International Journal of Mathematics and
Computer Science, **15**(2020), no. 1, 433–442

Certain Properties on Analytic p-Valent Functions

Suhila Elhaddad, Maslina Darus

Department of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600, Bangi
Selangor, Malaysia

email: suhila.e@yahoo.com , maslina@ukm.edu.my

(Received October 25, 2019, November 29, 2019)

Abstract

Owing to the importance and great interest of linear operators, a generalization of linear derivative operator $\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)$ is newly introduced in this study. Thus, a new subclass of p-valent functions $\mathcal{H}^m_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$ is defined by means of the aforementioned linear differential operator. In addition to this, different properties and characteristics were considered in the study for this subclass. Some of these properties include the coefficient inequalities and the growth and distortion properties.

1 Introduction and Preliminaries

The function class is denoted by $\mathcal{A}(p)$ which represented by the following form:

$$h(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic and p-valent in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For simplicity, we write $\mathcal{A}(1) = \mathcal{A}$. The Hadamard product (or convolution) $h * k$ for two

Key words and phrases: Analytic functions, p-valent functions, differential operator, Mittag-Leffler function.
AMS (MOS) Subject Classifications: 30C45.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

analytic functions h defined in (1.1) and

$$k(z) = z^p + \sum_{j=p+1}^{\infty} b_j z^j,$$

is given by

$$h(z) * k(z) = z^p + \sum_{j=p+1}^{\infty} a_j b_j z^j.$$

Dozik and Srivastava [6] defined the p-valent function $\mathcal{X}_p(a_1, ..., a_r, b_1, ..., b_s; z)$, which defined by generalized hypergeometric function as following

$$\mathcal{X}_p(a_1, ..., a_r, b_1, ..., b_s; z) = z^p + \sum_{j=p+1}^{\infty} \frac{\prod_{i=1}^{r} (a_i)_{j-p}}{\prod_{n=1}^{s} (b_n)_{j-p}} \frac{z^j}{(j-p)!}, \quad p \in \mathbb{N} \quad (1.2)$$

where $a_i \in \mathbb{C}, b_n \in \mathbb{C} \setminus \{0, -1, -2, ...\}, (i = 1, ..., r, n = 1, ..., s)$, and $r \leq s + 1; r, s \in \mathbb{N}_0$, and the Pochhammer symbol $(\nu)_j$ is defined by

$$(\nu)_j = \frac{\Gamma(\nu + j)}{\Gamma(\nu)} = \begin{cases} \nu(\nu+1)...(\nu+j-1), & j = 1, 2, 3, ..., \\ 1, & j = 0. \end{cases}$$

For convenience, we write $\mathcal{X}_p(a_1, ..., a_r, b_1, ..., b_s; z) = \mathcal{X}_p(a_1, b_1; z)$.

The well-known Mittag-Leffler function $E_\upsilon(z)$ that Mittag-Leffler introduces in [13] and [14] is described below. Similarly, Wiman [22] also introduces the generalization $E_{\upsilon,\varrho}(z)$ of the same function.

$$E_\upsilon(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\upsilon j + 1)},$$

and

$$E_{\upsilon,\varrho}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\upsilon j + \varrho)}, \quad (1.3)$$

where $\upsilon, \varrho \in \mathbb{C}$, $Re(\upsilon)>0$ and $Re(\varrho)>0$.

Due to its increasing prospective applications in applied problems, probability, statistical distribution theory, etc., the interest in the Mittag-Leffler type function has risen significantly over the past few years. For more information on Mittag Leffler function and its application, following works may be referred to [2], [3], [9], [10], [11], [16], [19], [20] and [21].

Geometric properties for the Mittag-Leffler function $E_{\upsilon,\varrho}(z)$ including starlikeness, convexity and close to convexity have been recently researched in [4]. In addition, findings were achieved on partial sums of $E_{\upsilon,\varrho}(z)$ in [17].

The function specified by (1.3) does not belong to the class $\mathcal{A}(p)$. Thus, the next normalization of the function $E_{\upsilon,\varrho}(z)$ is considered to be :

$$\begin{aligned}\Psi_{\upsilon,\varrho}(z) &= z^p\Gamma(\varrho)E_{\upsilon,\varrho}(z)\\ &= z^p+\sum_{j=p+1}^{\infty}\frac{\Gamma(\varrho)}{\Gamma(\upsilon(j-p)+\varrho)}z^j.\end{aligned}$$

Then, for $h\in\mathcal{A}(p)$, the operator $\mathcal{P}^{m,p}_{\lambda,\upsilon,\varrho}h(z):\mathcal{A}(p)\to\mathcal{A}(p)$ is defined by

$$\mathcal{P}^{0,p}_{\lambda,\upsilon,\varrho}f(z)=h(z)*\Psi_{\upsilon,\varrho}(z),$$

$$\mathcal{P}^{1,p}_{\lambda,\upsilon,\varrho}f(z)=(1-\lambda)(h(z)*\Psi_{\upsilon,\varrho}(z))+\frac{\lambda}{p}z(h(z)*\Psi_{\upsilon,\varrho}(z))' \tag{1.4}$$

$$\vdots$$

$$\mathcal{P}^{m,p}_{\lambda,\upsilon,\varrho}h(z)=\mathcal{P}^{1,p}_{\lambda,\upsilon,\varrho}(\mathcal{P}^{m-1,p}_{\lambda,\upsilon,\varrho}h(z)). \tag{1.5}$$

If h is given by (1.1), then from (1.4) and (1.5) we see that

$$\mathcal{P}^{m,p}_{\lambda,\upsilon,\varrho}h(z)=z^p+\sum_{j=p+1}^{\infty}\left[\frac{p+\lambda(j-p)}{p}\right]^m\frac{\Gamma(\varrho)}{\Gamma(\upsilon(j-p)+\varrho)}a_jz^j. \tag{1.6}$$

where $m\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\},\lambda\geq 0$.

Corresponding to $\mathcal{X}_p(a_1,b_1;z)$ which is is defined in (1.2), $\mathcal{P}^{m,p}_{\lambda,\upsilon,\varrho}h(z)$ defined in (1.6). Using the convolution, we define a new generalized derivative operator $\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)$ as follows:

Definition 1.1. Let $h\in\mathcal{A}(p)$, then the generalized derivative operator $\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z):\mathcal{A}(p)\to\mathcal{A}(p)$ is given by

$$\begin{aligned}&\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)=\mathcal{X}_p(a_1,b_1;z)*\mathcal{P}^{m,p}_{\lambda,\upsilon,\varrho}h(z)\\ &=z^p+\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Gamma(\varrho)}{\Gamma(\upsilon(j-p)+\varrho)}\left(\frac{\prod_{i=1}^{r}(a_i)_{j-p}}{\prod_{n=1}^{s}(b_n)_{j-p}}\right)\frac{a_jz^j}{(j-p)!}\end{aligned} \tag{1.7}$$

For the sake of simplicity, we write

$$\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)=z^p+\sum_{j=p+1}^{\infty}\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}a_jz^j, \tag{1.8}$$

where $m\in\mathbb{N}_0,\lambda\geq 0$ and $\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)$ is given by

$$\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)=\frac{\Gamma(\varrho)}{\Gamma(\upsilon(j-p)+\varrho)}\left(\frac{\prod_{i=1}^{r}(a_i)_{j-p}}{\prod_{n=1}^{s}(b_n)_{j-p}}\right) \tag{1.9}$$

Remark 1.1

- For $s=0,r=1,a_1=1$ and $p=1$, we get the operator studied by Elhaddad et al. [7], [8].
- For $s=0,r=1,a_1=1,\upsilon=0,\varrho=1$ and $p=1$, we get Al-Oboudi operator [1].
- For $s=0,r=1,a_1=1,\upsilon=0,\varrho=1$, $\lambda=1$ and $p=1$, we get Sălăgean operator [18].
- For $s=0,r=1,a_1=1,m=0$ and $p=1$, we get $\mathbb{E}_{\alpha,\beta}(z)$ [20].
- For $m=0,\upsilon=0$ and $\varrho=1$, we get the operator studied by Dozik and Srivastava [6].
- For $m=0,\upsilon=0,\varrho=1,r=1,s=0,a_1=\delta+1$ and $p=1$, we get the Ruscheweyh presented operator [15]
- For $m=0,\upsilon=0,\varrho=1,r=2,s=1$ and $p=1$, we obtain the operator which was given by Hohlov [12].
- For $m=0,\upsilon=0,\varrho=1,r=2,s=1,a_2=1$ and $p=1$, we get the operator that was given by Carlson and Shaffer [5].

Using the operator $\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)$, the new subclass $\mathcal{H}^{m}_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$ of the functions $h\in\mathcal{A}(p)$ is defined as follows:

Definition 1.2. Let $\mathcal{H}^{m}_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$ denote the subclass of $\mathcal{A}(p)$ composed of functions h that satisfy

$$Re\left\{1+\frac{1}{d}\left[p(1-\eta)\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)}{z}+\eta(\widetilde{\mathfrak{D}}^{m}_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z))'-p\right]\right\}>0, \tag{1.10}$$

where $\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)$ is given by (1.7).

This implies that it satisfies the following inequality

$$\left|\frac{p(1-\eta)\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)}{z}+\eta(\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z))'-p}{p(1-\eta)\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)}{z}+\eta(\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z))'-1+2d}\right|<1, \quad (1.11)$$

where $p\in\mathbb{N}, m\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\}, d\in\mathbb{C}\setminus\{0\}, \lambda\geq 0, \eta\geq 0$ and $z\in\mathbb{U}$.

The properties of the above subclass such as coefficient estimates, growth and distortion theorems are investigated.

2 Coefficient Inequalities

Theorem 2.1. *Let h given by (1.1). If*

$$\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}|a_j|\leq|d| \qquad (2.12)$$

where $m\in\mathbb{N}_0, \lambda\geq 0$ and $\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)$ is given by (1.9). Then $h\in\mathcal{H}^m_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$.

Proof. Supposed that the inequality (2.12) is true , we obtain

$$\left|p(1-\eta)\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)}{z}+\eta(\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z))'-p\right|$$
$$-\left|p(1-\eta)\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)}{z}+\eta(\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z))'-1+2d\right|$$
$$=\left|\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}\right|a_j|z^{j-1}|$$
$$-\left|2d+\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}\right|a_j|z^{j-1}|$$
$$\leq\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}|a_j||z^{j-1}|-2|d|$$
$$-\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}|a_j||z^{j-1}|$$
$$\leq\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}|a_j|-|d|\leq 0,$$

where $\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)h(z)$ is given by (1.7).
Then $h\in\mathcal{H}^m_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$.

3 Growth and Distortion Theorems

Theorem 3.1. *Let $h\in\mathcal{H}^m_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$. Then for $|z|=r<1$, we have*

$$r^p-\frac{|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^m\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}r^{p+1}\leq|h(z)|$$
$$\leq r^p+\frac{|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^m\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}r^{p+1}.$$

Proof. Having regard to Theorem 2.1, we have

$$\sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}|a_j|\leq|d|,$$

and

$$[p+\eta]\left[\frac{p+\lambda}{p}\right]^{m}\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)\sum_{j=p+1}^{\infty}|a_j| \\ \leq \sum_{j=p+1}^{\infty}[p+\eta(j-p)]\left[\frac{p+(j-p)\lambda}{p}\right]^{m}\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}|a_j| \leq |d|.$$

So, we have

$$\sum_{j=p+1}^{\infty}|a_j| \leq \frac{|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^{m}\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}.$$

From (1.1), we have

$$\begin{aligned} |h(z)| &= |z^p+\sum_{j=p+1}^{\infty}a_jz^j| \leq |z^p|+r^{p+1}\sum_{j=p+1}^{\infty}|a_j| \\ &\leq r^p+\frac{|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^{m}\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}r^{p+1}. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |h(z)| &= |z^p+\sum_{j=p+1}^{\infty}|a_j||z^j| \geq |z^p|-r^{p+1}\sum_{j=p+1}^{\infty}|a_j| \\ &\geq r^p-\frac{|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^{m}\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}r^{p+1}. \end{aligned}$$

This is the end of the proof.

In the same way, the following theorem can be proved.

Theorem 3.2. *If $h \in \mathcal{H}^m_{\upsilon,\varrho}(a_r,b_s,\lambda;\eta,d,p)$. Then for $|z|=r<1$, we have*

$$pr^{p-1}-\frac{(p+1)|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^{m}\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}r^{p} \leq |h'(z)| \\ \leq pr^{p-1}+\frac{(p+1)|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^{m}\Upsilon_{(1,\upsilon,\varrho)}(a_r,b_s)}r^{p}.$$

Proof. Let $h \in \mathcal{H}^m_{\upsilon,\varrho}(a_r, b_s, \lambda; \eta, d, p)$. Then from (2.12), we have

$$\sum_{j=p+1}^{\infty} |a_j| \leq \frac{|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^m \Upsilon_{(1,\upsilon,\varrho)}(a_r, b_s)}.$$

Also, from (1.1), we have

$$\begin{aligned} |h'(z)| &= |pz^{p-1} + \sum_{j=p+1}^{\infty} j a_j z^{j-1}| \leq pr^{p-1} + (p+1)r^p \sum_{j=p+1}^{\infty} |a_j| \\ &\leq pr^{p-1} + \frac{(p+1)|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^m \Upsilon_{(1,\upsilon,\varrho)}(a_r, b_s)} r^p. \end{aligned}$$

Similarly, we can prove that

$$|h'(z)| \geq pr^{p-1} - \frac{(p+1)|d|}{[p+\eta]\left[\frac{p+\lambda}{p}\right]^m \Upsilon_{(1,\upsilon,\varrho)}(a_r, b_s)} r^p.$$

This finishes the theorem assertion.

4 Acknowledgment

The work here is supported by Universiti Kebangsaan Malaysia under the grant no: GUP-2019-032.

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, *Internat. J. Math. Math. Sci.*, **27,** (2004), 1429–1436.

[2] M. K. Aouf, T. M. Seoudy, Some preserving sandwich results of certain operator containing a generalized Mittag-Leffler function, *T.M. Bol. Soc. Mat. Mex.*, (2018). https://doi.org/10.1007/s40590-018-0224-8.

[3] A. A. Attiya, Some applications of Mittag-Leffler function in the unit disk, *Filomat*, **30,** (2016), 2075–2081.

[4] D. Bansal, J. K. Prajapat, Certain geometric properties of the Mittag-Leffler functions, *Complex Var. Elliptic Eq.,* **61,** no. 3, (2016), 338-350.

[5] B. C. Carlson, D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM J. Math. Anal.*, **15,** (1984), 737–745.

[6] J. Dziok, H. S. Srivastava, Classes of analytic functions associated with the generalised hypergeometric function, *Appl. Math. Comput.,* **103,** (1999), 1–13.

[7] S. Elhaddad, H. Aldweby, M. Darus, On certain subclasses of analytic functions involving differential operator, *Jñānābha*, **48,** no. 1, (2018), 53–62.

[8] S. Elhaddad, H. Aldweby, M. Darus, Neighborhoods of certain classes of analytic functions defined by a generalized differential operator involving Mittag-Leffler function, *Acta Universitatis Apulensis*, **18,** no. 55, (2018), 1–10.

[9] S. Elhaddad and M. Darus, On meromorphic functions defined by a new operator containing the MittagLeffler function, *Symmetry*, **11,** no. 2, (2019), 210.

[10] S. Elhaddad, H. Aldweby, M. Darus, Majorization properties for subclass of analytic p-valent functions associated with generalized differential operator involving Mittag-Leffler function, *Nonlinear Functional Analysis and Applications*, **23,** no. 4,(2018), 743–753.

[11] I. S. Gupta, L. Debnath, Some properties of the Mittag-Leffler functions. *Integral Transform. Spec. Funct.*, **18,** no. 5, (2007), 329-336.

[12] J. E. Hohlov, Operators and operations on the class of univalent functions, *Izvestiya Vysshikh Uchebnykh Zavedenii Matematika*, **10,** (1978), 83–89.

[13] G. M. Mittag-Leffler, Sur la nouvelle fonction $E_\alpha(x)$, *CR Acad. Sci. Paris* **137,** no. 2, (1903), 554–558.

[14] G. M. Mittag-Leffler, Sur la representation analytique d'une branche uniforme d'une fonction monogene, *Acta Mathematica* , **29,** no. 1, (1905), 101–181.

[15] S. Ruscheweyh, New criteria for univalent functions, *Proceedings of the American Mathematical Society*, **49,** (1975), 109–115.

[16] D. Răducanu, Third-Order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions, *Mediterr. J. Math.*, **14,** (2017), 18 pages.

[17] D. Răducanu, Partial sums of normalized Mittag-Leffler functions, *An. Şt.Univ. Ovidius Constanţ,* **25,** no. 2, (2017), 8 pages.

[18] G. S. Sălăgean, Subclasses of univalent functions, *Lecture Notes in Math.*, *Springer-Verlag, Heidelberg*, **1013,** (1983), 362–372.

[19] J. Salah, M. Darus, A note on generalized Mittag-Leffler function and application, *Far East Journal of Mathematical Sciences(FJMS)*, **48,** no. 1, (2011), 33–46

[20] H. M. Srivastava, B. A. Frasin, V. Pescar, Univalence of integral operators involving Mittag- Leffler functions, A*ppl. Math. Inf. Sci.*, **11,** no. 3, (2017), 635–641.

[21] H. M. Srivastava, Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, *Appl. Math. Comput.*, **211,** (2009), 198–210.

[22] A. Wiman, Über den fundamentalsatz in der teorie der funktionen $E_\alpha(x)$, *Acta Mathematica*, **29,** no. 1, (1905), 191–201.