\textbf{\Gamma\text{-open Sets in Biclosure Spaces}}

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Abstract

A new class of open sets in a biclosure space, called \(\Gamma\)-open sets, is introduced and studied. Moreover, we give the notions of \(\Gamma\)-open maps, \(\Gamma\)-continuous maps and contra-\(\Gamma\)-continuous maps and study some of their properties.

1 Introduction and Preliminaries

The notion of closure spaces was first introduced and studied by Čech [3]. In 2003, Šlapal [5] introduced generalized Čech closure spaces which are called closure spaces.

\textbf{Definition 1.1.} A map \(u : P(X) \to P(X)\) is called a \textit{closure operator} on \(X\) and the pair \((X, u)\) is called a \textit{closure space} if the following axioms are satisfied:

\begin{align*}
(A1) & \quad u\emptyset = \emptyset, \\
(A2) & \quad A \subseteq uA \text{ for every } A \subseteq X, \\
(A3) & \quad A \subseteq B \Rightarrow uA \subseteq uB \text{ for all } A, B \subseteq X.
\end{align*}

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A subset $A \subseteq X$ is *closed* in the closure space $(X, u)$ if $uA = A$ and it is *open* if its complement in $X$ is closed. The empty set and the whole space are both open and closed.

A closure space $(Y, v)$ is said to be a *subspace* of $(X, u)$ if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$.

Let $(X, u)$ and $(Y, v)$ be closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is closed (resp. open) subset of $(Y, v)$ whenever $F$ is a closed (resp. open) subset of $(X, u)$.

If $(X, u)$ and $(Y, v)$ are closure spaces, then a map $f : (X, u) \rightarrow (Y, v)$ is called *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ of closure spaces, denoted by $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$ is the closure space $\left( \prod_{\alpha \in J} X_\alpha, u \right)$ where $\prod_{\alpha \in J} X_\alpha$ denotes the Cartesian product of the sets $X_\alpha$, $\alpha \in J$ and $u$ is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in J} (X_\alpha, u_\alpha) \rightarrow (X_\alpha, u_\alpha)$, $\alpha \in J$; i.e., defined by $uA = \prod_{\alpha \in J} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha \in J} X_\alpha$.

Boonpok [1] studied some fundamental properties of closure spaces. In 2009, the following statements were proven.

**Proposition 1.2.** Let $(X, u)$ and $(Y, v)$ be closure spaces. If a map $f : (X, u) \rightarrow (Y, v)$ is continuous, then $f^{-1}(H)$ is open in $(X, u)$ each open set $H$ in $(Y, v)$.

**Proposition 1.3.** Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces and let $\beta \in J$. Then the projection map $\pi_\beta : \prod_{\alpha \in J} (X_\alpha, u_\alpha) \rightarrow (X_\beta, u_\beta)$ is closed and continuous.

**Proposition 1.4.** Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces and let $\beta \in J$. Then $G_\beta$ is an open subset of $(X_\beta, u_\beta)$ if and only if $G_\beta \times \prod_{\alpha \neq \beta \atop \alpha \in J} X_\alpha$ is an open subset of $\prod_{\alpha \in J} (X_\alpha, u_\alpha)$.

In 2009, the concept of biclosure spaces was introduced by Boonpok [2]. Biclosure spaces are sets endowed with two closure operators.
Definition 1.5. A biclosure space is a triple $(X, u_1, u_2)$ where $X$ is a set and $u_1, u_2$ are two closure operators on $X$. A subset $A$ of a biclosure space $(X, u_1, u_2)$ is called closed if $u_1u_2A = A$. The complement of closed set is called open.

Let $(X, u_1, u_2)$ be a biclosure space. A biclosure space $(Y, v_1, v_2)$ is called a subspace of $(X, u_1, u_2)$ if $Y \subseteq X$ and $v_iA = u_iA \cap Y$ for all $i \in \{1, 2\}$ and every subset $A$ of $Y$.

Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be biclosure spaces and let $i \in \{1, 2\}$. Then a map $f : (X, u_i) \to (Y, v_i)$ is called:

(i) $i$-open (respectively $i$-closed) if $f : (X, u_i) \to (Y, v_i)$ is open (respectively closed) for all $i \in \{1, 2\}$.

(ii) open (respectively closed) if $f$ is $i$-open (respectively $i$-closed) for all $i \in \{1, 2\}$.

(iii) $i$-continuous if $f : (X, u_i) \to (Y, v_i)$ is continuous for all $i \in \{1, 2\}$.

(iv) continuous if $f$ is $i$-continuous for all $i \in \{1, 2\}$.

In 2009, Khampakdee and Boonpok [6] introduced the notion of semi-open sets in biclosure spaces.

Definition 1.6. A subset $A$ of a biclosure space $(X, u_1, u_2)$ is called semi-open, if there exists an open subset $G$ of $(X, u_1)$ such that $G \subseteq A \subseteq u_2G$. The complement of a semi-open set in $X$ is called semi-closed.

Moreover, the following statements were studied in [6].

Remark 1.7. Let $A$ be a subset of a biclosure space $(X, u_1, u_2)$. Then $A$ is open in $(X, u_1, u_2)$ if and only if $A$ is open in both $(X, u_1)$ and $(X, u_2)$.

Proposition 1.8. Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If $f$ is open, then $f(G)$ is open in $(Y, v_1, v_2)$ for every open subset $G$ of $(X, u_1, u_2)$.

Proposition 1.9. Let $(X, u_1, u_2)$ and $(Y, v_1, v_2)$ be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If $f$ is continuous, then $f^{-1}(H)$ is open in $(X, u_1, u_2)$ for every open subset $H$ of $(Y, v_1, v_2)$.
2 Γ-open Sets in Closure Spaces

In this section, we introduce a new type of open sets in closure spaces and study some of their properties.

**Definition 2.1.** Let \((X, u)\) be a closure space. A subset \(A\) of \(X\) is called a Γ-open set if there exists an open set \(G\) in \(X\) such that \(A \subseteq G \subseteq uA\). A subset \(A \subseteq X\) is called a Γ-closed set if its complement is Γ-open.

Clearly, if \(A\) is open (respectively closed) in \((X, u)\), then \(A\) is Γ-open (respectively Γ-closed) in \((X, u)\). The converse is not true as shown in the following example.

**Example 2.2.** Let \(X = \{1, 2, 3\}\) and define a closure operator \(u\) on \(X\) by \(u\emptyset = \emptyset, u\{1\} = X, u\{2\} = \{2, 3\}, u\{3\} = \{3\}\) and \(u\{1, 2\} = u\{1, 3\} = u\{2, 3\} = uX = X\). It is easy to see that \(\{1\}\) is Γ-open because there is an open set \(\{1\}\) such that \(\{1\} \subseteq \{1, 2\} \subseteq u\{1\}\). But \(\{1\}\) is not open. And we also see that \(\{2, 3\}\) is Γ-closed but not closed.

Regarding the union of Γ-open sets and the intersection of Γ-closed sets we have the following statements:

**Theorem 2.3.** Let \(\{A_\alpha\}_{\alpha \in J}\) be a collection of Γ-open sets in a closure space \((X, u)\). Then \(\bigcup_{\alpha \in J} A_\alpha\) is Γ-open in \((X, u)\).

*Proof.* Let \(A_\alpha\) be Γ-open in \((X, u)\) for all \(\alpha \in J\). Then, for each \(\alpha \in J\), we have an open set \(G_\alpha\) in \((X, u)\) such that \(A_\alpha \subseteq G_\alpha \subseteq uA_\alpha\). Thus, \(\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in J} G_\alpha \subseteq \bigcup_{\alpha \in J} uA_\alpha\). Since \(A_\alpha \subseteq \bigcup_{\alpha \in J} A_\alpha\) for each \(\alpha \in J\), \(uA_\alpha \subseteq u \bigcup_{\alpha \in J} A_\alpha\) for all \(\alpha \in J\). Thus, \(\bigcup_{\alpha \in J} uA_\alpha \subseteq \bigcup_{\alpha \in J} uA_\alpha\). Consequently, \(\bigcup_{\alpha \in J} uA_\alpha \subseteq \bigcup_{\alpha \in J} G_\alpha \subseteq \bigcup_{\alpha \in J} uA_\alpha\). As \(G_\alpha\) is open, \(X - G_\alpha\) is closed for all \(\alpha \in J\). Hence, \(u \cap_{\alpha \in J} (X - G_\alpha) \subseteq u(X - G_\alpha) = X - G_\alpha\) for each \(\alpha \in J\). Thus, \(u \cap_{\alpha \in J} (X - G_\alpha) \subseteq \bigcap_{\alpha \in J} (X - G_\alpha)\). But \(\bigcap_{\alpha \in J} (X - G_\alpha) \subseteq u \cap_{\alpha \in J} (X - G_\alpha)\). Then \(\bigcap_{\alpha \in J} (X - G_\alpha)\) is closed in \((X, u)\); i.e., \(\bigcup_{\alpha \in J} G_\alpha\) is open. Therefore, \(\bigcup_{\alpha \in J} A_\alpha\) is Γ-open in \((X, u)\). \(\square\)

**Theorem 2.4.** Let \(\{A_\alpha\}_{\alpha \in J}\) be a collection of Γ-closed sets in a closure space \((X, u)\). Then \(\bigcap_{\alpha \in J} A_\alpha\) is Γ-closed.

*Proof.* Let \(A_\alpha\) be a Γ-closed set in \((X, u)\) for all \(\alpha \in J\). Then \(X - A_\alpha\) is Γ-open for each \(\alpha \in J\). It follows that \(\bigcup_{\alpha \in J} (X - A_\alpha)\) is Γ-open. But \(\bigcup_{\alpha \in J} (X - A_\alpha) = X - \bigcap_{\alpha \in J} A_\alpha\). Thus \(\bigcap_{\alpha \in J} A_\alpha\) is Γ-closed. \(\square\)
Theorem 2.5. Let \((X, u)\) be a closure space and \(A\) be \(\Gamma\)-open in \((X, u)\). If \(B \subseteq A\) and \(uA \subseteq uB\), then \(B\) is \(\Gamma\)-open.

Proof. Let \(A\) be a \(\Gamma\)-open set in \((X, u)\). Then there exists an open set \(G\) in \((X, u)\) such that \(A \subseteq G \subseteq uA\). Since \(B \subseteq A\) and \(uA \subseteq uB\), \(B \subseteq A \subseteq G \subseteq uA \subseteq uB\). Thus, \(B\) is \(\Gamma\)-open. \(\square\)

Theorem 2.6. Let \((Y, v)\) be a closure subspace of \((X, u)\). If a subset \(A \subseteq Y\) is a \(\Gamma\)-open set in \((X, u)\), then \(A\) is a \(\Gamma\)-open set in \((Y, v)\).

Proof. Let \(A\) be \(\Gamma\)-open in \((X, u)\). Then there exists an open set \(G\) in \((X, u)\) such that \(A \subseteq G \subseteq uA\). Hence, \(A = A \cap Y \subseteq G \cap Y \subseteq uA \cap Y = vA\). Since \(v(Y - (G \cap Y)) = u(Y - (G \cap Y)) \cap Y \subseteq u(X - G) \cap Y = (X - G) \cap Y = Y - (G \cap Y)\), \(G \cap Y\) is open in \((Y, v)\). Therefore, \(A\) is a \(\Gamma\)-open set in \((Y, v)\). \(\square\)

Theorem 2.7. Let \((X, u)\) be a closure space and let \(A \subseteq X\). Then \(A\) is \(\Gamma\)-closed if and only if there exists a closed set \(F\) in \((X, u)\) such that \(X - u(X - A) \subseteq F \subseteq A\).

Proof. Let \(A\) be \(\Gamma\)-closed. Then there exists an open set \(G\) in \((X, u)\) such that \(X - A \subseteq G \subseteq u(X - A)\). Thus, there exists a closed set \(F\) in \((X, u)\) such that \(G = X - F\) and \(X - A \subseteq X - F \subseteq u(X - A)\). Hence, \(X - u(X - A) \subseteq F \subseteq A\). Conversely, by the assumption, there is a closed set \(F\) in \((X, u)\) such that \(X - u(X - A) \subseteq F \subseteq A\). Thus, there exists an open set \(G\) in \((X, u)\) such that \(F = X - G\) and \(X - u(X - A) \subseteq X - G \subseteq A\). It follows that \(X - A \subseteq G \subseteq u(X - A)\). Consequently, \(X - A\) is \(\Gamma\)-open in \((X, u)\). As a result, \(A\) is \(\Gamma\)-closed in \((X, u)\). \(\square\)

3 \(\Gamma\)-open Sets in Biclosure Spaces

Definition 3.1. A subset \(A\) of a biclosure space \((X, u_1, u_2)\) is called \(\Gamma\)-open if there exists an open set \(G\) in \((X, u_1)\) such that \(A \subseteq G \subseteq u_2A\). The complement of a \(\Gamma\)-open subset of \(X\) is called \(\Gamma\)-closed.

Remark 3.2. The independence of semi-open sets and \(\Gamma\)-open sets in biclosure spaces can be seen from the following examples.

Example 3.3. Let \(X = \{1, 2, 3\}\) and define a closure operator \(u_1\) on \(X\) by \(u_1\emptyset = \emptyset, u_1\{1\} = \{1, 2\}, u_1\{2\} = u_1\{3\} = u_1\{2, 3\} = \{2, 3\}\) and \(u_1\{1, 2\} = \{1, 2, 3\}\).
$u_1\{1,3\} = u_1X = X$. Define a closure operator $u_2$ on $X$ by $u_2\emptyset = \emptyset$, $u_2\{1\} = u_2\{3\} = u_2\{1,3\} = \{1,3\}$, $u_2\{2\} = \{1,2\}$ and $u_2\{1,2\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $\{1,3\}$ is semi-open because there is an open set $\{1\}$ in $(X, u_1)$ such that $\{1\} \subseteq \{1,3\} \subseteq u_2\{1\}$. But $\{1,3\}$ is not $\Gamma$-open in $(X, u_1, u_2)$.

**Example 3.4.** Let $X = \{1,2,3\}$ and define a closure operator $u_1$ on $X$ by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$, $u_1\{3\} = \{1,3\}$ and $u_1\{2\} = u_1\{1,2\} = u_1\{1,3\} = u_1\{2,3\} = u_1X = X$. Define a closure operator $u_2$ on $X$ by $u_2\emptyset = \emptyset$, $u_2\{1\} = \{1\}$ and $u_2\{2\} = u_2\{3\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $\{3\}$ is $\Gamma$-open because there is an open set $\{2,3\}$ in $(X, u_1)$ such that $\{3\} \subseteq \{2,3\} \subseteq u_2\{3\}$. But $\{3\}$ is not semi-open in $(X, u_1, u_2)$.

**Remark 3.5.** If $(X, u_1, u_2)$ is a biclosure space and $A$ is open (respectively closed) in $(X, u_1)$, then $A$ is $\Gamma$-open (respectively $\Gamma$-closed) in $(X, u_1, u_2)$. The converse is not true in general.

**Example 3.6.** Let $X = \{1,2,3\}$ and define a closure operator $u_1$ on $X$ by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1,2\}$, $u_1\{2\} = \{2,3\}$, $u_1\{3\} = \{3\}$ and $u_1\{1,2\} = u_1\{1,3\} = u_1\{2,3\} = u_1X = X$. Define a closure operator $u_2$ on $X$ by $u_2\emptyset = \emptyset$, $u_2\{2\} = \{2\}$, $u_2\{3\} = \{2,3\}$ and $u_2\{1\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $\{1\}$ is $\Gamma$-open because there is an open set $\{1,2\}$ in $(X, u_1)$ such that $\{1\} \subseteq \{1,2\} \subseteq u_2\{1\}$. But $\{1\}$ is not open in $(X, u_1)$. And we also see that $\{2,3\}$ is $\Gamma$-closed but not closed.

**Theorem 3.7.** Let $(X, u_1, u_2)$ be a biclosure space and let $A \subseteq X$. Then $A$ is $\Gamma$-closed in $(X, u_1, u_2)$ if and only if there exists a closed subset $F$ of $(X, u_1)$ such that $X - u_2(X - A) \subseteq F \subseteq A$.

**Proof.** Let $A$ be $\Gamma$-closed in $(X, u_1, u_2)$. Then there exists an open subset $G$ in $(X, u_1)$ such that $X - A \subseteq G \subseteq u_2(X - A)$. Hence, there exists a closed subset $F$ of $(X, u_1)$ such that $G = X - F$. It follows that $X - A \subseteq X - F \subseteq u_2(X - A)$. Therefore, $X - u_2(X - A) \subseteq F \subseteq A$.

Conversely, by the assumption, there is a closed subset $F$ of $(X, u_1)$ such that $X - u_2(X - A) \subseteq F \subseteq A$. Thus, there exists an open set $G$ in $(X, u_1)$ such that $F = X - G$, i.e. $X - u_2(X - A) \subseteq X - G \subseteq A$. It follows that $X - A \subseteq G \subseteq u_2(X - A)$. Therefore, $A$ is $\Gamma$-closed in $(X, u_1, u_2)$.

**Theorem 3.8.** Let $\{A_\alpha\}_{\alpha \in I}$ be a collection of $\Gamma$-open sets in a biclosure space $(X, u_1, u_2)$. Then $\cup_{\alpha \in I} A_\alpha$ is a $\Gamma$-open set in $(X, u_1, u_2)$.
Proof. Let $A_\alpha$ be $\Gamma$-open in $(X, u_1, u_2)$ for all $\alpha \in J$. Then, for each $\alpha \in J$, we have an open set $G_\alpha$ in $(X, u_1)$ such that $A_\alpha \subseteq G_\alpha \subseteq u_2A_\alpha$. Thus, $\bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in J} G_\alpha \subseteq \bigcup_{\alpha \in J} u_2A_\alpha$. As $G_\alpha$ is open in $(X, u_1)$ for all $\alpha \in J$, $u_1 \cap_{\alpha \in J} (X - G_\alpha) \subseteq u_1(X - G_\alpha) = X - G_\alpha$ for each $\alpha \in J$. Thus, $u_1 \cap_{\alpha \in J} (X - G_\alpha) \subseteq \cap_{\alpha \in J}(X - G_\alpha)$. It follows that $\cap_{\alpha \in J}(X - G_\alpha)$ is closed in $(X, u_1)$, i.e. $\cap_{\alpha \in J} G_\alpha$ is open in $(X, u_1)$. Since $A_\alpha \subseteq \cap_{\alpha \in J} A_\alpha$ for each $\alpha \in J$, $u_2A_\alpha \subseteq u_2 \cap_{\alpha \in J} A_\alpha$ for all $\alpha \in J$. Hence, $\cap_{\alpha \in J} u_2A_\alpha \subseteq u_2 \cap_{\alpha \in J} A_\alpha$ thus $\cap_{\alpha \in J} A_\alpha \subseteq \cup_{\alpha \in J} G_\alpha \subseteq u_2 \cup_{\alpha \in J} A_\alpha$. Therefore, $\cap_{\alpha \in J} A_\alpha$ is $\Gamma$-open in $(X, u_1, u_2)$. □

Corollary 3.9. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of $\Gamma$-closed sets in a biclosure space $(X, u_1, u_2)$. Then $\cap_{\alpha \in J} A_\alpha$ is $\Gamma$-closed.

Proof. Since $A_\alpha$ is $\Gamma$-closed in $(X, u_1, u_2)$ for each $\alpha \in J$, $X - A_\alpha$ is $\Gamma$-open for all $\alpha \in J$. But $\cup_{\alpha \in J} (X - A_\alpha) = X - \cup_{\alpha \in J} A_\alpha$ is a $\Gamma$-open sets. Therefore, $\cap_{\alpha \in J} A_\alpha$ is $\Gamma$-closed in $(X, u_1, u_2)$. □

Theorem 3.10. Let $(X, u_1, u_2)$ be a biclosure space and let $A$ be a $\Gamma$-open in $(X, u_1, u_2)$. If $B \subseteq A$ and $u_2A \subseteq u_2B$, then $B$ is $\Gamma$-open in $(X, u_1, u_2)$

Proof. Let $A$ be $\Gamma$-open in $(X, u_1, u_2)$. Then there exists an open set $G$ in $(X, u_1)$ such that $A \subseteq G \subseteq u_2A$. As $B \subseteq A$ and $u_2A \subseteq u_2B$, $B \subseteq A \subseteq G \subseteq u_2B$, then $B$ is $\Gamma$-open in $(X, u_1, u_2)$. □

Theorem 3.11. Let $(Y, v_1, v_2)$ be a biclosure subspace of $(X, u_1, u_2)$ and $A \subseteq Y$. If $A$ is $\Gamma$-open in $(X, u_1, u_2)$, then $A$ is $\Gamma$-open in $(Y, v_1, v_2)$.

Proof. Let $A$ be a $\Gamma$-open subset of $(X, u_1, u_2)$. Then there exists an open subset $G$ of $(X, u_1)$ such that $A \subseteq G \subseteq u_2A$. It follows that $A \subseteq A \cap Y \subseteq G \cap Y \subseteq u_2A \cap Y \subseteq v_2A$. Since $G$ is open in $(X, u_1)$, $v_1(Y - (G \cap Y)) = u_1(Y - (G \cap Y)) \cap Y \subseteq u_1(X - G) \cap Y = (X - G) \cap Y = Y - (G \cap Y)$. Thus, $G \cap Y$ is open in $(Y, v_1)$. Therefore, $A$ is $\Gamma$-open in $(Y, v_1, v_2)$. □

Theorem 3.12. Let $\{(X_\alpha, u_1^\alpha, u_2^\alpha) : \alpha \in J\}$ be a family of biclosure spaces and let $k \in J$. If $A_k$ is $\Gamma$-open in $(X_k, u_k^1, u_k^2)$, then $A_k \times \prod_{\alpha \in J} X_\alpha$ is $\Gamma$-open in $\prod_{\alpha \in J} (X_\alpha, u_1^\alpha, u_2^\alpha)$.

Proof. Let $k \in J$ and let $A_k$ be $\Gamma$-open in $(X_k, u_k^1, u_k^2)$. Then there exists an open set $G_k$ in $(X_k, u_k^1)$ such that $A_k \subseteq G_k \subseteq u_k^2A_k$. Hence, $A_k \times \prod_{\alpha \in J, \alpha \neq k} X_\alpha \subseteq G_k \times \prod_{\alpha \in J, \alpha \neq k} X_\alpha \subseteq u_k^2A_k \times \prod_{\alpha \in J} X_\alpha$. But $G_k \times \prod_{\alpha \in J} X_\alpha$ is open in
\[ \prod_{\alpha \in J} (X_{\alpha}, u_{\alpha}^1) \text{ and } u_{k}\, A_k \times \prod_{\alpha \neq k} X_{\alpha} = u_{k}\, A_k \times \prod_{\alpha \neq k} u_{\alpha}^2 X_{\alpha} = \prod_{\alpha \neq k} u_{\alpha}^2 \pi_{\alpha} (A_k \times \prod_{\alpha \in J} X_{\alpha}) \]

\[ = u_2(A_k \times \prod_{\alpha \in J} X_{\alpha}). \] Therefore, \( A_k \times \prod_{\alpha \in J} X_{\alpha} \) is \( \Gamma \)-open in \( \prod_{\alpha \in J} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2). \) \( \square \)

**Theorem 3.13.** Let \( \{(X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2) : \alpha \in J\} \) be a family of biclosure spaces and let \( k \in J. \) If \( A_k \) is \( \Gamma \)-closed in \( (X_k, u_k^1, u_k^2) \), then \( A_k \times \prod_{\alpha \neq k} X_{\alpha} \) is \( \Gamma \)-closed in \( \prod_{\alpha \in J} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2). \)

**Proof.** Let \( k \in J \) and let \( A_k \) be \( \Gamma \)-closed in \( (X_k, u_k^1, u_k^2) \). Then \( X_k - A_k \) is \( \Gamma \)-open in \( (X_k, u_k^1, u_k^2) \). Hence, \( (X_k - A_k) \times \prod_{\alpha \neq k} X_{\alpha} = \prod_{\alpha \in J} X_{\alpha} - (A_k \times \prod_{\alpha \neq k} X_{\alpha}) \) is \( \Gamma \)-open in \( \prod_{\alpha \in J} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2). \) Consequently, \( A_k \times \prod_{\alpha \in J} X_{\alpha} \) is \( \Gamma \)-closed in \( \prod_{\alpha \in J} (X_{\alpha}, u_{\alpha}^1, u_{\alpha}^2). \) \( \square \)

**Definition 3.14.** Let \( (X, u_1, v_2) \) and \( (Y, v_1, v_2) \) be biclosure spaces. Then a map \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) is called \( \Gamma \)-open (respectively \( \Gamma \)-closed) if \( f(A) \) is \( \Gamma \)-open (respectively \( \Gamma \)-closed) in \( (Y, v_1, v_2) \) for every open (respectively closed) subset \( A \) of \( (X, u_1, u_2). \)

Clearly, if \( f \) is open (respectively closed), then \( f \) is \( \Gamma \)-open (respectively \( \Gamma \)-closed). The converse need not be true as the following example shows.

**Example 3.15.** Let \( X = \{1, 2\} = Y \) and define a closure operator \( v_1 \) on \( X \) by \( u_1\emptyset = \emptyset, u_1\{1\} = \{1\}, u_1\{2\} = u_1 X = X \). Define a closure operator \( u_2 \) on \( X \) by \( u_2\emptyset = \emptyset, u_2\{1\} = \{1\}, u_2\{2\} = \{2\} \text{ and } u_2 X = X \). Define a closure operator \( v_1 \) on \( Y \) by \( v_1\emptyset = \emptyset, v_1\{1\} = \{1\}, v_1\{2\} = v_1 Y = Y \) and define a closure operator \( v_2 \) on \( Y \) by \( v_2\emptyset = \emptyset, v_2\{1\} = v_2\{2\} = v_2 Y = Y \). Let \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) be an identity map. It is easy to see that \( f \) is \( \Gamma \)-open but not open because \( f(\{2\}) \) is not open in \( (Y, v_1, v_2) \) while \( \{2\} \) is open in \( (X, u_1, u_2) \). Moreover, we can see that \( f \) is \( \Gamma \)-closed but not closed because \( f(\{1\}) \) is not closed in \( (Y, v_1, v_2) \) while \( \{1\} \) is closed in \( (X, u_1, u_2) \).

**Theorem 3.16.** Let \( (X, u_1, u_2), (Y, v_1, v_2) \) and \( (Z, w_1, w_2) \) be biclosure spaces and let \( f : (X, u_1, u_2) \to (Y, v_1, v_2) \) and \( g : (Y, v_1, v_2) \to (Z, w_1, w_2) \) be maps. Then \( g \circ f \) is \( \Gamma \)-open if \( f \) is open and \( g \) is \( \Gamma \)-open.

**Proof.** Let \( G \) be an open subset of \( (X, u_1, u_2) \) and let \( f \) be an open map. It
follows that \( f(G) \) is open in \((Y, v_1, v_2)\). Since \( g \) is \( \Gamma \)-open, \( g(f(G)) = g \circ f(G) \) is \( \Gamma \)-open in \((Z, w_1, w_2)\). Therefore, \( g \circ f \) is \( \Gamma \)-open.

\[\Box\]

**Theorem 3.17.** Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces and let \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) and \( g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2) \) be maps. If \( g \circ f \) is \( \Gamma \)-open and \( f \) is a continuous surjection, then \( g \) is \( \Gamma \)-open.

**Proof.** Let \( H \) be an open subset of \((Y, v_1, v_2)\) and let \( f \) be continuous. Then \( f^{-1}(H) \) is open in \((X, u_1, u_2)\). Since \( g \circ f \) is \( \Gamma \)-open, \( g \circ f(f^{-1}(H)) \) is \( \Gamma \)-open in \((Z, w_1, w_2)\). Since \( f \) is a surjection, \( g \circ f(f^{-1}(H)) = g(H) \). Thus, \( g(H) \) is \( \Gamma \)-open in \((Z, w_1, w_2)\). Therefore, \( g \) is \( \Gamma \)-open.

\[\Box\]

**Definition 3.18.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. Then a map \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) is called \( \Gamma \)-continuous if \( f^{-1}(H) \) is a \( \Gamma \)-open subset of \((X, u_1, u_2)\) for every open subset \( H \) of \((Y, v_1, v_2)\).

Clearly, if \( f \) is continuous, then \( f \) is \( \Gamma \)-continuous. The converse need not be true as can be seen from the following example.

**Example 3.19.** Let \( X = \{1, 2\} = Y \) and define a closure operator \( u_1 \) on \( X \) by \( u_1\emptyset = \emptyset \), \( u_1\{1\} = \{1\} \) and \( u_1\{2\} = u_1X = X \). Define a closure operator \( u_2 \) on \( X \) by \( u_2\emptyset = \emptyset \) and \( u_2\{1\} = u_2\{2\} = u_2X = X \). Define a closure operator \( v_1 \) on \( Y \) by \( v_1\emptyset = \emptyset \), \( v_1\{1\} = \{1\} \) and \( v_1\{2\} = \{2\} \) and \( v_1Y = Y \) and define a closure operator \( v_2 \) on \( Y \) by \( v_2\emptyset = \emptyset \), \( v_2\{1\} = \{1\} \) and \( v_2\{2\} = v_2Y = Y \). Let \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) be an identity map. It is easy to see that \( f \) is \( \Gamma \)-continuous but not continuous because \( f^{-1}(\{2\}) \) is not open in \((X, u_1, u_2)\) while \( \{2\} \) is open in \((Y, v_1, v_2)\).

**Theorem 3.20.** Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. Then a map \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) is \( \Gamma \)-continuous if and only if \( f^{-1}(F) \) is a \( \Gamma \)-closed subset of \((X, u_1, u_2)\) for every closed subset \( F \) of \((Y, v_1, v_2)\).

**Proof.** Let \( f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2) \) be \( \Gamma \)-continuous. Let \( F \) be a closed subset of \((Y, v_1, v_2)\). Then \( Y - F \) is open in \((Y, v_1, v_2)\). Since \( F \) is \( \Gamma \)-continuous, \( f^{-1}(Y - F) = X - f^{-1}(F) \) is \( \Gamma \)-open in \((X, u_1, u_2)\). Hence, \( f^{-1}(F) \) is \( \Gamma \)-closed. Conversely, suppose that \( H \) is an open set in \((Y, v_1, v_2)\). Then \( Y - H \) is closed in \((Y, v_1, v_2)\). By the assumption, \( f^{-1}(Y - H) = X - f^{-1}(H) \) is a \( \Gamma \)-closed set in \((X, u_1, u_2)\). Consequently, \( f^{-1}(H) \) is \( \Gamma \)-open in \((X, u_1, u_2)\). Therefore, \( f \) is \( \Gamma \)-continuous.

\[\Box\]
Definition 3.21. A biclosure space \((X, u_1, u_2)\) is said to be a \(\Gamma\)-open space if every \(\Gamma\)-open set in \((X, u_1, u_2)\) is open in \((X, u_1, u_2)\). Obviously, the biclosure space \((X, u_1, u_2)\) in Example 3.15 is a \(\Gamma\)-open space.

Theorem 3.22. Let \((X, u_1, u_2)\) and \((Z, w_1, w_2)\) be biclosure spaces and \((Y, v_1, v_2)\) be a \(\Gamma\)-open space and let \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) and \(g : (Y, v_1, v_2) \to (Z, w_1, w_2)\) be maps. If \(f\) and \(g\) are \(\Gamma\)-open, then \(g \circ f\) is \(\Gamma\)-open.

Proof. Let \(G\) be an open subset of \((X, u_1, u_2)\). Since \(f\) is \(\Gamma\)-open, \(f(G)\) is \(\Gamma\)-open in \((Y, v_1, v_2)\). As \((Y, v_1, v_2)\) is a \(\Gamma\)-open space, \(f(G)\) is open in \((Y, v_1, v_2)\). But \(g\) is \(\Gamma\)-open, hence \(g(f(G)) = (g \circ f)(G)\) is \(\Gamma\)-open in \((Z, w_1, w_2)\). Therefore, \(g \circ f\) is \(\Gamma\)-open. \(\square\)

Theorem 3.23. Let \((X, u_1, u_2)\) and \((Z, w_1, w_2)\) be biclosure spaces and \((Y, v_1, v_2)\) be a \(\Gamma\)-open space and let \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) and \(g : (Y, v_1, v_2) \to (Z, w_1, w_2)\) be maps. If \(f\) and \(g\) are \(\Gamma\)-continuous, then \(g \circ f\) is \(\Gamma\)-continuous.

Proof. Let \(H\) be an open subset of \((Z, w_1, w_2)\). Since \(g\) is \(\Gamma\)-continuous, \(g^{-1}(H)\) is \(\Gamma\)-open in \((Y, v_1, v_2)\). But \((Y, v_1, v_2)\) is a \(\Gamma\)-space, hence \(g^{-1}(H)\) is open in \((Y, v_1, v_2)\). As \(f\) is \(\Gamma\)-continuous, \(f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)\) is \(\Gamma\)-open in \((X, u_1, u_2)\). Therefore, \(g \circ f\) is \(\Gamma\)-continuous. \(\square\)

Definition 3.24. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces. A map \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) is called contra-\(\Gamma\)-continuous if the inverse image under \(f\) of every open set in \((Y, v_1, v_2)\) is \(\Gamma\)-closed in \((X, u_1, u_2)\).

Theorem 3.25. Let \((X, u_1, u_2)\) and \((Y, v_1, v_2)\) be biclosure spaces and let \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) be a map. Then \(f\) is contra-\(\Gamma\)-continuous if and only if the inverse image under \(f\) of every closed subset of \((Y, v_1, v_2)\) is \(\Gamma\)-open in \((X, u_1, u_2)\).

Proof. Let \(F\) be a closed subset of \((Y, v_1, v_2)\). Then \(Y - F\) is open in \((Y, v_1, v_2)\). Since \(f\) is contra-\(\Gamma\)-continuous, \(f^{-1}(Y - F)\) is \(\Gamma\)-closed in \((X, u_1, u_2)\). But \(f^{-1}(Y - F) = X - f^{-1}(F)\), thus \(f^{-1}(F)\) is \(\Gamma\)-open in \((X, u_1, u_2)\). Conversely, let \(G\) be an open set in \((Y, v_1, v_2)\). Then \(Y - G\) is closed in \((Y, v_1, v_2)\). Since the inverse image under \(f\) of every closed subset of \((Y, v_1, v_2)\) is \(\Gamma\)-open in \((X, u_1, u_2)\), \(f^{-1}(Y - G)\) is \(\Gamma\)-open in \((X, u_1, u_2)\). But \(f^{-1}(Y - G) = X - f^{-1}(G)\), thus \(f^{-1}(G)\) is \(\Gamma\)-closed. Therefore, \(f\) is contra-\(\Gamma\)-continuous. \(\square\)
Theorem 3.26. Let \((X, u_1, u_2)\), \((Y, v_1, v_2)\) and \((Z, w_1, w_2)\) be biclosure spaces. Let \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) and \(g : (Y, v_1, v_2) \to (Z, w_1, w_2)\) be maps. If \(g \circ f\) is contra-\(\Gamma\)-continuous and \(g\) is a closed injection, then \(f\) is contra-\(\Gamma\)-continuous.

Proof. Let \(H\) be a closed subset of \((Y, v_1, v_2)\). Since \(g\) is closed, \(g(H)\) is closed in \((Z, w_1, w_2)\). As \(g \circ f\) is contra-\(\Gamma\)-continuous, \((g \circ f)^{-1}(g(H)) = f^{-1}(g^{-1}(g(H)))\) is \(\Gamma\)-open in \((X, u_1, u_2)\). Since \(g\) is injective, \(f^{-1}(g^{-1}(g(H))) = f^{-1}(H)\). Therefore, \(f\) is contra-\(\Gamma\)-continuous.

Theorem 3.27. Let \((X, u_1, u_2)\) and \((Z, w_1, w_2)\) be biclosure spaces and \((Y, v_1, v_2)\) be a \(\Gamma\)-open space. If \(f : (X, u_1, u_2) \to (Y, v_1, v_2)\) and \(g : (Y, v_1, v_2) \to (Z, w_1, w_2)\) are contra-\(\Gamma\)-continuous maps, then \(g \circ f\) is \(\Gamma\)-continuous.

Proof. Let \(H\) be a closed subset in \((Z, w_1, w_2)\). Since \(g\) is contra-\(\Gamma\)-continuous, \(g^{-1}(H)\) is \(\Gamma\)-open in \((Y, v_1, v_2)\). But \((Y, v_1, v_2)\) is a \(\Gamma\)-open space, hence \(g^{-1}(H)\) is open in \((Y, v_1, v_2)\). As \(f\) is contra-\(\Gamma\)-continuous, \(f^{-1}(g^{-1}(g(H))) = (g \circ f)^{-1}(H)\) is \(\Gamma\)-closed in \((X, u_1, u_2)\). Therefore, \(g \circ f\) is \(\Gamma\)-continuous.

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References


