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Γ-open Sets in Biclosure Spaces

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Abstract

A new class of open sets in a biclosure space, called Γ-open sets, is introduced and studied. Moreover, we give the notions of Γ-open maps, Γ-continuous maps and contra-Γ-continuous maps and study some of their properties.

1 Introduction and Preliminaries

The notion of closure spaces was first introduced and studied by Čech [3]. In 2003, Šlapal [5] introduced generalized Čech closure spaces which are called closure spaces.

Definition 1.1. A map $u : P(X) \to P(X)$ is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied:

(A1) $u\emptyset = \emptyset$,

(A2) $A \subseteq uA$ for every $A \subseteq X$,

(A3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

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A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement in X is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$.

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

If (X, u) and (Y, v) are closure spaces, then a map $f : (X, u) \to (Y, v)$ is called *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ of closure spaces, denoted by $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$ is the closure space $\big(\prod_{\alpha\in J} X_\alpha, u\big)$ where $\prod_{\alpha\in J} X_\alpha$ denotes the Cartesian product of the sets X_α, $\alpha \in J$ and u is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha\in J}(X_\alpha, u_\alpha) \to (X_\alpha, u_\alpha)$, $\alpha \in J$; i.e., defined by $uA = \prod_{\alpha\in J} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha\in J} X_\alpha$.

Boonpok [1] studied some fundamental properties of closure spaces. In 2009, the following statements were proven.

Proposition 1.2. Let (X, u) and (Y, v) be closure spaces. If a map $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(H)$ is open in (X, u) each open set H in (Y, v).

Proposition 1.3. *Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces and let $\beta \in J$. Then the projection map $\pi_\beta : \prod_{\alpha\in J}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous.*

Proposition 1.4. *Let $\{(X_\alpha, u_\alpha) : \alpha \in J\}$ be a collection of closure spaces and let $\beta \in J$. Then G_β is an open subset of (X_β, u_β) if and only if $G_\beta \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in J}} X_\alpha$ is an open subset of $\prod_{\alpha\in J}(X_\alpha, u_\alpha)$.*

In 2009, the concept of biclosure spaces was introduced by Boonpok [2]. Biclosure spaces are sets endowed with two closure operators.

Definition 1.5. A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1u_2A = A$. The complement of closed set is called *open*.

Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_iA = u_iA \cap Y$ for all $i \in \{1, 2\}$ and every subset A of Y.

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called:

(i) *i-open* (respectively *i-closed*) if $f : (X, u_i) \to (Y, v_i)$ is open (respectively closed) for all $i \in \{1, 2\}$.

(ii) *open* (respectively *closed*) if f is i-open (respectively i-closed) for all $i \in \{1, 2\}$.

(iii) *i-continuous* if $f : (X, u_i) \to (Y, v_i)$ is continuous for all $i \in \{1, 2\}$.

(iv) *continuous* if f is *i-continuous* for all $i \in \{1, 2\}$.

In 2009, Khampakdee and Boonpok [6] introduced the notion of semi-open sets in biclosure spaces.

Definition 1.6. A subset A of a biclosure space (X, u_1, u_2) is called *semi-open*, if there exists an open subset G of (X, u_1) such that $G \subseteq A \subseteq u_2G$. The complement of a semi-open set in X is called *semi-closed*.

Moreover, the following statements were studied in [6].

Remark 1.7. Let A be a subset of a biclosure space (X, u_1, u_2). Then A is open in (X, u_1, u_2) if and only if A is open in both (X, u_1) and (X, u_2).

Proposition 1.8. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is open, then $f(G)$ is open in (Y, v_1, v_2) for every open subset G of (X, u_1, u_2).*

Proposition 1.9. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is continuous, then $f^{-1}(H)$ is open in (X, u_1, u_2) for every open subset H of (Y, v_1, v_2).*

2 Γ-open Sets in Closure Spaces

In this section, we introduce a new type of open sets in closure spaces and study some of their properties.

Definition 2.1. Let (X, u) be a closure space. A subset A of X is called a Γ-open set if there exists an open set G in X such that $A \subseteq G \subseteq uA$. A subset $A \subseteq X$ is called a Γ-closed set if its complement is Γ-open.

Clearly, if A is open (respectively closed) in (X, u) , then A is Γ-open (respectively Γ-closed) in (X, u). The converse is not true as shown in the following example.

Example 2.2. Let $X = \{1, 2, 3\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1\} = X$, $u\{2\} = \{2, 3\}$, $u\{3\} = \{3\}$ and $u\{1, 2\} = u\{1, 3\} = u\{2, 3\} = uX = X$. It is easy to see that $\{1\}$ is Γ-open because there is an open set $\{1, 2\}$ such that $\{1\} \subseteq \{1, 2\} \subseteq u\{1\}$. But$\{1\}$ is not open. And we also see that $\{2, 3\}$ is Γ-closed but not closed.

Regarding the union of Γ-open sets and the intersection of Γ-closed sets we have the following statements:

Theorem 2.3. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of Γ-open sets in a closure space (X, u). Then $\cup_{\alpha\in J} A_\alpha$ is Γ-open in (X, u).*

Proof. Let A_α be Γ-open in (X, u) for all $\alpha \in J$. Then, for each $\alpha \in J$, we have an open set G_α in (X, u) such that $A_\alpha \subseteq G_\alpha \subseteq uA_\alpha$. Thus, $\cup_{\alpha\in J} A_\alpha \subseteq \cup_{\alpha\in J} G_\alpha \subseteq \cup_{\alpha\in J} uA_\alpha$. Since $A_\alpha \subseteq \cup_{\alpha\in J} A_\alpha$ for each $\alpha \in J$, $uA_\alpha \subseteq u \cup_{\alpha\in J} A_\alpha$ for all $\alpha \in J$. Thus, $\cup_{\alpha\in J} uA_\alpha \subseteq u \cup_{\alpha\in J} A_\alpha$. Consequently, $\cup_{\alpha\in J} A_\alpha \subseteq \cup_{\alpha\in J} G_\alpha \subseteq u \cup_{\alpha\in J} A_\alpha$. As G_α is open, $X - G_\alpha$ is closed for all $\alpha \in J$. Hence, $u\cap_{\alpha\in J}(X-G_\alpha) \subseteq u(X-G_\alpha) = X-G_\alpha$ for each $\alpha \in J$. Thus, $u \cap_{\alpha\in J} (X - G_\alpha) \subseteq \cap_{\alpha\in J}(X - G_\alpha)$. But $\cap_{\alpha\in J}(X - G_\alpha) \subseteq u \cap_{\alpha\in J} (X - G_\alpha)$. Then $\cap_{\alpha\in J}(X - G_\alpha)$ is closed in (X, u); i.e., $\cup_{\alpha\in J} G_\alpha$ is open . Therefore, $\cup_{\alpha\in J} A_\alpha$ is Γ-open in (X, u). □

Theorem 2.4. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of Γ-closed sets in a closure space (X, u). Then $\cap_{\alpha\in J} A_\alpha$ is Γ-closed.*

Proof. Let A_α be a Γ-closed set in (X, u) for all $\alpha \in J$. Then $X - A_\alpha$ is Γ-open for each $\alpha \in J$. It follows that $\cup_{\alpha\in J}(X - A_\alpha)$ is Γ-open. But $\cup_{\alpha\in J}(X - A_\alpha) = X - \cap_{\alpha\in J} A_\alpha$. Thus $\cap_{\alpha\in J} A_\alpha$ is Γ-closed. □

Theorem 2.5. *Let (X, u) be a closure space and A be Γ-open in (X, u). If $B \subseteq A$ and $uA \subseteq uB$, then B is Γ-open.*

Proof. Let A be a Γ-open set in (X, u). Then there exists an open set G in (X, u) such that $A \subseteq G \subseteq uA$. Since $B \subseteq A$ and $uA \subseteq uB$, $B \subseteq A \subseteq G \subseteq uA \subseteq uB$. Thus, B is Γ-open. □

Theorem 2.6. *Let (Y, v) be a closure subspace of (X, u). If a subset $A \subseteq Y$ is a Γ-open set in (X, u), then A is a Γ-open set in (Y, v).*

Proof. Let A be Γ-open in (X, u). Then there exists an open set G in (X, u) such that $A \subseteq G \subseteq uA$. Hence, $A = A \cap Y \subseteq G \cap Y \subseteq uA \cap Y = vA$. Since $v(Y - (G \cap Y)) = u(Y - (G \cap Y)) \cap Y \subseteq u(X - G) \cap Y = (X - G) \cap Y = Y - (G \cap Y)$, $G \cap Y$ is open in (Y, v). Therefore, A is a Γ-open set in (Y, v). □

Theorem 2.7. *Let (X, u) be a closure space and let $A \subseteq X$. Then A is Γ-closed if and only if there exists a closed set F in (X, u) such that $X - u(X - A) \subseteq F \subseteq A$.*

Proof. Let A be Γ-closed. Then there exists an open set G in (X, u) such that $X - A \subseteq G \subseteq u(X - A)$. Thus, there exists a closed set F in (X, u) such that $G = X - F$ and $X - A \subseteq X - F \subseteq u(X - A)$. Hence, $X - u(X - A) \subseteq F \subseteq A$. Conversely, by the assumption, there is a closed set F in (X, u) such that $X - u(X - A) \subseteq F \subseteq A$. Thus, there exists an open set G in (X, u) such that $F = X - G$ and $X - u(X - A) \subseteq X - G \subseteq A$. It follows that $X - A \subseteq G \subseteq u(X - A)$. Consequently, $X - A$ is Γ-open in (X, u). As a result, A is Γ-closed in (X, u). □

3 Γ-open Sets in Biclosure Spaces

Definition 3.1. A subset A of a biclosure space (X, u_1, u_2) is called Γ-*open* if there exists an open set G in (X, u_1) such that $A \subseteq G \subseteq u_2A$. The complement of a Γ-open subset of X is called Γ-*closed*.

Remark 3.2. The independence of semi-open sets and Γ-open sets in biclosure spaces can be seen from the following examples.

Example 3.3. Let $X = \{1, 2, 3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1, 2\}$, $u_1\{2\} = u_1\{3\} = u_1\{2, 3\} = \{2, 3\}$ and $u_1\{1, 2\} =$

$u_1\{1,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = u_2\{3\} = u_2\{1,3\} = \{1,3\}$, $u_2\{2\} = \{1,2\}$ and $u_2\{1,2\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $\{1,3\}$ is semi-open because there is an open set $\{1\}$ in (X, u_1) such that $\{1\} \subseteq \{1,3\} \subseteq u_2\{1\}$. But $\{1,3\}$ is not Γ-open in (X, u_1, u_2).

Example 3.4. Let $X = \{1,2,3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$, $u_1\{3\} = \{1,3\}$ and $u_1\{2\} = u_1\{1,2\} = u_1\{1,3\} = u_1\{2,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = \{1\}$ and $u_2\{2\} = u_2\{3\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $\{3\}$ is Γ-open because there is an open set $\{2,3\}$ in (X, u_1) such that $\{3\} \subseteq \{2,3\} \subseteq u_2\{3\}$. But $\{3\}$ is not semi-open in (X, u_1, u_2).

Remark 3.5. If (X, u_1, u_2) is a biclosure space and A is open (respectively closed) in (X, u_1), then A is Γ-open (respectively Γ-closed) in (X, u_1, u_2). The converse is not true in general.

Example 3.6. Let $X = \{1,2,3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1,2\}$, $u_1\{2\} = \{2,3\}$, $u_1\{3\} = \{3\}$ and $u_1\{1,2\} = u_1\{1,3\} = u_1\{2,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{2\} = \{2\}$, $u_2\{3\} = \{2,3\}$ and $u_2\{1\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. It is easy to see that $\{1\}$ is Γ-open because there is an open set $\{1,2\}$ in (X, u_1) such that $\{1\} \subseteq \{1,2\} \subseteq u_2\{1\}$. But $\{1\}$ is not open in (X, u_1). And we also see that $\{2,3\}$ is Γ-closed but not closed.

Theorem 3.7. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is Γ-closed in (X, u_1, u_2) if and only if there exists a closed subset F of (X, u_1) such that $X - u_2(X - A) \subseteq F \subseteq A$.*

Proof. Let A be Γ-closed in (X, u_1, u_2). Then there exists an open subset G in (X, u_1) such that $X - A \subseteq G \subseteq u_2(X - A)$. Hence, there exists a closed subset F of (X, u_1) such that $G = X - F$. It follows that $X - A \subseteq X - F \subseteq u_2(X - A)$. Therefore, $X - u_2(X - A) \subseteq F \subseteq A$.

Conversely, by the assumption, there is a closed subset F of (X, u_1) such that $X - u_2(X - A) \subseteq F \subseteq A$. Thus, there exists an open set G in (X, u_1) such that $F = X - G$, i.e. $X - u_2(X - A) \subseteq X - G \subseteq A$. It follows that $X - A \subseteq G \subseteq u_2(X - A)$. Therefore, A is Γ-closed in (X, u_1, u_2). □

Theorem 3.8. *Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of Γ-open sets in a biclosure space (X, u_1, u_2). Then $\cup_{\alpha \in J} A_\alpha$ is a Γ-open set in (X, u_1, u_2).*

Proof. Let A_α be Γ-open in (X, u_1, u_2) for all $\alpha \in J$. Then, for each $\alpha \in J$, we have an open set G_α in (X, u_1) such that $A_\alpha \subseteq G_\alpha \subseteq u_2 A_\alpha$. Thus, $\cup_{\alpha\in J} A_\alpha \subseteq \cup_{\alpha\in J} G_\alpha \subseteq \cup_{\alpha\in J} u_2 A_\alpha$. As G_α is open in (X, u_1) for all $\alpha \in J$, $u_1 \cap_{\alpha\in J} (X - G_\alpha) \subseteq u_1(X - G_\alpha) = X - G_\alpha$ for each $\alpha \in J$. Thus, $u_1 \cap_{\alpha\in J} (X - G_\alpha) \subseteq \cap_{\alpha\in J}(X - G_\alpha)$. It follows that $\cap_{\alpha\in J}(X - G_\alpha)$ is closed in (X, u_1), i.e. $\cup_{\alpha\in J} G_\alpha$ is open in (X, u_1). Since $A_\alpha \subseteq \cup_{\alpha\in J} A_\alpha$ for each $\alpha \in J$, $u_2 A_\alpha \subseteq u_2 \cup_{\alpha\in J} A_\alpha$ for all $\alpha \in J$. Hence, $\cup_{\alpha\in J} u_2 A_\alpha \subseteq u_2 \cup_{\alpha\in J} A_\alpha$. Thus, $\cup_{\alpha\in J} A_\alpha \subseteq \cup_{\alpha\in J} G_\alpha \subseteq u_2 \cup_{\alpha\in J} A_\alpha$. Therefore, $\cup_{\alpha\in J} A_\alpha$ is Γ-open in (X, u_1, u_2). □

Corollary 3.9. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of Γ-closed sets in a biclosure space (X, u_1, u_2). Then $\cap_{\alpha\in J} A_\alpha$ is Γ-closed.*

Proof. Since A_α is Γ-closed in (X, u_1, u_2) for each $\alpha \in J$, $X - A_\alpha$ is Γ-open for all $\alpha \in J$. But $\cup_{\alpha\in J}(X - A_\alpha) = X - \cap_{\alpha\in J} A_\alpha$ is a Γ-open sets. Therefore, $\cap_{\alpha\in J} A_\alpha$ is Γ-closed in (X, u_1, u_2). □

Theorem 3.10. *Let (X, u_1, u_2) be a biclosure space and let A be a Γ-open in (X, u_1, u_2). If $B \subseteq A$ and $u_2 A \subseteq u_2 B$, then B is Γ-open in (X, u_1, u_2)*

Proof. Let A be Γ-open in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $A \subseteq G \subseteq u_2 A$. As $B \subseteq A$ and $u_2 A \subseteq u_2 B$, $B \subseteq A \subseteq G \subseteq u_2 A \subseteq u_2 B$, then B is Γ-open in (X, u_1, u_2). □

Theorem 3.11. *Let (Y, v_1, v_2) be a biclosure subspace of (X, u_1, u_2) and $A \subseteq Y$. If A is Γ-open in (X, u_1, u_2), then A is Γ-open in (Y, v_1, v_2).*

Proof. Let A be a Γ-open subset of (X, u_1, u_2). Then there exists an open subset G of (X, u_1) such that $A \subseteq G \subseteq u_2 A$. It follows that $A = A \cap Y \subseteq G \cap Y \subseteq u_2 A \cap Y = v_2 A$. Since G is open in (X, u_1), $v_1(Y - (G \cap Y)) = u_1(Y - (G \cap Y)) \cap Y \subseteq u_1(X - G) \cap Y = (X - G) \cap Y = Y - (G \cap Y)$. Thus, $G \cap Y$ is open in (Y, v_1). Therefore, A is Γ-open in (Y, v_1, v_2). □

Theorem 3.12. *Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in J\}$ be a family of biclosure spaces and let $k \in J$. If A_k is Γ-open in (X_k, u^1_k, u^2_k), then $A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha$ is Γ-open in $\prod_{\alpha\in J}(X_\alpha, u^1_\alpha, u^2_\alpha)$.*

Proof. Let $k \in J$ and let A_k be Γ-open in (X_k, u^1_k, u^2_k). Then there exists an open set G_k in (X_k, u^1_k) such that $A_k \subseteq G_k \subseteq u^2_k A_k$. Hence, $A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha \subseteq G_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha \subseteq u^2_k A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha$. But $G_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha$ is open in

$\prod_{\alpha\in J}(X_\alpha, u_\alpha^1)$ and $u_k^2 A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha = u_k^2 A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} u_\alpha^2 X_\alpha = \prod_{\alpha\in J} u_\alpha^2 \pi_\alpha(A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha)$ $= u_2(A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha)$. Therefore, $A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha$ is Γ-open in $\prod_{\alpha\in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$. □

Theorem 3.13. *Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in J\}$ be a family of biclosure spaces and let $k \in J$. If A_k is Γ-closed in (X_k, u_k^1, u_k^2), then $A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha$ is Γ-closed in $\prod_{\alpha\in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$.*

Proof. Let $k \in J$ and let A_k be Γ-closed in (X_k, u_k^1, u_k^2). Then $X_k - A_k$ is Γ-open in (X_k, u_k^1, u_k^2). Hence, $(X_k - A_k) \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha = \prod_{\alpha\in J} X_\alpha - (A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha)$ is Γ-open in $\prod_{\alpha\in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$. Consequently, $A_k \times \prod_{\substack{\alpha\neq k\\ \alpha\in J}} X_\alpha$ is Γ-closed in $\prod_{\alpha\in J}(X_\alpha, u_\alpha^1, u_\alpha^2)$. □

Definition 3.14. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called Γ-*open* (respectively Γ-*closed*) if $f(A)$ is Γ-open (respectively Γ-closed) in (Y, v_1, v_2) for every open (respectively closed) subset A of (X, u_1, u_2).

Clearly, if f is open (respectively closed), then f is Γ-open (respectively Γ-closed). The converse need not be true as the following example shows.

Example 3.15. Let $X = \{1, 2\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$, $u_1\{2\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = \{1\}$, $u_2\{2\} = \{2\}$ and $u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$, $v_1\{2\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{1\} = v_2\{2\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that f is Γ-open but not open because $f(\{2\})$ is not open in (Y, v_1, v_2) while $\{2\}$ is open in (X, u_1, u_2). Moreover, we can see that f is Γ-closed but not closed because $f(\{1\})$ is not closed in (Y, v_1, v_2) while $\{1\}$ is closed in (X, u_1, u_2).

Theorem 3.16. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. Then $g \circ f$ is Γ-open if f is open and g is Γ-open.*

Proof. Let G be an open subset of (X, u_1, u_2) and let f be an open map. It follows that $f(G)$ is open in (Y, v_1, v_2). Since g is Γ-open, $g(f(G)) = g \circ f(G)$ is Γ-open in (Z, w_1, w_2). Therefore, $g \circ f$ is Γ-open. □

Theorem 3.17. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If $g \circ f$ is Γ-open and f is a continuous surjection, then g is Γ-open.*

Proof. Let H be an open subset of (Y, v_1, v_2) and let f be continuous. Then $f^{-1}(H)$ is open in (X, u_1, u_2). Since $g \circ f$ is Γ-open, $g \circ f(f^{-1}(H))$ is Γ-open in (Z, w_1, w_2). Since f is a surjection, $g \circ f(f^{-1}(H)) = g(H)$. Thus, $g(H)$ is Γ-open in (Z, w_1, w_2). Therefore, g is Γ-open. □

Definition 3.18. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called Γ-*continuous* if $f^{-1}(H)$ is a Γ-open subset of (X, u_1, u_2) for every open subset H of (Y, v_1, v_2).

Clearly, if f is continuous, then f is Γ-continuous. The converse need not be true as can be seen from the following example.

Example 3.19. Let $X = \{1, 2\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$ and $u_1\{2\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$, $v_1\{2\} = \{2\}$, $v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{1\} = \{1\}$ and $v_2\{2\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that f is Γ-continuous but not continuous because $f^{-1}(\{2\})$ is not open in (X, u_1, u_2) while $\{2\}$ is open in (Y, v_1, v_2).

Theorem 3.20. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is Γ-continuous if and only if $f^{-1}(F)$ is a Γ-closed subset of (X, u_1, u_2) for every closed subset F of (Y, v_1, v_2)*

Proof. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be Γ-continuous. Let F be a closed subset of (Y, v_1, v_2). Then $Y - F$ is open in (Y, v_1, v_2). Since F is Γ-continuous, $f^{-1}(Y - F) = X - f^{-1}(F)$ is Γ-open in (X, u_1, u_2). Hence, $f^{-1}(F)$ is Γ-closed. Conversely, suppose that H is an open set in (Y, v_1, v_2). Then $Y - H$ is closed in (Y, v_1, v_2). By the assumption, $f^{-1}(Y - H) = X - f^{-1}(H)$ is a Γ-closed set in (X, u_1, u_2). Consequently, $f^{-1}(H)$ is Γ-open in (X, u_1, u_2). Therefore, f is Γ-continuous. □

Definition 3.21. A biclosure space (X, u_1, u_2) is said to be a Γ-*open space* if every Γ-open set in (X, u_1, u_2) is open in (X, u_1, u_2). Obviously, the biclosure space (X, u_1, u_2) in Example 3.15 is a Γ-open space.

Theorem 3.22. *Let (X, u_1, u_2) and (Z, w_1, w_2) be biclosure spaces and (Y, v_1, v_2) be a Γ-open space and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If f and g are Γ-open, then $g \circ f$ is Γ-open.*

Proof. Let G be an open subset of (X, u_1, u_2). Since f is Γ-open, $f(G)$ is Γ-open in (Y, v_1, v_2). As (Y, v_1, v_2) is a Γ-open space, $f(G)$ is open in (Y, v_1, v_2). But g is Γ-open, hence $g(f(G)) = (g \circ f)(G)$ is Γ-open in (Z, w_1, w_2). Therefore, $g \circ f$ is Γ-open. □

Theorem 3.23. *Let (X, u_1, u_2) and (Z, w_1, w_2) be biclosure spaces and (Y, v_1, v_2) be a Γ-open space and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If f and g are Γ-continuous, then $g \circ f$ is Γ-continuous.*

Proof. Let H be an open subset of (Z, w_1, w_2). Since g is Γ-continuous, $g^{-1}(H)$ is Γ-open in (Y, v_1, v_2). But (Y, v_1, v_2) is a Γ-space, hence $g^{-1}(H)$ is open in (Y, v_1, v_2). As f is Γ-continuous, $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is Γ-open in (X, u_1, u_2). Therefore, $g \circ f$ is Γ-continuous. □

Definition 3.24. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *contra*-Γ-*continuous* if the inverse image under f of every open set in (Y, v_1, v_2) is Γ-closed in (X, u_1, u_2).

Theorem 3.25. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. Then f is contra-Γ-continuous if and only if the inverse image under f of every closed subset of (Y, v_1, v_2) is Γ-open in (X, u_1, u_2).*

Proof. Let F be a closed subset of (Y, v_1, v_2). Then $Y - F$ is open in (Y, v_1, v_2). Since f is contra-Γ-continuous, $f^{-1}(Y - F)$ is Γ-closed in (X, u_1, u_2). But $f^{-1}(Y - F) = X - f^{-1}(F)$, thus $f^{-1}(F)$ is Γ-open in (X, u_1, u_2). Conversely, let G be an open set in (Y, v_1, v_2). Then $Y - G$ is closed in (Y, v_1, v_2). Since the inverse image under f of every closed subset of (Y, v_1, v_2) is Γ-open in (X, u_1, u_2), $f^{-1}(Y - G)$ is Γ-open in (X, u_1, u_2). But $f^{-1}(Y - G) = X - f^{-1}(G)$, thus $f^{-1}(G)$ is Γ-closed. Therefore, f is contra-Γ-continuous. □

Theorem 3.26. *Let* (X, u_1, u_2), (Y, v_1, v_2) *and* (Z, w_1, w_2) *be biclosure spaces. Let* $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ *and* $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ *be maps. If* $g \circ f$ *is contra-*Γ*-continuous and* g *is a closed injection, then* f *is contra-*Γ*-continuous.*

Proof. Let H be a closed subset of (Y, v_1, v_2). Since g is closed, $g(H)$ is closed in (Z, w_1, w_2). As $g \circ f$ is contra-Γ-continuous, $(g \circ f)^{-1}(g(H)) = f^{-1}(g^{-1}(g(H)))$ is Γ-open in (X, u_1, u_2). Since g is injective, $f^{-1}(g^{-1}(g(H))) = f^{-1}(H)$. Therefore, f is contra-Γ-continuous. □

Theorem 3.27. *Let* (X, u_1, u_2) *and* (Z, w_1, w_2) *be biclosure spaces and* (Y, v_1, v_2) *be a* Γ*-open space. If* $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ *and* $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ *are contra-*Γ*-continuous maps, then* $g \circ f$ *is* Γ*-continuous.*

Proof. Let H be a closed subset in (Z, w_1, w_2). Since g is contra-Γ-continuous, $g^{-1}(H)$ is Γ-open in (Y, v_1, v_2) . But (Y, v_1, v_2) is a Γ-open space, hence $g^{-1}(H)$ is open in (Y, v_1, v_2). As f is contra-Γ-continuous, $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is Γ-closed in (X, u_1, u_2). Therefore, $g \circ f$ is Γ-continuous. □

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