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On Existence and invariance of sphere, of solutions of constrained evolution equation

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Abstract

Hilbert manifolds are infinite dimensional manifolds modeled on Hilbert spaces. The aim of this study is twofold. First, we study the existence of the solution of a constrained nonlinear initial value problem involving a diagonal operator, with values on a particular submanifold (unit sphere) in a Hilbert space. Secondly, we investigate the invariance of the submanifold; i.e., we find a sufficient condition for the solutions to stay on the submanifold.

1 Introduction

The motivation for this work comes from the projected parabolic gradient flows studied by Rybka [24] and Cafferelli, Lin [16]. Rybka studied the heat equation in $L^2(\mathcal{O})$ projected on a manifold $\mathcal{M}$, where

$$\mathcal{M} = \left\{ u \in L^2(\mathcal{O}) \cap C(\mathcal{O}) : \int_\Omega u^k(x)dx = C_k, k = 1, 2, ..., N \right\},$$

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and D be a bounded, connected region in $\mathbb{R}^2$. Rybka proved the global existence and uniqueness of the solution of the following projected heat equation,

$$\begin{cases} \frac{du}{dt} = \Delta u - \sum_{k=1}^{N} \lambda_k u^{k-1} & \text{in } \Omega \subset \mathbb{R}^2 \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, & u(0,x) = u_0 \end{cases} \tag{1.1}$$

where $\lambda_k = \lambda_k(u)$ are such that u_t is orthogonal to Span $\left\{u^{k-1}\right\}$ for a more regular initial data. He also showed that the solutions to (1.1) converge to a steady state as time tends to ∞.
Secondly, in [16], Cafarelli and Lin proved global existence and uniqueness of energy-conserving solution to the heat equation. Their main result was to prove the strong convergence of the solutions of these perturbed systems to some weak-solutions of the limiting constrained non-local heat flows of maps into a singular space.
Recently, in [7], Zdzisaw Brzeźniak, Gaurav Dhariwal and Mauro Mariani studied 2D Navier–Stokes equations with a constraint forcing the conservation of the energy of the solution. They proved the existence and uniqueness of a global solution for the constrained Navier-Stokes equation on $\mathbb{R}^2$ and $\mathbb{T}^2$, by a fixed point argument. They also showed that the solution of the constrained equation converges to the solution of the Euler equation as the viscosity ν vanishes.
In this paper, we consider a problem which links the aforementioned works. Some of the classical and modern references on partial differential equations on manifolds are [3], [4], [8], [9], [10], [15], [17], [18]. An interesting paper is the recent work by Irem Akbulut and Cemil Tunç [20].

Let us assume that $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is an abstract Hilbert Space and let $\mathcal{M}$ be a unit sphere in it; i.e.,

$$\mathcal{M} = \left\{u \in \mathcal{H} : |u|_{\mathcal{H}} = 1\right\}.$$

Assume that f denotes a vector field on $\mathcal{H}$ such that there exists a unique solution of following class of constrained initial value problems (comprising of an abstract evolution equation)

$$\begin{aligned} \frac{dx}{dt} &= f\left(x\left(t\right)\right), t \in [0,\infty) \\ x(0) &= x_{0,} \end{aligned} \tag{1.2}$$

for every initial data $x \in \mathcal{H}$.

It is interesting to observe that the trajectories $(x(t))_{t\geq 0}$ of the solution of above initial value problem do not necessarily satisfy the constraint to stay on the sphere $\mathcal{M}$ even if we take $x_0 \in \mathcal{M}$. One of the key reasons that the vector field f involved in the above initial value problem is not necessarily a tangent vector to $\mathcal{M}$; i.e., it does not satisfy

$$f\,(x) \in T_x\mathcal{M}, \text{ for } x \in D(f) \cap \mathcal{M}. \tag{1.3}$$

To turn a vector field f into a tangent vector on $\mathcal{M}$ at point x, We propose to replace f with a new F in such a way that such a modification becomes tangent to $\mathcal{M}$ and satisfies the property (1.3). The modification that we propose is to orthogonally project the vector field f on $\mathcal{M}$ at x. For this we are going to use the orthogonal projection $\pi_x : \mathcal{H} \to T_x\mathcal{M}$ described by:

$$\pi_x(y) = y - \langle y, x\rangle\, x.$$

By applying the orthogonal projection to vector field f we infer that

$$F(x) \equiv \pi_x(f) = f - \langle f, x\rangle\, x \tag{1.4}$$

One can easily see that for every $x \in \mathcal{M} \cap D(f)$, we have

$$\langle F(x), x\rangle = \langle \pi_x(f), x\rangle = 0$$

Hence F satisfies property (1.3). We will verify the last fact in detail in section 4.

It is worth noting that for the existence of the solution (local at least) of the initial value problem, it is important that f must be Lipschitz. Moreover, whenever f is Lipschitz and $D(f) = \mathcal{H}$, its modification F (i.e. the orthogonal projection of f) will also be globally Lipschitz and $D(F) = \mathcal{H}$. The satisfaction of Lipschitz property guarantees that the modified equation

$$\begin{aligned} \frac{dx}{dt} &= F\left(x\left(t\right)\right), t \geq 0 \\ x(0) &= x_0 \end{aligned} \tag{1.5}$$

has at least a local solution for every $x \in \mathcal{M}$. In addition, this solution satisfies the invariance property; i.e., this solution stays on $\mathcal{M}$ whenever $x \in \mathcal{M}$.

The key question that we are going to address now is that what if the vector field f is not globally defined; i.e., $D(f) \subset \mathcal{H}$? Are we going to have invariance of the manifold? This aspect is completely unexplored. We will study this invariance problem in a very particular case where $f(x) = Ax$ (which is not globally defined; i.e. $D(f) \subset \mathcal{H}$), and $\mathcal{H} = \ell^2(\mathbb{R})$, where A is discrete diagonal operator. Thus we are interested in the existence and invariance of sphere, for the solutions of following initial values constrained problem

$$\begin{aligned} \frac{dx}{dt}(t) &= Ax(t), t \geq 0 \\ x(0) &= x_0, \end{aligned} \tag{1.6}$$

where $x_0 \in \mathcal{M} = \{u \in \mathcal{H} : |u|_{\mathcal{H}} = 1\}$.

2 Notation and preliminaries

Definition 2.1. *[29] Let $\ell^2(\mathbb{R})$ denote the space of the all infinite real sequences $u = (u_i)_{i=1}^n$ such that $\sum_{i=1}^{\infty} u_i^2 < \infty$. The norm on $\ell^2(\mathbb{R})$ is defined in the following manner,*

$$\|u\| := \sqrt{\sum_{i=1}^{\infty} u_i^2}.$$

Definition 2.2. *[29](Continuous Embedding) Let X and Y be normed spaces. Then a mapping $I : X \to Y : x \to x$ is a continuous embedding if the operator I is continuous; i.e., $\|x\|_Y \leq N \|x\|_X$, where N can be taken as $\|I\|_{\mathfrak{L}(X,Y)}$.*

Definition 2.3. *[29](Bochner Measurable) Let a Banach space Y and a bounded interval $I \subset \mathbb{R}$ be given. Then a function $x : I \to Y$ is a Bochner Measurable if it is point-wise limit of a sequence $\{x_i\}_{i\in\mathbb{N}}$ of simple functions; i.e., $x_i(t) \to x(t)$ for all $t \in I$.*

Definition 2.4. *[29](Bochner Integrable function) Let a Banach space Y and a bounded interval $I \subset \mathbb{R}$ be given. Then a function $x : I \to Y$ is called a Bochner Integrable functions if*

$$\lim_{i\to\infty} \int_0^T \|x(t) - x_i(t)\|_Y \, dt = 0$$

implies

$$\int_0^T x(t)dt = \lim_{i\to\infty} \int_0^T x_i(t),$$

where $\{x_i\}_{i\in\mathbb{N}}$ *is a sequence of simple functions.*

Definition 2.5. *[29](Bochner Spaces ,*$L^2(I,Y), C(I,Y)$*) Let a Banach space* Y *and a bounded interval* $I \subset \mathbb{R}$ *be given. Then a linear space Bochner Integrable functions* $x : I \to Y$ *satisfying* $\int_0^T \|x(t)\|_Y^p \, dt < \infty$*, for* $1 < p < \infty$*, is a Bochner Space, denoted by* $L^p(I;Y)$*, with norm is defined as follows*

$$\|x\|_{L^p(I;Y)} := \left(\int_0^T \|x(t)\|_Y^p \, dt\right)^{\frac{1}{p}}.$$

Definition 2.6. *[29] (Isometric Isomorphism) let* X *and* Y *be normed vector spaces. An operator* $T : X \to Y$ *which is both an isometry and an isomorphism, is called anisometric isomorphism.*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

Definition 2.7. *Let* $A : D(A) \to \mathcal{H}$ *with*

$$D(A) = \left\{ x \in \mathcal{H} : \sum_{k=1}^{\infty} \lambda_{k,}^2 x_k, x_k \in \mathbb{R} \right\},$$

where $0 < \lambda_1 < \lambda_2 ...$*, defined by*

$$Ax := (\lambda_1 x_1, \lambda_2 x_2, ...) ,$$

where $x \in D(A)$*, is called a diagonal operator.*

Lemma 2.8. *([26], Lemma III 1.2) Let* $\mathcal{V}, \mathcal{H}$ *be two Hilbert spaces,* $\mathcal{V}^*$ *and* $\mathcal{H}^*$ *be corresponding dual spaces. Assume that* $\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}^* \hookrightarrow \mathcal{V}^*$*, where the embedding i dense as well. If a function u belongs to* $L^2(0,T;\mathcal{V})$ *and its weak derivative* u' *belongs to* $L^2(0,T;\mathcal{V})$*, then u is a.e. equal to an absolutely continuous function from* $[0,T]$ *into* $\mathcal{H}$*, and the following equality holds in sense of distributions on* $(0,T)$*:*

$$\frac{d}{dt} \|u(t)\|_{\mathcal{H}}^2 = 2 \langle u', u \rangle . \tag{2.1}$$

Lemma 2.9. *[29] (Young's inequality) Let* $1 \leq p, q \leq \infty$ *and* r *satisfies* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$*. Suppose* $f \in L^p(\mathbb{R})$ *and* $g \in L^q(\mathbb{R})$ *then*

2.1 Important space and results

In this section, we assume that $\mathcal{V}$, $\mathcal{H}$ are two Hilbert spaces and $\mathcal{V}^*, \mathcal{H}^*$ be corresponding dual spaces respectively. Moreover, assume that embeddings $\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}^* \hookrightarrow \mathcal{V}^*$ are dense and continuous. Recall the following well-known sequence spaces:

$$\begin{aligned} \ell^2\mathbb{R}) &= \left\{ (x)_{i=1}^{\infty} \in \mathbb{R}^{\infty} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}, \\ \ell^2_{\lambda}(\mathbb{R}) &= \left\{ (x)_{i=1}^{\infty} \in \ell^2(\mathbb{R}) : \sum_{i=1}^{\infty} \lambda_i |x_i|^2 < \infty \right\}, \\ \ell^2_{\lambda^{-1}}(\mathbb{R}) &= \left\{ (x)_{i=1}^{\infty} \in \ell^2(\mathbb{R}) : \sum_{i=1}^{\infty} \lambda_i^{-1} |x_i|^2 < \infty \right\}. \end{aligned}$$

The aim of the section is to show that we can identify the $\mathcal{V}, \mathcal{H}, \mathcal{V}^*$ as $\ell^2_{\lambda}(\mathbb{R}), \ell^2(\mathbb{R})$ and $\ell^2_{\lambda^{-1}}(\mathbb{R})$ respectively, through isometric isomorphisms. We will do our analysis in sequence spaces and demonstrate that these spaces satisfy the assumptions of the Lemma 2.8.

Theorem 2.10. *Let $\mathcal{H}$ be a Hilbert Space with the standard basis $\{e_i\}_{=1}^{\infty}$. Then $\ell^2(\mathbb{R})$ is isometrically isomorphic to $\ell^2(\mathbb{R})$*

Proof. Since $\{e_k\}_{k=1}^{\infty}$ is a basis of $\mathcal{H}$, each $x \in \mathcal{H}$ can be uniquely written as $x = \sum_{i=1}^{\infty} x_i e_i$. Define a map $T : \ell^2(\mathbb{R}) \to \mathcal{H}$ by

$$T(x) = T\left((x_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} x_i e_i,$$

where $x = (x_i)_{i=1}^{\infty} \in \ell^2(\mathbb{R})$. Clearly T is linear and bijective. We will show that T is an isometry. If $x \in \ell^2(\mathbb{R})$, then

$$\|T(x)\|_{\mathcal{H}} = \left\| \sum_{i=1}^{\infty} x_i e_i \right\|_{\mathcal{H}} = \sum_{i=1}^{\infty} |x_i|^2 = \|x\|_{\ell^2(\mathbb{R})}$$

Thus T is an isometric isomorphism. ∎

Theorem 2.11. *If $\mathcal{V}$ be a Hilbert space and densely and continuously embedded in $\mathcal{H}$, then $\mathcal{V}$ is isometrically Iiomorphic to $\ell^2_{\lambda}(\mathbb{R})$.*

Proof. We begin by showing that $\mathcal{H} \simeq \ell^2_\lambda(\mathbb{R})$. Since $\mathcal{H}$ has basis $\{e_k\}_{k=1}^{\infty}$ each $x \in \mathcal{H}$ can be uniquely written as $x = \sum_{i=1}^{\infty} x_i e_i$. Define a map $T : \ell^2_\lambda(\mathbb{R}) \to \mathcal{H}$ by

$$T(x) = T\left((x_i)_{i=1}^{\infty}\right) = \sum_{i=1}^{\infty} \lambda_i x_i e_i,$$

Next we show that T is an isometry.

$$\|T(x)\|_{\mathcal{V}} = \left\|\sum_{i=1}^{\infty} x_i e_i\right\|_{\mathcal{V}} = \sum_{i=1}^{\infty} \lambda_i |x_i|^2 = \|x\|_{\ell^2_\lambda(\mathbb{R})}.$$

Thus $\mathcal{V}$ is isometrically isomorphic to $\ell^2_\lambda(\mathbb{R})$. ∎

Theorem 2.12. *Let $\mathcal{V}'$ be the Dual space of a Hilbert Space $\mathcal{V}$. Then $\mathcal{V}'$ is isometrically isomorphic to $\ell^2_{\lambda^{-1}}(\mathbb{R})$.*

Proof. To prove that $\mathcal{V}'$ is isometrically isomorphic to $\ell^2_{\lambda^{-1}}(\mathbb{R})$, it is sufficient to show that the dual of $\ell^2_\lambda(\mathbb{R})$ is $\ell^2_{\lambda^{-1}}(\mathbb{R})$. To do so, let $x \in \ell^2_{\lambda^{-1}}(\mathbb{R})$. Define $T_x : \ell^2_\lambda(\mathbb{R}) \to \mathbb{R}$ by

$$T_x(y) = \sum_{i=1}^{\infty} x_i y_i, \text{ where } y \in \ell^2_\lambda(\mathbb{R}),$$

Using Cauchy-Schwartz inequality,

$$\begin{aligned} |T_x(y)| &= \left|\sum_{i=1}^{\infty} x_i y_i\right| \leq \left(\sum_{i=1}^{\infty} \lambda_i |y_i|^2\right)^{1/2} \left(\sum_{i=1}^{\infty} \frac{|x_i|^2}{\lambda_i}\right)^{1/2} \\ &= \|y\|_{\ell^2_\lambda(\mathbb{R})} \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})} < \infty \end{aligned} \tag{2.2}$$

which shows that $T_x \in \left(\ell^2_\lambda(\mathbb{R})\right)^*$. Linearity of T_x is obvious. To show T_x is bounded we can use inequality (2.2),

$$\begin{aligned} |T_x(y)| &\leq \|y\|_{\ell^2_\lambda(\mathbb{R})} \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})} \\ \sup_{y \neq 0, y \in \ell^2_\lambda(\mathbb{R})} \left(\frac{|T_x(y)|}{\|y\|_{\ell^2_\lambda(\mathbb{R})}}\right) &\leq \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})} \\ \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} &\leq \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})}. \end{aligned} \tag{2.3}$$

Hence $T_x \in \left(\ell^2_\lambda(\mathbb{R})\right)^*$. This observation allows us to define a map $\phi : \ell^2_{\lambda^{-1}}(\mathbb{R}) \to \left(\ell^2_\lambda(\mathbb{R})\right)^*$ by

$$\phi(x) = T_x, x \in \ell^2_{\lambda^{-1}}(\mathbb{R}).$$

We show that ϕ is an isometric isomorphism. ϕ is well defined because for any $x, y \in \ell^2_{\lambda^{-1}}(\mathbb{R})$ such that $x = y$ we have $T_x = T_y$ and so $\phi(x) = \phi(y)$. In addition, linearity is evident. To show isometry it is sufficient to show that $\|T_x(y)\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} = \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})}$. In inequality (2) we have already shown that

$$\|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \leq \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})}.$$

It remains to show that $\|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \geq \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})}$. Let $y \in \ell^2_\lambda(\mathbb{R})$. Then $y = \sum_{i=1}^{\infty} y_i e_i$. Using linearity of T_x, we get

$$T_x(y) = T_x\left(\sum_{i=1}^{\infty} y_i e_i\right) = \sum_{i=1}^{\infty} y_i T_x(e_i) = \sum_{i=1}^{\infty} y_i \gamma_i,$$

where $\gamma_i = T_x(e_i)$. Now define $y_n = (y_i^n)_{n=1}^{\infty}$, where y_i^n is given by

$$\begin{aligned} y_i^n &= \frac{|\gamma_i|^2}{\gamma_i \lambda_i}, \text{if } i \leq n, \gamma_i \neq 0 \\ &= 0, \qquad \text{if } \ i > n. \end{aligned}$$

Consider the value T_x at y_i^n,

$$T_x(y_n) = \sum_{i=1}^{\infty} y_i^n \gamma_i = \sum_{i=1}^{n} \frac{|\gamma_i|^2}{\gamma_i \lambda_i} \gamma_i = \sum_{i=1}^{n} \frac{|\gamma_i|^2}{\lambda_i}. \tag{2.4}$$

Using continuity of T we get

$$T_x(y_n) \leq \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \left(\sum_{i=1}^{\infty} \lambda_i |y_i^n|^2\right)^{1/2} = \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \left(\sum_{i=1}^{\infty} \frac{|\gamma_i|^2}{\lambda_i}\right)^{1/2}$$

and now using (2.3) we get

$$\sum_{i=1}^{n} \frac{|\gamma_i|^2}{\lambda_i} \leq \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \left(\sum_{i=1}^{\infty} \frac{|\gamma_i|^2}{\lambda_i}\right)^{1/2}$$

Take $n \to \infty$ and divide by the last factor on both sides we get

$$\left(\sum_{i=1}^{\infty} \frac{|\gamma_i|^2}{\lambda_i}\right)^{1/2} \leq \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \text{ i.e. } \|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})} \leq \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*}.$$

Thus we conclude that $\|x\|_{\ell^2_{\lambda^{-1}}(\mathbb{R})} = \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*}$ and hence the dual of $\ell^2_\lambda(\mathbb{R})$ is $\ell^2_{\lambda^{-1}}(\mathbb{R})$. It remains to show that ϕ is surjective; i.e., for each $\eta \in \left(\ell^2_\lambda(\mathbb{R})\right)^*$ there exist $x \in \ell^2_{\lambda^{-1}}(\mathbb{R})$ such that $\phi(x) = \eta$ or $T_x = \eta$. For $n \in \mathbb{N}$, define a sequence $y_n = (y_i^n)$, where

$$\begin{aligned} y_i^n &= \frac{|\gamma_i|^2}{\lambda_i \gamma_i}, \text{if} i \leq n, \gamma_i \neq 0 \text{if} T_x(e_i) = \gamma_i \\ &= 0 \text{ otherwise.} \end{aligned}$$

Observe that $y_n = \sum_{i=1}^n y_i^n e_i$. So

$$T_x(y_n) = \sum_{i=1}^n y_i^n T_x(e_i) = \sum_{i=1}^n y_i^n \gamma_i = \sum_{i=1}^n \frac{|\gamma_i|^2}{\lambda_i}.$$

Therefore

$$\sum_{i=1}^n \frac{|\gamma_i|^2}{\lambda_i} = T_x(y_n) \leq \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \left(\sum_{i=1}^n \lambda_i |y_i^n|^2\right)^{1/2} = \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*} \left(\sum_{i=1}^n \frac{|\gamma_i|^2}{\lambda_i}\right)^{1/2}$$

which implies that

$$\left(\sum_{i=1}^n \frac{|\gamma_i|^2}{\lambda_i}\right)^{1/2} \leq \|T_x\|_{\left(\ell^2_\lambda(\mathbb{R})\right)^*}, \text{ forall } n.$$

It follows that $\sum_{i=1}^n \frac{|\gamma_i|^2}{\lambda_i} < \infty$ and so $(\gamma_i) \in \ell^2_{\lambda^{-1}}(\mathbb{R})$ as required. This completes the proof. ∎

3 Abstract Cauchy problem with diagonal operators

In this section we will show that the trajectories of the solution of the following abstract Cauchy operator with Diagonal operator satisfies the conditions of Lemma 2.8 . Consider

$$\begin{aligned} \frac{dx}{dt}(t) + Ax(t) &= f(t), t \geq 0 \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $x_0 \in V \simeq \ell^2_\lambda(\mathbb{R})$ and $f = (f_k) \in L^2(0, T; \mathcal{V})$. To do so, we show that $x = (x_k)_{k=0}^\infty \in L^2((0,T), \mathcal{V})$ and $x' \in L^2((0,T), \mathcal{V}')$, where $\mathcal{V} \simeq \ell^2_\lambda(\mathbb{R})$ and $\mathcal{V}' \simeq \ell^2_{\lambda^{-1}}(\mathbb{R})$. The solution of the above problem can be given as,

$$x(t) = x_0 + \int_0^t e^{-(t-s)A} f(s) ds,$$

which is equivalent to

$$x_k(t) = x_{0,k} + \int_0^t e^{-(t-s)\lambda_k} f_k(s) ds = x_{0,k} + (\varphi_k * f_k)(t).$$

Here

$$\begin{aligned} \varphi_k(t) &= e^{-t\lambda_k}, \text{if } t > 0 \\ &= 0, \quad\quad \text{if } t \leq 0 \end{aligned}$$

and

$$\begin{aligned} f_k(t) &= f_k(t), \text{if } t > 0 \\ &= 0, \quad\quad \text{if } t \leq 0 \end{aligned}$$

and

$$\begin{aligned} (\varphi_k * f_k)(t) &= \int_{-\infty}^t \varphi(t-s) f(s) ds = 0, \text{if } t \leq 0 \\ &= \int_{-\infty}^t \varphi(t-s) f(s) ds, \ \text{IF } t > 0. \end{aligned}$$

Let us begin by showing that $x = (x_k)_{k=0}^\infty \in L^2((0,T), \mathcal{V})$ and compute $\|x\|^2_{L^2((0,T),\mathcal{V})}$.

$$\begin{aligned} \|x\|^2_{L^2((0,T),\mathcal{V})} &= \int_0^T \|x(t)\|^2_{\mathcal{V}} dt \\ &= \int_0^T \left(\sum_{k=1}^\infty \lambda_k |x_k(t)|^2 \right) dt \\ &= \sum_{k=1}^\infty \lambda_k \int_0^T |x_k(t)|^2 dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \lambda_k \int_0^T |x_{0,k} + (\varphi_k * f_k)(t)|^2 \, dt \\
&\leq 2 \sum_{k=1}^{\infty} \lambda_k \int_0^T \left(|x_{0,k}|^2 + |(\varphi_k * f_k)(t)|^2 \right) dt \\
&\leq 2 \sum_{k=1}^{\infty} \lambda_k |x_{0,k}|^2 + \sum_{k=1}^{\infty} \lambda_k \int_0^T |(\varphi_k * f_k)(t)|^2 \, dt \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + \sum_{k=1}^{\infty} \lambda_k \|\varphi_k * f_k\|_{L^2(0,T)}
\end{aligned}
$$

Moreover, by using Young's inequality 2.9, for $p = 1, q = 2$ and $r = 2$, we get

$$
\begin{aligned}
\|x\|_{L^2((0,T),\mathcal{V})}^2 &\leq 2 \|x_0\|_{\mathcal{V}}^2 + 2 \sum_{k=1}^{\infty} \lambda_k \|\varphi_k(t)\|_{L^1(0,T)} \|f_k(t)\|_{L^2(0,T)}^2 \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + 2 \sum_{k=1}^{\infty} \lambda_k \left(\int_0^T e^{-t\lambda_k} dt \right)^2 \|f_k(t)\|_{L^2(0,T)}^2 \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + 2 \sum_{k=1}^{\infty} \lambda_k \left(\frac{1 - e^{-T\lambda_k}}{T\lambda_k} \right)^2 \|f_k(t)\|_{L^2(0,T)}^2 \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + \frac{2}{T} \sum_{k=1}^{\infty} \frac{\left(1 - e^{-T\lambda_k}\right)^2}{\lambda_k} \|f_k(t)\|_{L^2(0,T)}^2
\end{aligned}
$$

Using the elementary inequality, $(1 - e^x)^2 \leq x^2$, for all $x \leq 0$ and Tenolli's Theorem

$$
\begin{aligned}
\|x\|_{L^2((0,T),\mathcal{V})}^2 &\leq 2 \|x_0\|_{\mathcal{V}}^2 + \frac{1}{T} \sum_{k=1}^{\infty} \frac{T^2 \lambda_k^2}{\lambda_k} \|f_k(t)\|_{L^2(0,T)}^2 \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + T \sum_{k=1}^{\infty} \lambda_k \|f_k(t)\|_{L^2(0,T)}^2 \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + T \sum_{k=1}^{\infty} \int_0^T \lambda_k |f_k(t)|^2 \, dt \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + T \int_0^T \left(\sum_{k=1}^{\infty} \lambda_k |f_k(t)|^2 \right) dt \\
&= 2 \|x_0\|_{\mathcal{V}}^2 + T \int_0^T \|f_k(t)\|_{\mathcal{V}}^2 \, dt
\end{aligned}
$$

$$= \quad 2\left\|x_0\right\|_{\mathcal{V}}^2 + T\left\|f\right\|_{L^2(0,T;\mathcal{V})}^2$$

Since $x_0 \in V$ and $f \in L^2(0,T;\mathcal{V})$, $\|x\|_{L^2((0,T),\mathcal{V})} < \infty$; i.e., $x \in L^2(0,T;\mathcal{V})$. Next we show $x' \in L^2(0,T;\mathcal{V})$. Let us compute $\|x'\|_{L^2(0,T;\mathcal{V}')}$.

$$\begin{aligned}\|x'\|_{L^2(0,T;\mathcal{V}')}^2 &= \int_0^T \|x'(t)\|_{\mathcal{V}}^2\,dt = \int_0^T \left(\sum_{k=1}^{\infty} \frac{|x_k'(t)|^2}{\lambda_k}\right) dt = \int_0^T \left(\sum_{k=1}^{\infty} \frac{|(\varphi_k' * f_k)(t)|^2}{\lambda_k}\right) dt \\ &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_0^T |(\varphi_k' * f_k)(t)|^2\,dt = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \|\varphi_k' * f_k\|_{L^2(0,T)}^2\end{aligned}$$

Using Young's inequality for $p = 1, q = 2$ and $r = 2$,

$$\|x'\|_{L^2(0,T;\mathcal{V}')}^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \|\varphi_k'\|_{L^1} \|f_k\|_{L^2(0,T)} \tag{3.2}$$

Since

$$\varphi_k'(t) = \frac{d}{dt}\left(e^{-t\lambda_k}\right) = -\lambda_k e^{-t\lambda_k},$$

$$\begin{aligned}\|\varphi_k'\|_{L^1} &= \int_0^T |\varphi_k'(t)|\,dt = \int_0^T \left|-\lambda_k e^{-t\lambda_k}\right| dt = \lambda_k \int_0^T e^{-t\lambda_k} dt = \frac{1 - e^{-T\lambda_k}}{T}. \\ \|\varphi_k'\|_{L^1}^2 &\le \left(\frac{1 - e^{-T\lambda_k}}{T}\right)^2 \le T^2\lambda_k^2.\end{aligned}$$

Therefore, using last inequality in inequality (3.2), we infer,

$$\begin{aligned}\|x'\|_{L^2(0,T;\mathcal{V}')}^2 &\le \sum_{k=1}^{\infty} \frac{T^2\lambda_k^2}{\lambda_k} \|f_k\|_{L^2(0,T)} = T^2 \sum_{k=1}^{\infty} \lambda_k \int_0^T |f_k(t)|^2 \\ &= T^2 \int_0^T \sum_{k=1}^{\infty} \lambda_k |f_k(t)|^2\,dt = T^2 \int_0^T \|f_k(t)\|_{\mathcal{V}}^2\,dt \\ &= T\|f\|_{L^2(0,T;\mathcal{V})} < \infty.\end{aligned}$$

Since $f \in L^2(0,T;\mathcal{V})$, $\|x'\|_{L^2(0,T;\mathcal{V}')} < \infty$. Thus $x' \in L^2((0,T),\mathcal{V}')$.

4 Invariance of Sphere

Let $\mathcal{H}$ be an abstract Hilbert-space. Consider the evolution equation

$$\begin{aligned}\frac{dx}{dt} &= f(x), x \in \mathcal{H} \\ x(0) &= x_0,\end{aligned} \tag{4.1}$$

where f is a vector field on $\mathcal{H}$ and $x_0 \in V \cap M$. Assume that

$$\mathcal{M} = \{x \in \mathcal{H} : \varphi(x) = 0\},$$

and $\varphi : \mathcal{H} \to \mathbb{R}$ is a fixed smooth function and chosen in such a way that $\mathcal{M}$ can be a Hilbert submanifold of $\mathcal{H}$.

Our aim is to study the sufficient conditions on the vector field f that assures the invariance of submanifold $\mathcal{M}$; i.e., if $x_0 \in \mathcal{M}$ and all $x(t)$ of the solution of constrained problem, belong of M, for all $t \geq 0$

Remark 4.1. *1. A vector field f is said to satisfy the property $(*\ )$ on manifold $\mathcal{M}$ if for each $x \in \mathcal{M}$ and $f(x) \in T_xM$.*

2. For each $x \in \mathcal{M}$, we have $\nabla_x\varphi \in \mathcal{H}$ defined as

$$d_x\varphi(y) = \langle \nabla_x\varphi, y\rangle, \text{where } y \in \mathcal{H}. \tag{4.2}$$

where $d_x\varphi(y)$ is the Frećhet derivative of φ at x evaluated at y.

3. One can see easily that $d_x\varphi \in L(H, \mathbb{R})$ for all $x \in \mathcal{M}$, and

$$T_x\mathcal{M} = \{y \in \mathcal{H} : d_x\varphi(y) = 0\}.$$

Proposition 4.2. *For all $x \in \mathcal{M}$,*

$$T_x\mathcal{M} = \{y \in \mathcal{H} : \langle d_x\varphi, y\rangle = 0\}.$$

Proof. Set $A = \{y \in \mathcal{H} : \langle d_x\varphi, y\rangle = 0\}$. We show that $T_x\mathcal{M} = A$. Let $y \in T_xM$. Then $d_x\varphi(y) = 0$ which implies that $\langle d_x\varphi, y\rangle = \langle 0, y\rangle = 0$ so $y \in B$ and $T_xM \subset A$. For the converse, take an arbitrary $y \in A$.

$$\langle d_x\varphi, y\rangle = 0.$$

Since $d_x\varphi(y) = \langle \nabla_x\varphi, y\rangle$

$$\begin{aligned} \langle \langle \nabla_x\varphi, y\rangle, y\rangle &= 0 \\ \langle \nabla_x\varphi, y\rangle \langle 1, y\rangle &= 0 \end{aligned}$$

Clearly $\langle \nabla_x\varphi, y\rangle$ is scalar and y is arbitrary. So $\langle 1, y\rangle$ is not necessarily zero. It follows that

$$\langle \nabla_x\varphi, y\rangle = 0, \text{forall } y \in A.$$

Hence, by equation (4.2),

$$d_x\varphi(y) = \langle \nabla_x\varphi, y\rangle = 0, \text{forall } y \in A.$$

Hence $A \subset T_xM$. Since the inclusion is two sided, $T_x\mathcal{M} = A$. ∎

Consider now the case when the f involved in equation (4.1) does not satisfy the property $(*)$. In this case, we can modify f so that F satisfies the property $(*)$. We can achieve this by projecting vector field onto tangent space at a point on $\mathcal{M}$ i.e. $\pi_x : \mathcal{H} \to T_x M$

$$\pi_x(f) \equiv F(x) := f(x) - \langle f(x), x \rangle \, x. \tag{4.3}$$

We now state the key result of this section.

Theorem 4.3. *Let* $\mathcal{M} = \{x \in \mathcal{H} : \varphi(x) = 0\}$ *and* f *does not satisfy* $(*)$*. For a fixed* $x \in \mathcal{M}$ *,we define* $\pi_x : \mathcal{H} \to T_x M$ *as*

$$\pi_x(y) = y - \left\langle y, \frac{\nabla_x \varphi}{\|\nabla_x \varphi\|} \right\rangle \frac{\nabla_x \varphi}{\|\nabla_x \varphi\|}.$$

Then $\pi_x(y) \in T_x M$*, for all* $y \in \mathcal{M}$*, and* $f(x)$ *is modified as* $F(x) = \pi_x(f(x))$ *such that it satisfies property* $(*)$*.*

Proof. Set $\widehat{\nabla}_x \varphi(y) = \frac{\nabla_x \varphi(y)}{\|\nabla_x \varphi(y)\|}$. For $y \in \mathcal{H}$, consider

$$\begin{aligned}
\langle \pi_x(y), \nabla_x \varphi(y) \rangle &= \left\langle y - \left\langle y, \widehat{\nabla}_x \varphi(y) \right\rangle \widehat{\nabla}_x \varphi(y), \nabla_x \varphi(y) \right\rangle \\
&= \langle y, \nabla_x \varphi(y) \rangle - \left\langle \left\langle y, \widehat{\nabla}_x \varphi(y) \right\rangle \widehat{\nabla}_x \varphi(y), \nabla_x \varphi(y) \right\rangle \\
&= \langle y, \nabla_x \varphi(y) \rangle - \left\langle y, \widehat{\nabla}_x \varphi(y) \right\rangle \left\langle \widehat{\nabla}_x \varphi(y), \nabla_x \varphi(y) \right\rangle \\
&= \langle y, \nabla_x \varphi(y) \rangle - \frac{\langle y, \nabla_x \varphi \rangle \langle \nabla_x \varphi(y), \nabla_x \varphi(y) \rangle}{\|\nabla_x \varphi(y)\|^2} \\
&= \langle y, \nabla_x \varphi(y) \rangle - \frac{\langle y, \nabla_x \varphi(y) \rangle \|\nabla_x \varphi(y)\|^2}{\|\nabla_x \varphi(y)\|^2} \\
&= \langle y, \nabla_x \varphi(y) \rangle - \langle y, \nabla_x \varphi(y) \rangle = 0
\end{aligned}$$

so that $\pi_x(y) \in T_x M$. In particular, if we choose $y = f(x)$, then

$$F(x) := \pi_x(f(x)) \in T_x M, \text{forall } x \in \mathcal{M}.$$

∎

4.1 Key result regarding invariance of Sphere in $\ell^2_\lambda(\mathbb{R})$

In this section, we assume that $\mathcal{H} = \ell^2(\mathbb{R})$, $\mathcal{V} = \ell^2_\lambda(\mathbb{R})$ and $\mathcal{V}' = \ell^2_{\lambda^{-1}}(\mathbb{R})$. Keeping in view the notation of the last section and choosing $\varphi(x) = \|x\|^2_{\mathcal{H}} - 1$, we infer that

$$\begin{aligned} M &= \{x \in \mathcal{H} : \varphi(x) = 0\} \\ &= \{x \in \mathcal{H} : \|x\|_{\mathcal{H}} = 1\}. \end{aligned}$$

In the next lemma, we compute the Frećhet derivative of $\varphi(x) = \|x\|^2_{\mathcal{H}} - 1$.

Lemma 4.4. *Let the map $\varphi : \mathcal{H} \to R$ be defined by*

$$\varphi(u) = \|u\|^2_{\mathcal{H}} - 1.$$

Then φ has Frećhet derivative $d_x\varphi$ given by

$$d_u\varphi(v) \equiv \langle \nabla_u \varphi(x), v\rangle = \langle u, v\rangle,$$

for all $v \in \mathcal{H}$.

Proof. Since φ is a real-valued polynomial, it is smooth. Let us first calculate the first order Frećhet derivatives of φ. For any u and $h \in \mathcal{H}$,

$$\begin{aligned} \varphi(u+th) - \varphi(u) &= \frac{1}{2}\|u+th\|^2_{\mathcal{H}} - \frac{1}{2}\|u\|^2_{\mathcal{H}}, \\ &= \frac{1}{2}\|u\|^2_{\mathcal{H}} + \frac{1}{2}\|th\|^2_{\mathcal{H}} + \langle u, th\rangle - \frac{1}{2}\|u\|^2_{\mathcal{H}}, \\ &= \frac{t^2}{2}\|h\|^2_{\mathcal{H}} + t\langle u, h\rangle \\ \lim_{t\to 0}\frac{\varphi(u+th) - \varphi(u)}{t} &= \lim_{t\to 0}\frac{t}{2}\|h\|^2_{\mathcal{H}} + \langle u, h\rangle \\ &= \langle u, h\rangle, \end{aligned}$$

for all$h \in \mathcal{H}$.
Define a functional $d_u\varphi : \mathcal{H} \to \mathbb{R}$ by

$$d_u\varphi(h) := \langle u, h\rangle, \tag{4.4}$$

for all $h \in \mathcal{H}$.
Clearly, $d_u\varphi$ is linear. Moreover, to prove continuity, it is sufficient to prove boundedness. For $h \in \mathcal{H}$, consider

$$|d_u\varphi(h)| = |\langle u, h\rangle| \le \|u\|_{\mathcal{H}}\|h\|_{\mathcal{H}} = M\|h\|_{\mathcal{H}},$$

where $\mathcal{M} = |u|_{\mathcal{H}} < \infty$. Thus

$$\frac{|\varphi(u+h) - \varphi(u) - d_u\varphi(h)|}{\|h\|_{\mathcal{H}}} = \frac{o(\|h\|_{\mathcal{H}})}{\|h\|_{\mathcal{H}}} \to 0 \text{ as } \|h\|_{\mathcal{H}} \to 0.$$

Hence, we conclude the first order continuous Frećhet derivative of φ, given by

$$\langle \nabla_u \varphi(u), v\rangle \equiv d_u\varphi(h) = \langle u, h\rangle\,, \tag{4.5}$$

for all $h \in \mathcal{H}$. ∎

Remark 4.5. *From the last lemma, it follows that* $\nabla_u\varphi(x) = u$, *for all* $u \in \mathcal{H}$.

We are now interested in studying the evolution equation (4.1) for the case $f(x) := Ax$, where A is diagonal operator on $\mathcal{H}$, defined in Definition 2.7.

$$\begin{aligned} \frac{dx}{dt} &= -Ax, \\ x(0) &= x_0. \end{aligned} \tag{4.6}$$

Note that $D(A)$ is not globally defined in $\mathcal{H}$; i.e., $D(A) \subset \mathcal{H}$. We claim that if we project $f(x) := Ax$ orthogonally onto the tangent space of the sphere $\mathcal{M}$ in $\mathcal{H}$ and modify f by F (i.e., $F = \pi_x(f)$) in problem 4.6 in the following manner,

$$\begin{aligned} \frac{dx}{dt} &= \pi_x(-Ax), \\ x(0) &= x_0, \end{aligned} \tag{4.7}$$

where $x_0 \in V \cap M$. We mainly want to show that in the above projected problem if we assume that initial data of the problem $x_0 \in \mathcal{M}$, then the all trajectories of the solution of problem 4.7 are going to also belong to $\mathcal{M}$; i.e., we have invariance of sphere. Before getting into the proof of the invariance of sphere consider the following important observation

Proposition 4.6. *If* $\pi_x : \mathcal{H} \to T_xM$ *be the orthogonal projection and* $f : \mathcal{H} \to \mathcal{H}$ *be any vector field on* $\mathcal{H}$, *then* $F(x) = \pi_x(f(x))$ *satisfies the property* $(*)$ *on* $\mathcal{M} = \{x \in \mathcal{M} : \|x\|_{\mathcal{H}} = 1\}$.

Proof. Recall property $(*)$: f is said to satisfy the property $(*)$ on manifold $\mathcal{M}$ if for each $x \in \mathcal{M}$ and $f(x) \in T_xM$. We will verify this for $F(x) = \pi_x(f(x))$

For $x \in \mathcal{M}$ so

$$\begin{aligned}
\langle F(x), x\rangle_{\mathcal{H}} &= \langle \pi_x(f(x)), x\rangle_{\mathcal{H}} \\
&= \langle f(x) - \langle f(x), x\rangle x, x\rangle_{\mathcal{H}} \\
&= \langle f(x), x\rangle_{\mathcal{H}} - \langle\langle f(x), x\rangle x, x\rangle_{\mathcal{H}} \\
&= \langle f(x), x\rangle_{\mathcal{H}} - \langle f(x), x\rangle_{\mathcal{H}} \langle x, x\rangle_{\mathcal{H}} \\
&= \langle f(x), x\rangle_{\mathcal{H}} - \langle f(x), x\rangle_{\mathcal{H}} \|x\|_{\mathcal{H}} \\
&= \langle f(x), x\rangle_{\mathcal{H}} - \langle f(x), x\rangle_{\mathcal{H}} = 0.
\end{aligned}$$

Hence $F(x) \perp x$, for all $x \in \mathcal{M}$; i.e., $F(x) \in T_xM$, for all $x \in \mathcal{M}$. ∎

Remark 4.7. *Recall that we have chosen $\mathcal{H} = \ell^2(\mathbb{R})$, $\mathcal{V} = \ell^2_\lambda(\mathbb{R})$, for this section. Let $x = (x_i)_{i=1}^\infty \in D(A)\ \mathcal{H}$, where A is diagonal operator defined in Definition 2.7. We want to explicitly compute the expression for the $\pi_x(-Ax)$. Consider*

$$\begin{aligned}
\pi_x(-Ax) &= -Ax - \langle -Ax, x\rangle_{\mathcal{H}} x \\
&= -Ax + \langle (\lambda_i x_i)_{i=1}^\infty, (x_i)_{i=1}^\infty\rangle_{\mathcal{H}} x \\
&= Ax - \left(\sum_{i=1}^\infty \lambda_i x_i^2\right) x \\
&= -Ax + \|x\|^2_{\mathcal{V}} x
\end{aligned}$$

Hence problem 4.7 can be reformulated as

$$\begin{aligned}
\frac{dx}{dt} &= \pi_x(-Ax) = -Ax + \|x\|^2_{\mathcal{V}} x, \qquad (4.8)\\
x(0) &= x_0.
\end{aligned}$$

We intend to prove our key result.

Theorem 4.8. *If $x_0 \in V \cap M$, then all trajectories of solution of 4.8 belong to $\mathcal{M}$; i.e., $x(t) \in \mathcal{M}$, for all $t \geq 0$.*

Proof. Assume that $x_0 \in V \cap M$. We show that all trajectories of the solution belong to M. We employ Lemma 2.1. Note that problem 4.8 becomes a special case of the initial value problem 3.1, when $f(t) = \|x\|^2_{\mathcal{V}}\, x(t)$. In the last section, we have already shown that $x \in L^2((0,T),\mathcal{V})$ (i.e., $L^2((0,T),\ell^2_\lambda(\mathbb{R}))$) and $x' \in L^2((0,T),\mathcal{V}')$ (i.e., $L^2((0,T),\ell^2_{\lambda^{-1}}(\mathbb{R}))$). Hence the conditions of the Lemma are satisfied. so we begin by using the identity (2.1) i.e.

$$\frac{1}{2}\frac{d}{dt}\left(\|x(t)\|^2_{\mathcal{H}} - 1\right) = \langle x'(t), x(t)\rangle_{\mathcal{H}}$$

From equation (4.8) we infer that, for $t \geq 0$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|x(t)\|^2_{\mathcal{H}} - 1\right) &= \left\langle -Ax(t) + \|x(t)\|^2_{\mathcal{V}}\, x(t), x(t)\right\rangle_{\mathcal{H}} \\
&= -\langle Ax(t), x(t)\rangle + \left\langle \|x(t)\|^2_{\mathcal{V}}\, x(t), x(t)\right\rangle_{\mathcal{H}} \\
&= -\langle Ax(t), x(t)\rangle + \|x(t)\|^2_{\mathcal{V}} \langle x(t), x(t)\rangle_{\mathcal{H}} \\
&= -\langle Ax(t), x(t)\rangle + \|x(t)\|^2_{\mathcal{V}} \|x(t)\|^2_{\mathcal{H}}
\end{aligned}$$

From Remark 4.7, we know that $\langle -Ax(t), x(t)\rangle_{\mathcal{H}} = \|x(t)\|^2_{\mathcal{V}}$, and using this fact in the last equation it follows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|x(t)\|^2_{\mathcal{H}} - 1\right) &= \|x(t)\|^2_{\mathcal{V}} \|x(t)\|^2_{\mathcal{H}} - \|x(t)\|^2_{\mathcal{V}} \\
&= \|x(t)\|^2_{\mathcal{V}} \left(\|x(t)\|^2_{\mathcal{H}} - 1\right)
\end{aligned}$$

Set $u(t) = \|x(t)\|^2_{\mathcal{H}} - 1$

$$\frac{du}{dt}(t) = \|x(t)\|^2_{\mathcal{V}}\, u(t),$$

Since $\|x(t)\|^2_{\mathcal{V}}$ is polynomial, it must be continuous in time. Hence the solution of the above evolution equation is given as

$$u(t) = u(0)e^{\|x(t)\|^2_{\mathcal{V}}}$$

Since $x(0) = x_0 \in \mathcal{M}$ so $u(0) = \|x(0)\|^2_{\mathcal{H}} - 1 = \|x_0\|^2_{\mathcal{H}} - 1 = 0$, it follows from the last equation, that

$$\begin{aligned}
u(t) &= 0, \\
\|x(t)\|^2_{\mathcal{H}} - 1 &= 0, \\
\|x(t)\|^2_{\mathcal{H}} &= 1.
\end{aligned}$$

Thus we show that $x(t) \in \mathcal{M}$, for all $t \geq 0$. This completes the proof of the invariance of the sphere $\mathcal{M}$. ∎

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