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# Triangle equivalences and Gorenstein schemes

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**Abstract**

Singularity category is an important invariant for rings of infinite global dimension and for singular varieties. It is quite an active subject widely studied by a number of authors. Let $(X, \mathcal{O}_X)$ be a Gorenstein scheme satisfying (ELF) condition below. We will show that the singularity category $\mathbf{D}_{sg}(X)$ is triangle equivalent to the stable category $\underline{\mathbf{MCM}}(X)$ of maximal Cohen-Macaulay sheaves over $X$.

## 1 Introduction

Singularity category is an important invariant for rings of infinite global dimension and for singular varieties. Singularity category is quite an active subject widely studied by a number of authors (see, for example, [1], [2], [3], [4] and [5]). In [1], Buchweitz states a remarkable theorem:

**Theorem 1.1.** *(Buchweitz) Assume $R$ is a Gorenstein ring. The singularity category $\mathbf{D}_{sg}(R)$ is triangle equivalent to the stable module category $\underline{\mathbf{MCM}}(R)$.*

Buchweitz's theorem is a beautiful result. Our primary aim is to prove Buchweitz's result for a Gorenstein scheme $(X, \mathcal{O}_X)$.

Let $X$ be a scheme. We will say that it satisfies condition (**ELF**) if it is separated, Noetherian, of finite Krull dimension, and the category of coherent sheaves $Coh(X)$ has enough locally free sheaves.

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**We always assume that $X$ satisfies condition (ELF).**

For example, any quasi-projective scheme satisfies this condition.
Let $X$ be a scheme. We say $X$ is Gorenstein if $X$ is locally Noetherian and $\mathcal{O}_{X,x}$ is Gorenstein for all $x \in X$.

The stable category $\underline{\mathbf{MCM}}(X)$ of maximal Cohen-Macaulay sheaves over $X$ is defined as follows. The objects are the maximal Cohen-Macaulay $\mathcal{O}_X$-sheaves, i.e., $\mathcal{F} \in Coh(X)$ with $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i > 0$. The hom-set $Hom_{\underline{\mathbf{MCM}}(X)}(\mathcal{F}, \mathcal{G})$ is the quotient of $Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ by the Abelian group consisting of homomorphisms $\mathcal{F} \to \mathcal{G}$ factoring through some locally free $\mathcal{O}_X$-sheaves of finite type.

**Main Theorem 1.** *Let $(X, \mathcal{O}_X)$ satisfy (ELF) and be Gorenstein. The singularity category $\mathbf{D}_{sg}(X)$ is triangle equivalent to the stable category $\underline{\mathbf{MCM}}(X)$ of maximal Cohen-Macaulay sheaves over $X$.*

We now give a brief outline of the paper. In section 2, we set notation and review some basics of the triangulated categories, the truncation of complexes and the locally free resolution. In section 3 we will prove our Main Theorem.

# 2 Preliminaries

## 2.1 Triangulated categories

For the definition of the triangulated category see, for example, [6], [7], [8].

Let $\mathbf{K}(X) = \mathbf{K}(Coh(X))$ be the homotopy category of unbounded chain complexes of coherent sheaves. Let $\mathbf{K}$ be the homotopy category of unbounded chain complexes of locally free sheaves of finite type. The following triangulated subcategories of $\mathbf{K}$ are of our concern.

$$\mathbf{K}^{\infty,b} = \{\mathcal{F}^\bullet \in \mathbf{K} | H^i(\mathcal{F}^\bullet) = 0 \ (except\ for\ finite\ i)\}. \tag{2.1}$$

$$\mathbf{K}^{-,b} = \{\mathcal{F}^\bullet \in \mathbf{K}^{\infty,b} | \mathcal{F}^i = 0 \ (for\ sufficiently\ large\ i)\}. \tag{2.2}$$

$$\mathbf{K}^{b} = \{\mathcal{F} \in \mathbf{K} | \mathcal{F}^i = 0 \ (except\ for\ finite\ i)\}. \tag{2.3}$$

The singularity category $\mathbf{D}_{sg}(X)$ of $X$ is defined to be the Verdier quotient of $\mathbf{D}^b(Coh(X))$ by the subcategory consisting of all perfect $\mathcal{O}_X$-complexes, i.e. $\mathbf{D}_{sg}(X) = \mathbf{K}^{-,b}/\mathbf{K}^b$.

Let $\underline{\mathbf{APC}}(X)$ be the full subcategory of $\mathbf{K}$ consisting of those chain complexes that are isomorphic to an acyclic complex of locally free sheaves of finite type.

**Proposition 2.1.** *([9], Proposition 3.5.7) Let $\mathcal{X} \subset \mathbf{K}(X)$ be a triangulated subcategory, and let $\mathcal{F}^\bullet \in \mathbf{K}(X)$.*

*If $\mathcal{H}om_{\mathbf{K}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = 0$ for every $\mathcal{G}^\bullet \in \mathbf{X}$, then the canonical functor $Q : \mathbf{K}(X) \to \mathbf{K}(X)/\mathcal{X}$ induces an isomorphism*

$$\mathcal{H}om_{\mathbf{K}(X)}(\mathcal{F}^\bullet, -) \to \mathcal{H}om_{\mathbf{K}(X)/\mathcal{X}}(\mathcal{F}^\bullet, -). \tag{2.4}$$

## 2.2 Truncation of complexes

Let $\mathcal{F}^\bullet$ be a complex. There are several ways to truncate the complex $\mathcal{F}^\bullet$:

- (1) The **left brutal truncation** $\sigma_{\leq n}$ is the subcomplex $\sigma_{\leq n}\mathcal{F}^\bullet$ defined by the rule

$$(\sigma_{\leq n}\mathcal{F}^\bullet)^i = \begin{cases} 0 \ (i > n) \\ \mathcal{F}^i \ (i \leq n). \end{cases} \tag{2.5}$$

- (2) The **right brutal truncation** $\sigma_{\geq n}$ is the subcomplex $\sigma_{\geq n}\mathcal{F}^\bullet$ defined by the rule

$$(\sigma_{\geq n}\mathcal{F}^\bullet)^i = \begin{cases} 0 \ (i < n) \\ \mathcal{F}^i \ (i \geq n). \end{cases} \tag{2.6}$$

**Proposition 2.2 ([9], Lemma 2.6.1).** *Let $\mathcal{F}^\bullet$ be a chain complex. Then*

$$0 \to \sigma_{\geq n}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to \sigma_{\leq n-1}\mathcal{F}^\bullet \to 0 \tag{2.7}$$

*is an exact sequence.*

**Proposition 2.3.** *Let $\mathcal{P}^\bullet \in \mathbf{K}^{-,b}$, then $\mathcal{P}^\bullet \to \sigma_{\leq n}\mathcal{P}^\bullet$ is an isomorphism in $\mathbf{D}_{sg}(X)$.*

*Proof.* By Proposition 2.2

$$\sigma_{\geq n+1}\mathcal{P}^\bullet \to \mathcal{P}^\bullet \to \sigma_{\leq n}\mathcal{P}^\bullet \to \sigma_{\geq n+1}P^\bullet[1] \tag{2.8}$$

is a distinguished triangle. Since $\mathcal{P}^\bullet \in \mathbf{K}^{-,b}$, we have $\sigma_{\geq n+1}\mathcal{P}^\bullet \in \mathbf{K}^b$, therefore $\mathcal{P}^\bullet \to \sigma_{\leq n}\mathcal{P}^\bullet$ is an isomorphism in $\mathbf{D}_{sg}(X)$. □

## 2.3 Locally free resolution

A bounded above locally free complex $\mathcal{P}^\bullet$ together with a quasi-isomorphism $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$ is called a locally free resolution of $\mathcal{F}^\bullet$.

**Proposition 2.4 ([9], Theorem 4.2.2).** *Let $X$ satisfy* (**ELF**)*, and let $Coh(X)$ be the category of coherent sheaves on $X$. In the category $\mathbf{Ch}^-(Coh(X))$ of bounded above coherent $\mathcal{O}_X$-complexes, every object has a locally free resolution.*

**Proposition 2.5.** *Given any $\mathcal{O}_X$-complex $\mathcal{G}^\bullet$ in $\mathbf{D}^b(Coh(X))$ and integers $s = sup\{i|H^i(\mathcal{G}^\bullet) \neq 0\}$. If $H^i(\mathcal{G}^\bullet) = 0$ for all $i < m$, then there is a distinguished triangle*

$$\mathcal{P}^\bullet \to \mathcal{G}^\bullet \to \mathcal{F}[1-m] \to \mathcal{P}^\bullet[1] \tag{2.9}$$

*in $\mathbf{D}^b(Coh(X))$ where $\mathcal{F}$ is a coherent $\mathcal{O}_X$-sheaf, and $\mathcal{P}^\bullet$ is a perfect complex with $\mathcal{P}^i = 0$ for $i > s$.*

*Proof.* By Proposition 2.4, we can assume that each $\mathcal{G}^i$ is a locally free sheaves of finite type, and $\mathcal{G}^i = 0$ for $i > s$:

$$\mathcal{G}^\bullet = \dots \xrightarrow{d^{-2}} \mathcal{G}^{-1} \xrightarrow{d^{-1}} \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \mathcal{G}^2 \xrightarrow{d^2} \dots \xrightarrow{d^{s-1}} \mathcal{G}^s \xrightarrow{0} 0 \to 0 \dots . \tag{2.10}$$

There is an exact sequence

$$0 \to \sigma_{\geq m}\mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \sigma_{\leq m-1}\mathcal{G}^\bullet \to 0, \tag{2.11}$$

so there is a distinguished triangle in $\mathbf{D}^b(Coh(X))$:

$$\sigma_{\geq m}\mathcal{G}^\bullet \to \mathcal{G}^\bullet \to \sigma_{\leq m-1}\mathcal{G}^\bullet \to \sigma_{\geq m}\mathcal{G}^\bullet[1], \tag{2.12}$$

where $\sigma_{\geq m}\mathcal{G}^\bullet$ is a perfect complex, Let $\mathcal{P}^\bullet = \sigma_{\geq m}\mathcal{G}^\bullet$ and $\mathcal{F} = Imd^{m-1}$, then there is a distinguished triangle

$$\mathcal{P}^\bullet \to \mathcal{G}^\bullet \to \mathcal{F}[1-m] \to \mathcal{P}^\bullet[1] \tag{2.13}$$

in $\mathbf{D}^b(Coh(X))$. □

**Proposition 2.6.** *([3], Proposition 1.19) Let $X$ satisfy* (**ELF**) *and be Gorenstein. Then the following conditions on a coherent sheaf $\mathcal{F}$ are equivalent.*

- *(1) The sheaves $\underline{\mathcal{E}xt}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ are trivial for all $i > 0$.*
- *(2) There is a unique right locally free resolution in $\mathbf{K}(X)$*

$$0 \to \mathcal{F} \to \mathcal{Q}^0 \to \mathcal{Q}^1 \to \mathcal{Q}^2 \to \dots . \tag{2.14}$$

# 3 Main Theorem

Next we define the functor $\Omega^k : \underline{\mathbf{APC}}(X) \to \underline{\mathbf{MCM}}(X)$ and show that it is an equivalence of categories.

**Definition 3.1.** *For any integer $k$ let $\Omega^k : \underline{\mathbf{APC}}(X) \to \underline{\mathbf{MCM}}(X)$ be the functor that sends a complex $\mathcal{F}^\bullet$ to the kernel of its $k$-th differential $d^k : \mathcal{F}^k \to \mathcal{F}^{k+1}$.*

**Lemma 3.2.** *The functor $\Omega^k : \underline{\mathbf{APC}}(X) \to \underline{\mathbf{MCM}}(X)$ is an equivalence of categories.*

*Proof.* We prove that it is dense and fully faithful. To see that it is dense let $\mathcal{F} \in Coh(X)$ be a maximal Cohen-Macaulay sheaf. There is a complex $\mathcal{P}^\bullet$ of locally free sheaves, such that $\mathcal{F} = Imd^{-1}$. Shifting so that $\mathcal{F}$ is the kernel of $d^k$ gives an object $\mathcal{G}^\bullet \in \underline{\mathbf{APC}}(X)$ such that $\Omega^k(\mathcal{G}^\bullet) = \mathcal{F}$.

To see that $\Omega^k$ is fully faithful let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \underline{\mathbf{APC}}(X)$ and let $f : \Omega^k(\mathcal{F}^\bullet) \to \Omega^k(\mathcal{G}^\bullet)$. Note that the complex $\mathcal{F}^\bullet$ gives a left locally free and right locally free resolution of $\Omega^k(\mathcal{F}^\bullet)$, and similarly for $\mathcal{G}^\bullet$ and $\Omega^k(\mathcal{G}^\bullet)$. Thus any map $\Omega^k(\mathcal{F}^\bullet) \to \Omega^k(\mathcal{G}^\bullet)$ can be lifted to a chain map $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and this map is unique up to chain homotopy. □

Next we define the functor $\sigma_{\leq k} : \underline{\mathbf{APC}}(X) \to \mathbf{D}_{sg}(X)$ and show that it is an equivalence of categories.

**Definition 3.3.** *For any integer $k$ let $\sigma_{\leq k} : \underline{\mathbf{APC}}(X) \to \mathbf{D}_{sg}(X)$ be the left brutal truncation.*

**Lemma 3.4.** *The functor $\sigma_{\leq k}$ is dense for all $k$.*

*Proof.* Given any $\mathcal{O}_X$-complex $\mathcal{G}^\bullet$ in $\mathbf{D}^b(Coh(X))$ and integers $s = sup\{i | H^i(\mathcal{G}^\bullet) \neq 0\}$. If $H^i(\mathcal{G}^\bullet) = 0$ for all $i < m$, then by Proposition 2.5 there is a distinguished triangle

$$\mathcal{P}^\bullet \to \mathcal{G}^\bullet \to \mathcal{F}[1-m] \to \mathcal{P}^\bullet[1] \tag{3.15}$$

in $\mathbf{D}^b(Coh(X))$ where $\mathcal{F}$ is a coherent $\mathcal{O}_X$-sheaf, and $\mathcal{P}^\bullet$ is a perfect complex with $\mathcal{P}^i = 0$ for $i > s$. Moreover,we get that $\mathcal{G}^\bullet = \mathcal{F}[1-m]$ in $\mathbf{D}_{sg}(X)$.

Since $\mathcal{G}^\bullet$ is bounded and $X$ is Gorenstein we know that the complex $\mathbf{R}\underline{\mathcal{H}om^\bullet}(\mathcal{G}^\bullet, \mathcal{O}_X)$ is bounded. This implies that if $m \ll 0$ then $\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i > 0$.

By Proposition 2.6 there is a right locally free resolution

$$0 \to \mathcal{F} \to \mathcal{Q}^0 \to \mathcal{Q}^1 \to \mathcal{Q}^2 \to \dots. \tag{3.16}$$

Hence there is a complex $\mathcal{Q}^\bullet$ of locally free sheaves, such that $\mathcal{F} = Imd^{-1}$. Shifting so that $\mathcal{F}$ is the kernel of $d^k$ gives an object $\mathcal{H}^\bullet \in \underline{\mathbf{APC}}(X)$ such that $\sigma_{\leq k}(\mathcal{H}^\bullet) = \mathcal{G}^\bullet$. □

**Proposition 3.5.** *Let* $\mathcal{H}^\bullet \in \underline{\mathbf{APC}}(X)$. *Then* $\mathcal{H}om_{\mathbf{K}(X)}(\sigma_{\leq k}(\mathcal{H}^\bullet), \mathbf{K}^b) = 0$ *for all* $k$.

*Proof.* Let $\mathcal{P}^\bullet \in \mathbf{K}^b$. As the $\sigma_{\leq k}(\mathcal{H}^\bullet)$ are isomorphic for all $k$ it suffices to show that there exists a $k$ such that

$$\mathcal{H}om_{\mathbf{K}(X)}(\sigma_{\leq k}(\mathcal{H}^\bullet), \mathcal{P}^\bullet) = 0. \tag{3.17}$$

For this take $k$ to be below the lower bound of the homology of $\mathcal{P}^\bullet$. □

**Lemma 3.6.** *The functor* $\sigma_{\leq k}$ *is fully faithful for all* $k$.

*Proof.* Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \underline{\mathbf{APC}}(X)$. Since $\mathbf{D}_{sg}(X) = \mathbf{K}^{-,b}/\mathbf{K}^b$, by Proposition 2.1 and Proposition 3.5 we have

$$\mathcal{H}om_{\mathbf{K}(X)}(\sigma_{\leq k}(\mathcal{F}^\bullet), \sigma_{\leq k}(\mathcal{G}^\bullet)) = \mathcal{H}om_{\mathbf{D}_{sg}(X)}(\sigma_{\leq k}(\mathcal{F}^\bullet), \sigma_{\leq k}(\mathcal{G}^\bullet)). \tag{3.18}$$

Hence it suffices to show that

$$\mathcal{H}om_{\mathbf{K}(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathcal{H}om_{\mathbf{K}(X)}(\sigma_{\leq k}(\mathcal{F}^\bullet), \sigma_{\leq k}(\mathcal{G}^\bullet)). \tag{3.19}$$

By Proposition 2.6, we are done. □

Now we can prove our main result.

*Proof of Main Theorem 1.* Let $\alpha : \underline{\mathbf{MCM}}(X) \to \mathbf{D}_{sg}(X)$ be the functor that sends a sheaf $\mathcal{F}$ to the complex

$$\ldots \to 0 \to \mathcal{F} \to 0 \to 0 \to 0 \to \ldots \tag{3.20}$$

concentrated in degree zero. By Lemmas 3.2, 3.4 and 3.6, $\alpha$ is an equivalence. □

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