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On the essential singularities for positive currents

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Abstract

In this paper we study the existence of essential singularities for a positive plurisubharmonic and plurisupeharmonic current T defined outside a Cauchy-Riemann sub-manifold A of $\mathbb{C}^n$. We prove first the existence of a current S greater that T with some information on its dd^c. Then we give an example of a current which has an essential singularities on a point of A.

1 Introduction

The study of the extension of currents over a closed subset was the subject of several works in different cases of currents (closed, plurisubharmonic psh, plurisuperharmonic prh, and so on). The properties of the trivial extension was the main goal of these works. One of the well known problems in this

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direction is the existence of the essential singularities for positive current T defined on an open set Ω of $\mathbb{C}^n$ and outside a Cauchy-Riemann (CR) sub-manifold A. In the case of an analytic set X of pure dimension 1 and $A = \mathbb{R}^n$, H. Alexander[1], J. Beker[3] and B. Shiffman[11], proved that X has no essential singularities. This result was extended by Chirka [4, 5] for the case where A is a real analytic set of Ω. In the case of currents, there remains the problem of singularities for a current S defined on Ω greater than T (i.e., $S - T \geq 0$) which inherits the same characteristics of T. This problem was studied by many authors (see [10],[12],[8]). The main goal of their works was to find sufficient conditions on T and A so that the current T has no essential singularities across all the points of A. In 1971, King [10] proved that the current of integration over an analytic set X defined on $\Omega \smallsetminus \mathbb{R}^n$ has no essential singularity. This result has been extended by Sibony [12], in 1985, for positive closed currents of bidimension $(1, 1)$, where A is a totally real analytic manifold and Ω is an open set invariant under the complex conjugation. Sibony constructed a global "closed" current S greater than T. This work has been improved by Giret [8] in higher dimension but with a condition on dT.

In this paper, we treat the same problem but in the case of positive plurisubhamronic (psh) and plurisuperharmonic (prh) currents of bidimension (p, p) defined on $\Omega \smallsetminus A$, where A is a Cauchy-Riemann sub-manifold of complex dimension k. More precisely, when $dd^cT \geq 0$ (respectively $dd^cT \leq 0$) we study the existence of a current S defined on Ω such that $dd^cS \geq 0$ (respectively $dd^cS \leq 0$).

In the case $k \leq p - 2$, the trivial extension $\widetilde{T}$ of T exists and we can take $S = \widetilde{T}$ to solve the problem. But in the limit case $k = p - 1$, we construct first a current S greater than T but with some information on its dd^c, this represents the first result in this paper which is given by the following theorem

Theorem 1. *(Main Theorem)*

Let Ω be an open subset of $\mathbb{C}^n$ invariant under the complex conjugation and T a positive current of bidimension $(1, 1)$ on $\Omega \smallsetminus \mathbb{R}^n$ such that $dd^cT \leq 0$. There exist a positive current S of bidimension $(1, 1)$ defined on Ω and a current R supported by $\mathbb{R}^n \cap \Omega$ such that:

$$T_{|\Omega \smallsetminus \mathbb{R}^n} \leq S_{|\Omega \smallsetminus \mathbb{R}^n} \quad \text{and} \quad dd^cS \leq -R - \sigma_\star(dd^c\widetilde{T}).$$

In the last part we show that such current S may not exist in the general case if we want dd^cS to have the same sign as dd^cT. This is the second result which we establish establish in the following example:

Take $T = h[X]$, where

$$X := \{(z_1, 0) \in \mathbb{C}^2 \ / \ |z_1| < 1\} \ , \ h(z_1, z_2) = \Re e \left(\frac{1}{1 - z_1} \right)$$

and

$$A := \{(z_1, 0) \in \mathbb{C}^2, \ |z_1| = 1\}.$$

We show that this current has an essential singularity on the point $z_0 = (1, 0)$.

2 Singularities for currents

Throughout this paper Ω will be a domain of $\mathbb{C}^n$ $\iota : \mathbb{R}^n \longrightarrow \mathbb{C}^n$, the canonical injection and σ the complex conjugation given by $\sigma(z) = \overline{z}$.

We give first the definition of singularities for analytic sets.

Definition 1. Let A be a closed subset of Ω, X a complex analytic subset of $\Omega \setminus A$ of pure dimension equal to p and $a \in A \cap \overline{X}$. The point a is called a non-essential singularity for X if there exist a neighborhood U of a in Ω and a complex analytic subset Y of U such that $dim\ Y = p$ and

$$X \cap (U \setminus A) \subset Y \cap (U \setminus A).$$

Remark 1. 1. A necessary condition on the analytic set X not to have an essential singularity on all points of A, is that X must have a local finite mass near A. However, this condition is not sufficient as we can see in the following example:

$$X = \{(x, y, z) \in \mathbb{C}^3; \ |x| \leq \Re e(y) \ and \ z = \sin(\frac{x^2}{y})\}.$$

and $A = \{(x, y, z) \in \mathbb{C}^3; \ |x| = \Re e(y)\}$.

In fact, the set A is a real analytic sub-manifold of $\mathbb{C}^3$ of real dimension 5 and of Cauchy-Riemann dimension equal to 2 and X is a complex analytic set of $\mathbb{C} \setminus A$ of complex dimension equal to 2.

2. If $\overline{X}$ is a complex analytic set of Ω than one can take $Y = \overline{X}$.

We extend this definition to positive closed currents as follows:

Definition 2. Let A be a closed subset of Ω, T a positive closed current defined on $\Omega \setminus A$ of bidimension (p,p) and $a \in A$. We say that a is a non-essential singularity for T if there exist a neighborhood U of a in Ω and a positive closed current S defined on U of bidimension (p,p) such that

$$T_{|U \setminus A} \leq S_{|U \setminus A}.$$

Remark 2.

1. The Definition 2 is a generalization of Definition 1.

   In fact, let X be an analytic set of $\Omega \setminus A$ of pure dimension p and a a non-essential singularity for X. The current $T := [X]$ is a positive closed current of bidimension (p,p) on $\Omega \setminus A$. If $a \in A$ and U the neighborhood of a given by Definition 1, then taking $S = [Y]$ we have $T_{|U \setminus A} \leq S_{|U \setminus A}$.

   Conversely, if T is a positive closed current of bidimension (p,p), U, A and S are as in the Definition 2, then

   $$X \cap (U \setminus A) = E_1(T_{|U \setminus A}) \subset E_1(S_{|U \setminus A}) \subset E_1(S),$$

   where E_1 is the Siu [13] set. Now, using Siu [13] Theorem, the set $E_1(S)$ is a complex analytic set of pure dimension p that contains $X \cap (U \setminus A)$.

2. One can extend Definition 2 to the case of psh (respectively prh) currents and we just require the current S to be psh (respectively prh).

Definition 3. Let A be a closed sub-manifold in Ω of class $\mathcal{C}^1$ and $T_z(A)$ the tangent space to A at z. We say that A is a Cauchy-Riemann sub-manifold, if for all $z \in A$ the complex vector space

$$H_z(A) = T_z A \cap iT_z A$$

is of complex dimension equal to k, independent of z.

For more details about CR manifolds one can refer to [2, 4]. In what follows the subset A will be a CR sub-manifold of complex dimension k. In 1971, King [10] proved that if X is an analytic subset of $\Omega \setminus \mathbb{R}^n$, then for all $a \in X$ there exist a neighborhood U of a and an analytic subset Y of U such that $(U \cap X) \setminus \mathbb{R}^n \subset (U \cap Y) \setminus \mathbb{R}^n$. So an analytic subset does not have any essential singularity. In the case of positive closed currents, we have the following theorem due to ElMir [7].

Theorem 2. *[7]*

Let T be a positive closed current of bidimension (p,p) on $\Omega \setminus A$, where A is a smooth Cauchy-Riemann sub-manifold of complex dimension k.

- *If $p \geq k+1$, then $\widetilde{T}$ exists but is not necessary closed.*
- *If $p \geq k+2$, then $\widetilde{T}$ is a positive closed current.*

As a consequence of the previous theorem, if $p \geq k+2$, then T has no essential singularity (we just take $S = \widetilde{T}$). In the case $p = k+1$, such a reason is not true in general. But Sibony [12] proved that every positive closed current of bidimension $(1,1)$, defined outside a totally real manifold invariant by the complex conjugation σ, has no essential singularity on this manifold. More precisely, Sibony showed the following result:

Theorem 3. *[12]*
Let $A = \mathbb{R}^n \times \{0\} \subset \mathbb{C}^n$. Let Ω be an open subset of $\mathbb{C}^n$ invariant by σ and let T be a positive closed current of bidimension $(1,1)$ defined $\Omega \smallsetminus A$. For all $a \in A$ there exist a neighborhood U of a and a positive closed current S on U such that

$$S \geq T \qquad \text{on} \qquad U \smallsetminus A.$$

The previous result has been improved by Giret by taking a weaker condition on dT instead of the closeness condition. Giret proved the following theorem:

Theorem 4. *[8]*
Let A be a real analytic Levi-flat CR-manifold of Ω with complex dimension equal to $p-1$ and let T be a positive current of bidimension (p,p) defined on $\Omega \smallsetminus A$. Assume that the current $\widetilde{dT}$ exists and $b(\widetilde{dT})$ is a current od bidimension $(p-1,p-1)$. Then, for all $a \in A$, there exist a neighborhood U of a in Ω and a positive current S of bidimension (p,p) on U such that

$$T_{|U\smallsetminus A} \leq S_{|U\smallsetminus A} \quad \text{and} \quad bS = (\widetilde{bT} + (-1)^p \lambda_*(\widetilde{bT}))_{|U},$$

where λ is a proper application on U.

3 Main result (Case of plurisubharmonic and plurisuperharmonic current)

In this paper we study the problem of singularities in the case of plurisubharmonic (respectively plurisuperharmonic) currents. So we will solve the following problem

(P): 'If we assume that $dd^cT \leq 0$, is there a positive current S defined on Ω such that $dd^cS \leq 0$ and $S \geq T$ on $\Omega \smallsetminus A$?"

To solve the previous problem, we will establish some lemmas:

Lemma 1. *Let φ be a positive smooth differentiable form of degree r on $\mathbb{R}^n$. Then, for all $(p,q) \in \mathbb{N}^2$ such that $p+q=r$, there exists a smooth differentiable form ψ of bidegree (p,q) on $\mathbb{C}^n$, such that*

$$\iota^\star \psi = \varphi$$

Proof. A positive smooth differentiable form of degree r can be written as follows

$$\varphi = \sum_{|I|=r} \varphi_I dx_I$$

Let $p,q \in \mathbb{N}$ such that $p+q=r$. If $I=(i_1,...,i_r)$ is a multi-index of length equal to r, then we extend the form dx_I as follows

$$dx_I = (dz_{i_1} \wedge ... \wedge dz_{i_p})_{|\mathbb{R}^n} + (d\overline{z}_{i_{p+1}} \wedge ... \wedge d\overline{z}_{i_{p+q}})_{|\mathbb{R}^n}.$$

Then we obtain the required form ψ. □

Proposition 1. *If T is a positive current of dimension r on $\mathbb{R}^n$, then for all $(p,q) \in \mathbb{N}^2$ such that $p+q=r$, one has*

$$Supp\ T = Supp\ T_{p,q},$$

where $T_{p,q}$, is the component of bidimension (p,q) of the current $\iota_\star T$.

Proof. Let T be a current on $\mathbb{R}^n$ of dimension r. Assume that there exist $(p,q) \in \mathbb{N}^2$ such that $p+q=r$ and $T_{p,q}$ vanishes. If we take $\varphi \in \mathcal{D}_r(\mathbb{R}^n)$, then by the previous lemma,there exists a form $\psi \in \mathcal{D}_{p,q}(\mathbb{C}^n)$ such that $\varphi = \iota^\star \psi$. It follows that

$$\langle T, \varphi \rangle = \langle T, \iota^\star \psi \rangle = \langle \iota_\star T, \psi \rangle = \langle (\iota_\star T)_{p,q}, \psi \rangle = 0.$$

□

Note that the problem (P) is solved in the case $p \geq k+2$ by Dabbek, ElKhadhra and El Mir [6]:

Proposition 2. *Let A be a CR sub-manifold of dimension k on Ω, and T a positive current of bidimension (p,p) on $\Omega \setminus A$ such that $dd^cT \leq 0$. Then one has*

- *If $p \geq k+1$, then $\widetilde{T}$ exists.*
- *If $p \geq k+2$, then $\widetilde{dd^cT} = dd^c\widetilde{T}$.*

In the case $p = k + 1$, we give first a current S (different from $\widetilde{T}$) greater than T with some information on its dd^c. This represents the first result in this paper.

Theorem 5. *(Main Theorem)*
Let Ω be an open subset of $\mathbb{C}^n$ invariant under the complex conjugation and T a positive current of bidimension $(1,1)$ on $\Omega \smallsetminus \mathbb{R}^n$ such that $dd^cT \leq 0$. There exist a positive current S of bidimension $(1,1)$ defined on Ω and a current R supported by $\mathbb{R}^n \cap \Omega$ such that:

$$T_{|\Omega\smallsetminus\mathbb{R}^n} \leq S_{|\Omega\smallsetminus\mathbb{R}^n} \quad \textit{and} \quad dd^cS \leq -R - \sigma_\star(dd^c\widetilde{T}).$$

Proof. Using [6], the current T has a finite mass on a neighborhood of all points of $\mathbb{R}^n \cap \Omega$, so its trivial extension $\widetilde{T}$ on Ω exists. The current $-\sigma_\star T$ is of bidimension $(1,1)$ on $\Omega \smallsetminus \mathbb{R}^n$. Let's prove that it is a positive current. For this, let $\varphi \in \mathcal{D}(\Omega \smallsetminus \mathbb{R}^n)$ be a positive function and α a differentiable form of bidegree $(1,0)$ on $\Omega \smallsetminus \mathbb{R}^n$. Then

$$\begin{aligned}
\langle -\sigma_\star T \wedge i\alpha \wedge \overline{\alpha}\ ,\ \varphi\rangle &= \langle -\sigma_\star T, \varphi i\alpha \wedge \overline{\alpha}\rangle \\
&= -\langle T\ ,\ \sigma^\star(\varphi\ i\ \alpha \wedge \overline{\alpha})\rangle \\
&= -\langle T\ ,\ i(\varphi \circ \sigma)\sigma^\star\alpha \wedge \sigma^\star\overline{\alpha}\rangle \\
&= -\langle T \wedge\ i\ \sigma^\star\alpha \wedge \sigma^\star\overline{\alpha}\ , \varphi \circ \sigma\rangle \\
&= \langle T \wedge i\sigma^\star\overline{\alpha} \wedge \overline{\sigma^\star\overline{\alpha}}\ ,\ \varphi \circ \sigma\rangle \geq 0.
\end{aligned}$$

We have used the fact that $\varphi \circ \sigma$ is positive, $\sigma^\star\overline{\alpha}$ is of bidegree $(1,0)$ and T is positive.

We define the current S as follows

$$S := \widetilde{T} - \sigma_\star\widetilde{T}.$$

The current $\sigma_\star\widetilde{T} - \widetilde{\sigma_\star T}$, is a current of bidegree $(1,1)$ with support in $\mathbb{R}^n \cap \Omega$. So, by Proposition 1, it vanishes (its components of bidegree $(0,2)$ and $(2,0)$ are equal to 0). As a result $\sigma_\star\widetilde{T} = \widetilde{\sigma_\star T}$. So $S = \widetilde{T - \sigma_\star T}$. Consequently,

$$T_{|\Omega\smallsetminus\mathbb{R}^n} \leq S_{|\Omega\smallsetminus\mathbb{R}^n}.$$

Now we will compute dd^cS.

We have

$$dd^cS = dd^c\widetilde{T} - \sigma_\star(dd^c\widetilde{T})$$

If we take the current $R := \widetilde{dd^cT} - dd^c\widetilde{T}$, then R is supported by $\mathbb{R}^n \cap \Omega$ and one has

$$dd^cS = -R + \widetilde{dd^cT} - \sigma_\star(dd^c\widetilde{T})$$

Hence

$$dd^cS \leq -R - \sigma_\star(dd^c\widetilde{T}).$$

□

4 Example of current with an essential singularity

The problem (P) is still open for the case of plurisuperhamonic and plurisubharmonic current although the Theorem 5 gives a current S with some information on its dd^c. In this section we give an example of a current to prove that the answer of the problem is, in general, false for both cases. The example of such current is given by the following theorem.

Theorem 6. *Let*

$$X := \{(z_1,0) \in \mathbb{C}^2 \ / \ |z_1| < 1\}\ ,\ h(z_1,z_2) = \Re e\left(\frac{1}{1-z_1}\right)$$

and

$$A := \{(z_1,0) \in \mathbb{C}^2,\ |z_1| = 1\}.$$

The current $T := h[X]$ *is a pluriharmonic current of bidimension* $(1,1)$ *on* $\mathbb{C}^2 \smallsetminus A$ *which has the point* $z_0 = (1,0)$ *as singularity.*

Proof.

Case of psh current

Assume that there exists a positive psh current S on $\mathbb{C}^2$ such that $T \leq S$ on $\mathbb{C}^2 \smallsetminus A$. Let $M := \{(z_1,z_2) \in \mathbb{C}^2; z_2 = 0\}$, H the restriction of S on $\mathbb{C}^2 \smallsetminus M$ and $\widetilde{H} = S - \mathbb{1}_{\mathrm{M}}\mathrm{S}$. Thanks to Dabbek, Elkhadhra and Elmir [6] the current $\mathbb{1}_{\mathrm{M}}\mathrm{S}$ is a $\mathbb{C}-$normal positive current, it follows that

$$dd^c\widetilde{H} = dd^cS - dd^c(\mathbb{1}_{\mathrm{M}}\mathrm{S}).$$

Using Theorem 2 in [6], there exist a positive current R supported on M such that

$$\begin{aligned} R &= \widetilde{dd^cH} - dd^c\widetilde{H} \\ &= (dd^cS - \mathbb{1}_{\mathrm{M}}\mathrm{dd^cS}) - \mathrm{dd^c}\widetilde{\mathrm{H}} \\ &= dd^c(\mathbb{1}_{\mathrm{M}}\mathrm{S}) - \mathbb{1}_{\mathrm{M}}\mathrm{dd^cS}. \end{aligned}$$

So the current $dd^c(\mathbb{1}_{\mathrm{M}}\mathrm{S}) = \mathrm{R} + \mathbb{1}_{\mathrm{M}}\mathrm{dd^cS}$ is positive. It follows that $\mathbb{1}_{\mathrm{M}}\mathrm{S}$ is a positive psh current of bidimension $(1,1)$ supported on M. As $\dim M = 1$, by the support theorem with respect for $\mathbb{C}$-flats current, there exists a psh function v such that $\mathbb{1}_{\mathrm{M}}\mathrm{S} = v[M]$. For all $p \geq 1$, one has $v \in L^p_{loc}(X)$ and $h \leq v$. We get a contradiction since $h \notin L^p_{loc}(\vartheta(1,0))$ for $p \geq 2$.

Case of prh current

Assume that there exists a positive prh current S on $\mathbb{C}^2$ such that $T \leq S$ on $\mathbb{C}^2 \smallsetminus A$. Then the current $\mathbb{1}_{\mathrm{M}}\mathrm{S}$ is positive with support on M and $dd^c(\mathbb{1}_{\mathrm{M}}\mathrm{S}) = \mathrm{R} + \mathbb{1}_{\mathrm{M}}\mathrm{dd^c}, \mathrm{S}$ where R is a positive closed current and the current $\mathbb{1}_{\mathrm{M}}\mathrm{dd^cS}$ is negative and closed (See El Mir [7]). So the current $dd^c(\mathbb{1}_{\mathrm{M}}\mathrm{S})$ is the difference of two positive closed currents. Hence there exists a positive function $u \in L^p_{loc}(X)$ for all $p \geq 2$ such that $\mathbb{1}_{\mathrm{M}}\mathrm{S}{=}u[M]$. We get a contradiction to the fact that $h \leq u$. □

Open Problem:

We have proved by the example established in the above theorem that the current S does not exist in the general case if we want $dd^cS \leq 0$. So the problem is still open in the following sense

Under the same assertion of theorem 5, are there a current S and a (positive) current R such that:

$$T_{|\Omega\smallsetminus\mathbb{R}^n} \leq S_{|\Omega\smallsetminus\mathbb{R}^n} \quad and \quad dd^cS \leq R?$$

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