alpha = 1$, we have
$Tx = \frac{1}{4}$, $Ty = \frac{1}{2}$, $d(y, Tx) = \frac{1}{12}$, $d(x, Ty) = 0$, $d(x, Tx) = \frac{1}{4}$, $d(y, Ty) = \frac{1}{3}$, $d(x, y) = \frac{1}{3}$, $d(Tx, Ty) = \frac{1}{4}$.
Thus

$$\begin{aligned} \frac{1}{4} &= d(Tx, Ty) \\ &\leq \frac{d(x, Tx)d(y, TY)d(y, Tx)}{d(y, TX) + d(x, TY) + d(x, y)} + \beta d(x, y) \\ &= \frac{\frac{1}{4} \times \frac{1}{3} \times \frac{1}{12}}{\frac{1}{12} + 0 + \frac{1}{3}} + \frac{1}{3}\beta \\ &= \frac{1}{60} + \frac{1}{3}\beta, \end{aligned}$$

It follows that $\beta \geq \frac{7}{10}$. Hence $\beta \in [0, 1)$. Therefore T satisfies the contractive condition (1) and other hypotheses of Theorem 2.2. The unique fixed point of T is $\frac{1}{2}$.

3 Conclusion

In this research the existence and uniqueness of certain rational-type contractive mappings is established in an ordered metric spaces. This result is validated by an example . The potentality of this work is that it can be prove in different abstract spaces and can also be used to find the solution of integral equations.

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