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Soft bi-ideals of soft near-rings

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Abstract

In this article, we introduce the notions of soft bi-ideal of a soft near-ring and soft bi-ideal over a near-ring which are generalizations of soft quasi-ideal of a soft near-ring and soft quasi-ideal over a near-ring respectively. We explore the properties of these notions with illustrative examples. We provide the condition for a soft subgroup of a soft near-ring to be a soft bi-ideal. We show that every soft left ideal (resp. soft right ideal, soft ideal, soft quasi-ideal, soft left N-subgroup, soft right N-subgroup, soft invariant subnear-ring) of a soft near-ring is a soft bi-ideal but the converses are not true in general. We justify this by means of a counter example. Further we obtain the characterization of soft bi-ideals of soft zero-symmetric near-rings.

1 Introduction

Ideal theory plays an essential role in modern studies and manipulation of algebraic structures. The generalization of ideals explore the results necessary for the characterization of algebraic structures. In view of their importance, Yakabe defined the notions of quasi-ideals in near-rings and characterized those near-rings which are near-fields in terms of quasi-ideals in [18]. In [17],

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Tamizh Chelvam and Ganesan initiated the concept of bi-ideal and minimal bi-ideal in a near-ring. Manikantan defined the concept of fuzzy bi-ideal of a near-ring in [10]. The results concerning quasi-ideals, bi-ideals and fuzzy bi-ideals in the literature can be unified under the broad frame-work "soft sets". In [12], Molodtsov proposed the novel concept of soft set as a modern mathematical tool to understand the concepts of high complexity in daily life. In theoretical aspects, Maji et al. [9] presented several kinds of soft sets and soft binary operations. In [2], Ali et al. defined the operations such as the restricted intersection, the restricted union and the restricted difference of two soft sets. Sezgin and Atagün [15] examined the essential properties of these operations. Aktaş and Çağman [1] presented the concept of soft groups and derived some of its fundamental properties. In [4, 5, 6, 14, 16], Sezgin et al. defined and studied the notions of normalistic soft group, soft substructures of rings, fields and modules, soft near-ring, soft ideal and soft N-subgroup. In [11], Manikantan, Ramasamy and Sezgin introduced the concepts of soft quasi-ideal, soft minimal quasi-ideal, soft left (resp. right) N-subgroup and soft invariant subnear-ring of a soft near-ring. They also introduced the notions of soft zero-symmetric near-ring, soft constant near-ring, soft near-field and soft Q-simple near-ring over a near-ring. They investigated the properties of these notions. In this article, we define the notions of soft bi-ideal of a soft near-ring and soft bi-ideal over a near-ring as a generalization of soft quasi-ideal of a soft near-ring and soft quasi-ideal over a near-ring respectively. We explore the properties of these notions with illustrative examples. We provide the condition for a soft subgroup of a soft near-ring to be a soft bi-ideal. We show that every soft left ideal (resp. soft right ideal, soft ideal, soft quasi-ideal, soft left N-subgroup, soft right N-subgroup, soft invariant subnear-ring) of a soft near-ring is a soft bi-ideal but the converses are not true in general. We justify this by means of a counter example. Moreover, we obtain the characterization of soft bi-ideals of soft zero-symmetric near-rings. Finally, we prove that a soft right ideal of a soft left ideal of a soft zero-symmetric near-ring is a soft bi-ideal.

2 Preliminaries

A *near-ring* [13] is an algebraic structure $(N, +, \cdot)$ such that $(N, +)$ is a group with zero element 0, $(N, \cdot)$ is a semigroup and the right distributive law holds: $(u + v) \cdot z = u \cdot z + v \cdot z$ for all $u, v, z \in N$. In other words, it is a right near-ring. We will use the word "near-ring" to mean "right near-ring".

Throughout this article, N stands for a near-ring. We write uv for $u \cdot v$. Note that $0u = 0\ \forall u \in N$, while there may exist $u \in N$ such that $u0 \neq 0$. The set $N_0 = \{n \in N / n0 = 0\}$ is called the *zero-symmetric* part of N. N is called *zero-symmetric near-ring* if $N = N_0$. An element $d \in N$ is called distributive if $\forall n, n' \in N$: $d(n + n') = dn + dn'$.

Let A and B be two non-empty subsets of N. Then $A * B = \{a(a' + b) - aa'/a, a' \in A, b \in B\}$.

A subgroup M of $(N, +)$ with $MM \subseteq M$ (resp. $NM \subseteq M$, $MN \subseteq M$) is called a *subnear-ring* (resp. *left N-subgroup*, *right N-subgroup*) of N. A subnear-ring M of N with $MN \subseteq M$ and $NM \subseteq M$ is called an *invariant subnear-ring* of N [13].

A normal subgroup G of $(N, +)$ is called an *ideal* of N, if it satisfies: (i) $GN \subseteq G$ and (ii) $N * G \subseteq G$. A normal subgroup G of $(N, +)$ with (i) is called a *right-ideal* of N and with (ii) is called a *left-ideal* of N [13].

A subgroup G of $(N, +)$ is called a *quasi-ideal* of N, denoted by $G \lhd_q N$, if $GN \cap NG \cap N * G \subseteq G$. For a zero-symmetric near-ring N, a subgroup G of $(N, +)$ is a *quasi-ideal* of N if $GN \cap NG \subseteq G$ [18].

A subgroup G of $(N, +)$ is called a *bi-ideal* of N, denoted by $G \lhd_b N$, if $GNG \cap (GN) * G \subseteq G$. For a zero-symmetric near-ring N, a subgroup G of $(N, +)$ is a *bi-ideal* of N if $GNG \subseteq G$ [17].

In what follows U is a basic universal set and E is a set of parameters, $\mathbb{P}(U)$ being the power set of U and $A \subseteq E$.

Definition 2.1. *[12] A pair $(\widetilde{H}, A)$ is called a soft set (briefly, SS) over U, where $\widetilde{H}$ is a mapping given by $\widetilde{H} : A \to \mathbb{P}(U)$.*

Definition 2.2. *[9] Let $(\widetilde{H}_1, A_1)$ and $(\widetilde{H}_2, A_2)$ be SSs over U. Then $(\widetilde{H}_1, A_1)$ is called a soft subset of $(\widetilde{H}_2, A_2)$ if $A_1 \subseteq A_2$ and $\widetilde{H}_1(\rho) \subseteq \widetilde{H}_2(\rho)$, $\forall\ \rho \in A_1$.*

Definition 2.3. *[8] For a SS $(\widetilde{H}, A)$ over U, the set $Supp(\widetilde{H}, A) = \{\rho \in A / \widetilde{H}(\rho) \neq \emptyset\}$ is called a support of the SS $(\widetilde{H}, A)$. If $Supp(\widetilde{H}, A) \neq \emptyset$, then the SS $(\widetilde{H}, A)$ is called non-null.*

Definition 2.4. *[3] The SS $(\widetilde{H}, A)$ over U is said to be a relative whole SS (with respect to the parameter set A), denoted by $W_{(U,A)}$, if $\widetilde{H}(\rho) = U$, $\forall\ \rho \in A$. The relative whole SS with respect to the universe set of parameters E is called the absolute SS over U and is denoted by A_U.*

Definition 2.5. *[2] Let $(\widetilde{H}_1, A_1)$ and $(\widetilde{H}_2, A_2)$ be SSs over U such that $A_1 \cap A_2 \neq \emptyset$. The restricted intersection of $(\widetilde{H}_1, A_1)$ and $(\widetilde{H}_2, A_2)$, denoted by*

$(\widetilde{H}_1, A_1)\widetilde{\cap}_R(\widetilde{H}_2, A_2)$, is defined as $(\widetilde{H}_1, A_1)\widetilde{\cap}_R(\widetilde{H}_2, A_2) = (\widetilde{H}, A)$, where $A = A_1 \cap A_2$ and $\widetilde{H}(\rho) = \widetilde{H}_1(\rho) \cap \widetilde{H}_2(\rho)$, $\forall\ \rho \in A$.

Definition 2.6. *[15] Let $(\widetilde{H}_i, A_i)_{i\in\Omega} \neq \emptyset$ be a family of SSs over U. The restricted intersection of these SSs, denoted by $(\widetilde{\cap}_R)_{i\in\Omega}(\widetilde{H}_i, A_i)$, is defined to be the SS $(\widetilde{H}, A)$ such that $A = \cap_{i\in\Omega} A_i \neq \emptyset$ and $\widetilde{H}(\rho) = \cap_{i\in\Omega}\widetilde{H}_i(\rho)$, $\forall\ \rho \in A$.*

Definition 2.7. *[7] Let $(\widetilde{H}_1, A_1)$ and $(\widetilde{H}_2, A_2)$ be SSs over U. The product of $(\widetilde{H}_1, A_1)$ and $(\widetilde{H}_2, A_2)$ is denoted by $(\widetilde{H}_1, A_1)\,\widetilde{\circ}\,(\widetilde{H}_2, A_2)$, and is defined as the SS $(\widetilde{H}, A)$, where $A = A_1 \cap A_2 \neq \emptyset$, $\widetilde{H}(\rho) = \widetilde{H}_1(\rho)\,\widetilde{H}_2(\rho)$, $\forall\ \rho \in A$ and $\widetilde{H}_1(\rho)\widetilde{H}_2(\rho) = \{x_1\, x_2/x_1 \in \widetilde{H}_1(\rho), x_2 \in \widetilde{H}_2(\rho)\}$. Similarly, we define the product of an element $x \in N$ with the SS $(\widetilde{H}_1, A_1)$ as the SSs $x\,\widetilde{\circ}\,(\widetilde{H}_1, A_1) = \{x\widetilde{H}_1(\rho)/\rho \in A_1\}$ and $(\widetilde{H}_1, A_1)\,\widetilde{\circ}\,x = \{\widetilde{H}_1(\rho)x/\rho \in A_1\}$.*

From now on, let P be a nonempty set and $\mathcal{R}$ be an arbitrary binary relation between an element of P and an element of N, that is $\mathcal{R}$ is a subset of $P \times N$. A set-valued function $\widetilde{H} : P \to \mathbb{P}(N)$ can be defined as $\widetilde{H}(\rho) = \{\sigma \in N/(\rho, \sigma) \in \mathcal{R}\}$, $\forall\ \rho \in P$. Then the pair $(\widetilde{H}, P)$ is a SS over N, which is obtained from the relation $\mathcal{R}$.

Definition 2.8. *[11] Let $(\widetilde{H}_1, P_1)$ and $(\widetilde{H}_2, P_2)$ be two SSs over N. The $\widetilde{*}$-product of $(\widetilde{H}_1, P_1)$ and $(\widetilde{H}_2, P_2)$ is denoted by $(\widetilde{H}_1, P_1)\,\widetilde{*}\,(\widetilde{H}_2, P_2)$, and is defined as the SS $(\widetilde{H}, P)$, where $P = P_1 \cap P_2 \neq \emptyset$, $\widetilde{H}(\rho) = \widetilde{H}_1(\rho) * \widetilde{H}_2(\rho)$, $\forall\ \rho \in P$ and $\widetilde{H}_1(\rho) * \widetilde{H}_2(\rho) = \{n_1(n_2 + n_3) - n_1 n_2/n_1, n_2 \in \widetilde{H}_1(\rho), n_3 \in \widetilde{H}_2(\rho)\}$.*

Definition 2.9. *[1] Let $(\widetilde{H}, P)$ be a non-null SS over a group $\mathcal{G}$. Then $(\widetilde{H}, P)$ is called a soft group (briefly, SG) over $\mathcal{G}$ if and only if $\widetilde{H}(\rho)$ is a subgroup of $\mathcal{G}$, $\forall\ \rho \in P$.*

Definition 2.10. *[1] Let $(\widetilde{H}_1, P_1)$ and $(\widetilde{H}_2, P_2)$ be SGs over $\mathcal{G}$. Then the SG $(\widetilde{H}_2, P_2)$ is called a soft subgroup (briefly, SSG) of $(\widetilde{H}_1, P_1)$ if $P_2 \subseteq P_1$ and $\widetilde{H}_2(\rho)$ is a subgroup of $\widetilde{H}_1(\rho)$, $\forall\ \rho \in P_2$.*

Definition 2.11. *[14] Let $(\widetilde{H}, P)$ be a non-null SS over N. Then $(\widetilde{H}, P)$ is called a soft near-ring (SN, for short) over N if $\widetilde{H}(\rho)$ is a subnear-ring of N, $\forall\ \rho \in P$.*

Definition 2.12. *[14] Let $(\widetilde{H}_1, P_1)$ and $(\widetilde{H}_2, P_2)$ be SNs over N. Then the SN $(\widetilde{H}_2, P_2)$ is called a soft subnear-ring (SSN, for short) of $(\widetilde{H}_1, P_1)$ if $P_2 \subseteq P_1$ and $\widetilde{H}_2(\rho)$ is a subnear-ring of $\widetilde{H}_1(\rho)$, $\forall\ \rho \in P_2$.*

Definition 2.13. *[14] Let $(\widetilde{H}, P)$ be a SN over N. A non-null SS $(\widetilde{K}, I)$ over N is called a soft left (resp. right) ideal of $(\widetilde{H}, P)$ if $I \subseteq P$ and $\widetilde{K}(\rho)$ is a left (resp. right) ideal of $\widetilde{H}(\rho)$, $\forall\ \rho \in I$.*

If $(\widetilde{K}, I)$ is both soft left ideal and soft right ideal of $(\widetilde{H}, P)$, then we say that $(\widetilde{K}, I)$ is a soft ideal of $(\widetilde{H}, P)$.

Definition 2.14. *[11] Let $(\widetilde{H}, P)$ be a SN over N. A non-null SS $(\widetilde{K}, Q)$ over N is called a soft quasi-ideal (briefly, SQI) of $(\widetilde{H}, P)$ if $Q \subseteq P$ and $\widetilde{K}(\rho) \lhd_q \widetilde{H}(\rho)$, $\forall\ \rho \in Q$.*

Definition 2.15. *[11] A non-null SS $(\widetilde{K}, Q)$ over N is called a soft quasi-ideal (briefly, SQI) over N if $\widetilde{K}(\rho) \lhd_q N$, $\forall\ \rho \in Q$.*

3 Soft bi-ideals of soft near-rings

In this section, we introduce the notions of soft bi-ideal of a soft near-ring and soft bi-ideal over a near-ring. We discuss the properties of these notions with illustrative examples. We obtain the condition for a soft subgroup of a soft near-ring to be a soft bi-ideal.

Definition 3.1. *Let $(\widetilde{H}, P)$ be a SN over N. A non-null SS $(\widetilde{K}, B)$ over N is called a soft bi-ideal (briefly, SBI) of $(\widetilde{H}, P)$, denoted by $(\widetilde{K}, B) \mathrel{\widetilde{\lhd}_b} (\widetilde{H}, P)$, if it satisfies:*

(i) $B \subseteq P$ and

(ii) $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$.

Theorem 3.1. *A SSG $(\widetilde{K}, B)$ of a SN $(\widetilde{H}, P)$ is a SBI of $(\widetilde{H}, P)$ if and only if $((\widetilde{K}, B)\widetilde{\circ}(\widetilde{H}, P)\widetilde{\circ}(\widetilde{K}, B)) \mathbin{\widetilde{\cap}_R} ((\widetilde{K}, B)\ \widetilde{\circ}(\widetilde{H}, P))\widetilde{*}(\widetilde{K}, B) \subseteq (\widetilde{K}, B)$.*

Proof. $(\widetilde{K}, B)$ *is a SBI of* $(\widetilde{H}, P)$
$\Leftrightarrow B \subseteq P$ *and each* $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$
$\Leftrightarrow B \subseteq P$ *and* $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$
$\Leftrightarrow ((\widetilde{K}, B)\widetilde{\circ}(\widetilde{H}, P)\widetilde{\circ}(\widetilde{K}, B))\mathbin{\widetilde{\cap}_R}((\widetilde{K}, B)\widetilde{\circ}(\widetilde{H}, P))\widetilde{*}(\widetilde{K}, B)) \subseteq (\widetilde{K}, B)$.

Definition 3.2. *A non-null SS $(\widetilde{K}, B)$ over N is called a soft bi-ideal (briefly, SBI) over N if $\widetilde{K}(\rho) \lhd_b N$, $\forall\ \rho \in B$.*

Theorem 3.2. *A SSG $(\widetilde{K}, B)$ over N is a SBI over N if and only if* $((\widetilde{K}, B) \,\widetilde{\circ}\, A_N \,\widetilde{\circ}\, (\widetilde{K}, B)) \;\widetilde{\cap}_R\, ((\widetilde{K}, B) \,\widetilde{\circ}\, A_N) \,\widetilde{*}\, (\widetilde{K}, B) \subseteq (\widetilde{K}, B)$.

Proof. $(\widetilde{K}, B)$ *is a SBI over* N

$\Leftrightarrow$ *each* $\widetilde{K}(\rho) \lhd_b N$, $\forall\ \rho \in B$

$\Leftrightarrow (\widetilde{K}(\rho) N \widetilde{K}(\rho)) \cap (\widetilde{K}(\rho) N) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$

$\Leftrightarrow ((\widetilde{K}, B) \,\widetilde{\circ}\, A_N \,\widetilde{\circ}\, (\widetilde{K}, B)) \,\widetilde{\cap}_R\, ((\widetilde{K}, B) \,\widetilde{\circ}\, A_N) \,\widetilde{*}\, (\widetilde{K}, B) \subseteq (\widetilde{K}, B)$.

Example 3.1. *Consider a near-ring N over the Dihedral group $D_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the following operations table: (Scheme 76: (15,1,35,5,15,1,35,5) see[13], p. 415)*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	4	1	6	3	4	1	6	3
2	0	2	0	2	0	2	0	2
3	4	3	6	1	4	3	6	1
4	4	4	4	4	4	4	4	4
5	0	5	2	7	0	5	2	7
6	4	6	4	6	4	6	4	6
7	0	7	2	5	0	7	2	5

Let $P = N$ and $(\widetilde{H}, P)$ be a SS over N, where $\widetilde{H} : P \to \mathbb{P}(N)$ is a set-valued function defined by $\widetilde{H}(\rho) = \{\sigma \in N/\rho\mathcal{R}\sigma \Leftrightarrow \rho\sigma \in \{0, 2, 4, 6\}\}$, $\forall\ \rho \in P$. Then $\widetilde{H}(0) = \widetilde{H}(2) = \widetilde{H}(4) = \widetilde{H}(6) = N$ and $\widetilde{H}(1) = \widetilde{H}(3) = \widetilde{H}(5) = \widetilde{H}(7) = \{0, 2, 4, 6\}$ which are all subnear-rings of N. Hence $(\widetilde{H}, P)$ is a SN over N.

Let $(\widetilde{K}, B)$ be a SS over N defined by $\widetilde{K}(\rho) = \{\sigma \in N/\rho\mathcal{R}\sigma \Leftrightarrow \rho\sigma \in \{0, 4\}\}$, where $B = \{1, 2, 3\}$. Then $\widetilde{K}(1) = \{0, 4\}, \widetilde{K}(2) = \{0, 2, 4, 6\}$ and $\widetilde{K}(3) = \{0, 4\}$ are bi-ideals of $\widetilde{H}(1), \widetilde{H}(2)$ and $\widetilde{H}(3)$, respectively. So, $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$. Hence $(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$ over N.

Remark 3.1. *The SBI of a SN as in Definition 3.1 is different from the SBI over a near-ring N as in Definition 3.2. The following example illustrates this situation.*

Example 3.2. *Consider a near-ring N over the Dihedral group $D_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with $(N, +)$ and $(N, \cdot)$ as defined below (Scheme 132: (15,35,15,35,15,1,15,1) see[13], p. 415).*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	4	6	4	6	4	1	4	1
2	0	0	0	0	0	2	0	2
3	4	6	4	6	4	3	4	3
4	4	4	4	4	4	4	4	4
5	0	2	0	2	0	5	0	5
6	4	4	4	4	4	6	4	6
7	0	2	0	2	0	7	0	7

Let $(\widetilde{H}, P)$ be a SS over N, where $P = N$ and let $\widetilde{H}$ be defined as $\widetilde{H}(0) = \widetilde{H}(2) = \widetilde{H}(4) = N$, $\widetilde{H}(1) = \widetilde{H}(3) = \widetilde{H}(6) = \{0, 2, 4, 6\}$ and $\widetilde{H}(5) = \widetilde{H}(7) = \{0, 2, 5, 7\}$ which are all subnear-rings of N. Then $(\widetilde{H}, P)$ is a SN over N.

Let $(\widetilde{K}, B)$ be a SS over N defined by $\widetilde{K}(\rho) = \{0\} \cup \{\sigma \in N/\rho\mathcal{R}\sigma \Leftrightarrow \rho + \sigma = 0\}$, where $B = \{5, 6, 7\}$. Then $\widetilde{K}(5) = \{0, 5\}, \widetilde{K}(6) = \{0, 6\}$ and $\widetilde{K}(7) = \{0, 7\}$ are bi-ideals of $\widetilde{H}(5), \widetilde{H}(6)$ and $\widetilde{H}(7)$ respectively. Hence $(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$ over N. Whereas $\widetilde{K}(5) = \{0, 5\}$ and $\widetilde{K}(7) = \{0, 7\}$ are not bi-ideals of N, $(\widetilde{K}, B)$ is not a SBI over N.

Theorem 3.3. *The restricted intersection of family of SBIs of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$ if it is non-null.*

Proof. *Let $(\widetilde{K}_i, B_i)_{i\in\Omega}$ be a family of SBIs of a SN $(\widetilde{H}, P)$. Let $(\widetilde{K}, B) = (\widetilde{\cap}_R)_{i\in\Omega}\,(\widetilde{K}_i, B_i)$, where $B = \cap_{i\in\Omega} B_i \neq \emptyset$ and $\widetilde{K}(\rho) = \cap_{i\in\Omega} \widetilde{K}_i(\rho)$ for $\rho \in Supp(\widetilde{K}, B)$. Suppose that $(\widetilde{K}, B)$ is a non-null soft set. Since each $(\widetilde{K}_i, B_i)$ is a SSG of $(\widetilde{H}, P)$, by Theorem 24 of [1], $(\widetilde{K}, B)$ is also a SSG of $(\widetilde{H}, P)$. Since $\widetilde{K}(\rho) = \cap_{i\in\Omega} \widetilde{K}_i(\rho) \subseteq \widetilde{K}_i(\rho)$ for all $i \in \Omega$ and $\rho \in Supp(\widetilde{K}, B)$, we have $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq (\widetilde{K}_i(\rho)\widetilde{H}(\rho)\widetilde{K}_i(\rho)) \cap (\widetilde{K}_i(\rho)\widetilde{H}(\rho)) * \widetilde{K}_i(\rho) \subseteq \widetilde{K}_i(\rho)$, $\forall\ i \in \Omega$. It follows that $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \cap_{i\in\Omega} \widetilde{K}_i(\rho) = \widetilde{K}(\rho)$, $\forall\ \rho \in Supp(\widetilde{K}, B)$. Hence $(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$.*

Corollary 3.1. *The restricted intersection of two SBIs of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$ if it is non-null.*

Proof. *The proof is straightforward using Theorem 3.3.*

Theorem 3.4. *Let $(\widetilde{K}_1, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$ and $(\widetilde{K_2}, S)$ be a SSN of $(\widetilde{H}, P)$ over N. Then $(\widetilde{K}_1, B)\widetilde{\cap}_R\,(\widetilde{K_2}, S) \,\widetilde{\lhd}_b\, (\widetilde{K_2}, S)$ over N if it is non-null.*

Proof. *Using Definition 2.5, we can write $(\widetilde{K}_1, B)\widetilde{\cap}_R\,(\widetilde{K_2}, S) = (\widetilde{J}, C)$, where $C = B \cap S \neq \emptyset$ and $\widetilde{J}(\rho) = \widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho)$, $\forall\ \rho \in Supp(\widetilde{J}, C)$. Since*

$B \subseteq P$ and $S \subseteq P, B \cap S = C \subseteq P$. Suppose that $(\widetilde{J}, C)$ is non-null. If $\rho \in Supp(\widetilde{J}, C)$, then $\widetilde{J}(\rho) = \widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho) \neq \emptyset$. Since $\widetilde{K}_1(\rho)$ and $\widetilde{K_2}(\rho)$ are subgroups of $\widetilde{H}(\rho)$, $\widetilde{J}(\rho) = \widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho)$ is also a subgroup of $\widetilde{K_2}(\rho)$, $\forall$ $\rho \in Supp(\widetilde{J}, C)$. For any $\rho \in Supp(\widetilde{J}, C)$,

$$\begin{aligned}
&(\widetilde{J}(\rho)\widetilde{K_2}(\rho)\widetilde{J}(\rho)) \cap (\widetilde{J}(\rho)\widetilde{K_2}(\rho)) * \widetilde{J}(\rho) \\
&\quad = ((\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))\widetilde{K_2}(\rho)(\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))) \cap \\
&\qquad ((\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))\widetilde{K_2}(\rho)) * (\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho)) \\
&\quad \subseteq (\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))\widetilde{K_2}(\rho)(\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho)) \\
&\quad \subseteq \widetilde{K_2}(\rho)\widetilde{K_2}(\rho)\widetilde{K_2}(\rho), as \widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho) \subseteq \widetilde{K_2}(\rho) \\
&\quad \subseteq \widetilde{K_2}(\rho)\widetilde{K_2}(\rho) \subseteq \widetilde{K_2}(\rho) \ \textit{for all } \rho \in Supp(\widetilde{J}, C)
\end{aligned}$$

and

$$\begin{aligned}
&(\widetilde{J}(\rho)\widetilde{K_2}(\rho)\widetilde{J}(\rho)) \cap (\widetilde{J}(\rho)\widetilde{K_2}(\rho) * \widetilde{J}(\rho) \\
&\quad = ((\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))\widetilde{K_2}(\rho)(\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))) \cap \\
&\qquad ((\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho))\widetilde{K_2}(\rho)) * (\widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho)) \\
&\quad \subseteq (\widetilde{K}_1(\rho)\widetilde{H}(\rho)\widetilde{K}_1(\rho)) \cap (\widetilde{K}_1(\rho)\widetilde{H}(\rho)) * \widetilde{K}_1(\rho) \subseteq \widetilde{K}_1(\rho), \\
&\qquad as\, \widetilde{K_2}(\rho) \subseteq \widetilde{H}(\rho) \ and\ \widetilde{K}_1(\rho) \lhd_b \widetilde{H}(\rho), \forall \rho \in Supp(\widetilde{J}, C).
\end{aligned}$$

Therefore $(\widetilde{J}(\rho)\widetilde{K_2}(\rho)\widetilde{J}(\rho)) \cap (\widetilde{J}(\rho)\widetilde{K_2}(\rho) * \widetilde{J}(\rho) \subseteq \widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho) = \widetilde{J}(\rho)$ $\forall$ $\rho \in Supp(\widetilde{J}, C)$. Consequently, $\widetilde{J}(\rho) = \widetilde{K}_1(\rho) \cap \widetilde{K_2}(\rho) \lhd_b \widetilde{K_2}(\rho)$ $\forall$ $\rho \in Supp(\widetilde{J}, C)$. Hence $(\widetilde{K}, Q) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$ over N.

Theorem 3.5. *Every soft left ideal (resp. soft right ideal, soft ideal) of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$ if it is non-null.*

Proof. *Let $(\widetilde{K}, I)$ be a soft left ideal of $(\widetilde{H}, P)$. Then, $\widetilde{H}(\rho) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall$ $\rho \in I$. For each $\rho \in I$, we have $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq (\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap \widetilde{H}(\rho) * \widetilde{K}(\rho) \subseteq (\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$. This shows that $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall$ $\rho \in I$. Hence $(\widetilde{K}, I) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$. Similarly, the other cases follow.*

Theorem 3.6. *The restricted intersection of soft left ideal and soft right ideal of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$ if it is non-null.*

Proof. *The proof follows from Theorem 3.5 and Corollary 3.1.*

Theorem 3.7. *Every SQI of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$.*
Proof. *Let $(\widetilde{K}, Q)$ be a SQI of $(\widetilde{H}, P)$. Then, $\widetilde{K}(\rho)$ is a subgroup of $\widetilde{H}(\rho)$ and $\widetilde{K}(\rho)\widetilde{H}(\rho) \cap \widetilde{H}(\rho)\widetilde{K}(\rho) \cap \widetilde{H}(\rho) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in Q$. For any $\rho \in Q$,*

$$\begin{aligned}
&(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \\
&\quad = (\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \\
&\quad \subseteq (\widetilde{K}(\rho)\widetilde{H}(\rho)) \cap (\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap \widetilde{H}(\rho) * \widetilde{K}(\rho) \\
&\quad \subseteq \widetilde{K}(\rho), \, as \, \widetilde{K}(\rho)\widetilde{H}(\rho) \subseteq \widetilde{H}(\rho) \text{ and } \widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{H}(\rho).
\end{aligned}$$

This implies that $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in Q$. Hence $(\widetilde{K}, Q) \mathrel{\widetilde{\lhd}_b} (\widetilde{H}, P)$ over N.

Theorem 3.8. *Every SQI over a near-ring N is a SBI over N.*
Proof. *Let $(\widetilde{K}, Q)$ be a SQI over N. Then, $\widetilde{K}(\rho)$ is a subgroup of N and $\widetilde{K}(\rho)N \cap N\widetilde{K}(\rho) \cap N * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in Q$. For any $\rho \in Q$,*

$$\begin{aligned}
&(\widetilde{K}(\rho)N\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)N) * \widetilde{K}(\rho) \\
&\quad = (\widetilde{K}(\rho)N\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)N\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)N) * \widetilde{K}(\rho) \\
&\quad \subseteq (\widetilde{K}(\rho)N) \cap (N\widetilde{K}(\rho)) \cap N * \widetilde{K}(\rho) \\
&\quad \subseteq \widetilde{K}(\rho), \text{ as } \widetilde{K}(\rho)N \subseteq N \text{ and } N\widetilde{K}(\rho) \subseteq N.
\end{aligned}$$

This implies that $\widetilde{K}(\rho) \lhd_b N$, $\forall\ \rho \in Q$. Hence $(\widetilde{K}, Q)$ is a SBI over N.

Definition 3.3. *[11] Let $(\widetilde{K}, B)$ be a SSG of a SN $(\widetilde{H}, P)$ over N. Then $(\widetilde{K}, B)$ is called a soft left N-subgroup (resp. soft right N-subgroup) of $(\widetilde{H}, P)$ if $B \subseteq P$ and $\widetilde{K}(\rho)$ is a left N-subgroup (resp. right N-subgroup) of $\widetilde{H}(\rho)$, $\forall\ \rho \in B$.*

Theorem 3.9. *Every soft left N-subgroup (resp. right N-subgroup) of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$ over N.*

Proof. *Let $(\widetilde{K}, B)$ be a soft left N-subgroup of $(\widetilde{H}, P)$ over N. Then $\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. Since $\widetilde{K}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$, we have $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq (\widetilde{K}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, and so $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$. Hence $(\widetilde{K}, B) \mathrel{\widetilde{\lhd}_b} (\widetilde{H}, P)$. Similarly, we can prove every soft right N-subgroup of $(\widetilde{H}, P)$ is a SBI of $(\widetilde{H}, P)$.*

Definition 3.4. *[11] Let $(\widetilde{K}, B)$ be a SSN of a SN $(\widetilde{H}, P)$ over N. Then $(\widetilde{K}, B)$ is called a soft invariant subnear-ring of $(\widetilde{H}, P)$ if $B \subseteq P$ and $\widetilde{K}(\rho)$ is an invariant subnear-ring of $\widetilde{H}(\rho)$, $\forall\ \rho \in B$.*

Theorem 3.10. *Every soft invariant subnear-ring of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$ over N.*

Proof. *Let $(\widetilde{K}, B)$ be a soft invariant subnear-ring of a SN $(\widetilde{H}, P)$. Then, $\widetilde{K}(\rho)$ is an invariant subnear-ring of $\widetilde{H}(\rho)$; that is $\widetilde{K}(\rho)$ is a subnear-ring of $\widetilde{H}(\rho)$, $\widetilde{K}(\rho)\widetilde{H}(\rho) \subseteq \widetilde{K}(\rho)$ and $\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$.*

Now $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)\widetilde{K}(\rho) \cap \widetilde{K}(\rho) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho) \cap \widetilde{K}(\rho) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$. Thus $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$. Hence $(\widetilde{K}, B)\, \widetilde{\lhd}_b\, (\widetilde{H}, P)$.

Remark 3.2. *The converses of Theorem 3.5, Theorem 3.7, Theorem 3.8, Theorem 3.9 and Theorem 3.10 are not true in general. This is illustrated in the following example.*

Example 3.3. *Consider a near-ring N over $Z_2 \times Z_2 \times Z_2 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with $(N, +)$ and $(N, \cdot)$ as defined below (Scheme 5: (4,6,7,1,4,4,7,7) see[13], p. 420).*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	1	1	0	1	1	1	0	0
2	2	0	2	2	2	2	2	2
3	3	1	2	3	3	3	2	2
4	4	4	4	0	4	4	4	4
5	5	5	4	1	5	5	4	1
6	6	4	6	2	6	6	6	6
7	7	5	6	3	7	7	6	6

Let $(\widetilde{H}, P)$ be a SS over N, where $P = N$ and $\widetilde{H}$ is defined as $\widetilde{H}(0) = \widetilde{H}(2) = \widetilde{H}(6) = N$, $\widetilde{H}(1) = \widetilde{H}(3) = \{0, 1, 2, 3\}$ and $\widetilde{H}(4) = \widetilde{H}(5) = \widetilde{H}(7) = \{0, 2, 4, 6\}$, which are all subnear-rings of N. Then $(\widetilde{H}, P)$ is a SN over N.

Let $(\widetilde{K}, B)$ be a SS over N defined by $\widetilde{K}(\rho) = \{0\} \cup \{\rho \in N / \rho \mathcal{R} \sigma \Leftrightarrow \rho + \sigma = 0\}$, where $B = \{1, 2, 4, 6\}$. Then $\widetilde{K}(1) = \{0, 1\}$, $\widetilde{K}(2) = \{0, 2\}$, $\widetilde{K}(4) = \{0, 4\}$ and $\widetilde{K}(6) = \{0, 6\}$ are bi-ideals of $\widetilde{H}(1)$, $\widetilde{H}(2)$, $\widetilde{H}(4)$ and $\widetilde{H}(6)$ respectively. Hence $(\widetilde{K}, B)\, \widetilde{\lhd}_b\, (\widetilde{H}, P)$ over N. Also $\widetilde{K}(1) = \{0, 1\}$, $\widetilde{K}(2) = \{0, 2\}$, $\widetilde{K}(4) = \{0, 4\}$ and $\widetilde{K}(6) = \{0, 6\}$ are bi-ideals of N. Hence $(\widetilde{K}, B)$ is a SBI over N. Whereas $\widetilde{K}(6) = \{0, 6\}$ is not a quasi-ideal of $\widetilde{H}(6) = N$, $(\widetilde{K}, B)$ is not a SQI of $(\widetilde{H}, P)$ as well as $(\widetilde{K}, B)$ is not a SQI over N. Further $\widetilde{K}(1) = \{0, 1\}$, $\widetilde{K}(2) = \{0, 2\}$, $\widetilde{K}(4) = \{0, 4\}$ are not left N-subgroup (resp. invariant subnear-ring) of $\widetilde{H}(1)$, $\widetilde{H}(2)$ and $\widetilde{H}(4)$ respectively. And $\widetilde{K}(6) = \{0, 6\}$ is not a left ideal (resp. right ideal, left N-subgroup, right

N-subgroup and invariant subnear-ring) of $\widetilde{H}(6) = N$. Hence $(\widetilde{K}, B)$ is not a soft left ideal (resp. soft right ideal, soft left N-subgroup, soft right N-subgroup and soft invariant subnear-ring) of $(\widetilde{H}, P)$.

Theorem 3.11. *Let $(\widetilde{H}, P)$ be a SN over N. Then $m' \widetilde{\circ} (\widetilde{H}, P) \widetilde{\lhd}_b (\widetilde{H}, P)$ and $(\widetilde{H}, P) \widetilde{\circ} m \widetilde{\lhd}_b (\widetilde{H}, P)$ where $m, m' \in \widetilde{H}(\rho)$ and m' is a distributive element in $\widetilde{H}(\rho)$, $\forall\ \rho \in P$.*

Proof. *First we prove that $m' \widetilde{\circ} (\widetilde{H}, P) \widetilde{\lhd}_b (\widetilde{H}, P)$. Since m' is a distributive element in $\widetilde{H}(\rho)$, clearly $m' \widetilde{H}(\rho)$ is a subgroup of $\widetilde{H}(\rho)$, $\forall\ \rho \in P$. Then,*

$$
\begin{aligned}
&(m'\widetilde{H}(\rho)\widetilde{H}(\rho)m'\widetilde{H}(\rho)) \cap (m'\widetilde{H}(\rho)\widetilde{H}(\rho)) * m'\widetilde{H}(\rho)\\
&\subseteq (m'\widetilde{H}(\rho)(\widetilde{H}(\rho)\widetilde{H}(\rho)) \cap (m'\widetilde{H}(\rho)\widetilde{H}(\rho)) * m'\widetilde{H}(\rho)\\
&\subseteq (m'\widetilde{H}(\rho)\widetilde{H}(\rho)) \cap m'\widetilde{H}(\rho) * m'\widetilde{H}(\rho)\\
&\subseteq m'\widetilde{H}(\rho) \cap m'\widetilde{H}(\rho) * m'\widetilde{H}(\rho)\\
&\subseteq m'\widetilde{H}(\rho).
\end{aligned}
$$

Thus $m' \widetilde{H}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in P$. Hence $m' \widetilde{\circ} (\widetilde{H}, P) \widetilde{\lhd}_b (\widetilde{H}, P)$. Similarly, we can prove $(\widetilde{H}, P) \widetilde{\circ} m \widetilde{\lhd}_b (\widetilde{H}, P)$.

Theorem 3.12. *Let $(\widetilde{K}, B)$ be a soft left N-subgroup of a SN $(\widetilde{H}, P)$ over N. Then*
$e \widetilde{\circ} (\widetilde{K}, B) \widetilde{\lhd}_b (\widetilde{H}, P)$, where e is a distributive idempotent element of $\widetilde{H}(\rho)$, $\forall$ $\rho \in B$.

Proof. *Since each $\widetilde{K}(\rho)$ is a left N-subgroup of $\widetilde{H}(\rho)$, we have $\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. Now, Since $e \widetilde{K}(\rho) \subseteq \widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$ and $e \widetilde{K}(\rho) \subseteq e \widetilde{H}(\rho)$, we have $e \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho) \cap e \widetilde{H}(\rho)$, $\forall\ \rho \in B$. Let u be any element of $\widetilde{K}(\rho) \cap e\widetilde{H}(\rho)$, then we write $u = v = en$, where $v \in \widetilde{K}(\rho)$ and $n \in \widetilde{H}(\rho)$, $\forall\ \rho \in B$. From where $u = en = een = ev \in e\widetilde{K}(\rho)$, we have $\widetilde{K}(\rho) \cap e \widetilde{H}(\rho) \subseteq e\widetilde{K}(\rho)$, $\forall\ \rho \in B$. Therefore $e \widetilde{\circ} (\widetilde{K}, B) = (\widetilde{K}, B) \widetilde{\cap}_R (e \widetilde{\circ} (\widetilde{H}, P))$. By Theorem 3.11, $e \widetilde{\circ} (\widetilde{H}, P) \widetilde{\lhd}_b (\widetilde{H}, P)$. Hence by Corollary 3.1, $e \widetilde{\circ} (\widetilde{K}, B) = (\widetilde{K}, B) \widetilde{\cap}_R\, e \widetilde{\circ} (\widetilde{H}, P) \widetilde{\lhd}_b (\widetilde{H}, P)$.*

4 Characterizations of soft bi-ideals of soft zero-symmetric near-rings

In this section, we obtain the characterizations of soft bi-ideals of a soft zero-symmetric near-ring and discuss some of their properties. We prove that a soft right ideal of a soft left ideal of a soft zero-symmetric near-ring is a soft bi-ideal.

Definition 4.1. *[11] Let $(\widetilde{H}, P)$ be a SN over N. Then a non-null SS $(\widetilde{H}, P)_0$ over N is called a soft zero-symmetric part of $(\widetilde{H}, P)$ if $\widetilde{H}(\rho)_0$ is a zero-symmetric part of $\widetilde{H}(\rho)$, $\forall\ \rho \in P$, where $\widetilde{H}(\rho)_0 = \{n \in \widetilde{H}(\rho)/n0 = 0\}$. The SN $(\widetilde{H}, P)$ is called a soft zero-symmetric near-ring (briefly, SZN) over N if $\widetilde{H}(\rho) = \widetilde{H}(\rho)_0$; that is, $\widetilde{H}(\rho)$ is a zero-symmetric subnear-ring of N, $\forall$ $\rho \in P$.*

Definition 4.2. *[11] Let $(\widetilde{H}, P)$ be a SN over N. Then a non-null SS $(\widetilde{H}, P)_c$ over N is called a soft constant part of $(\widetilde{H}, P)$ if $\widetilde{H}(\rho)_c$ is a constant part of $\widetilde{H}(\rho)$, $\forall\ \rho \in P$, where $\widetilde{H}(\rho)_c = \{n \in \widetilde{H}(\rho)/n0 = n\}$. The SN $(\widetilde{H}, P)$ is called a soft constant near-ring (briefly, SCN) over N if $\widetilde{H}(\rho) = \widetilde{H}(\rho)_c$, that is, $\widetilde{H}(\rho)$ is a constant subnear-ring of N, $\forall\ \rho \in P$.*

Example 4.1. *Consider the SN $(\widetilde{H}, P)$ over N as defined in Example 3.1. Then $\widetilde{H}(0)_0 = \widetilde{H}(2)_0 = \widetilde{H}(4)_0 = \widetilde{H}(6)_0 = \{0, 2, 5, 7\}$ and $\widetilde{H}(1)_0 = \widetilde{H}(3)_0 = \widetilde{H}(5)_0 = \widetilde{H}(7)_0 = \{0, 2\}$ are zero-symmetric parts of $\widetilde{H}(0)$,$\widetilde{H}(2)$,$\widetilde{H}(4)$,$\widetilde{H}(6)$,$\widetilde{H}(1)$,$\widetilde{H}(3)$,$\widetilde{H}(5)$ and $\widetilde{H}(7)$, respectively. Also $\widetilde{H}(\rho)_c = \{0, 4\}$ is a constant part of $\widetilde{H}(\rho)$, $\forall$ $\rho \in P$. Suppose we define $\widetilde{H}(\rho) = \{0, 2, 5, 7\}$, $\forall\ \rho \in P$, then $(\widetilde{H}, P)$ is a SZN over N. If we define $\widetilde{H}(\rho) = \{0, 4\}$, $\forall\ \rho \in P$, then $(\widetilde{H}, P)$ is a SCN over N.*

Remark 4.1. *In Definitions 4.1 and 4.2, since the zero-symmetric part and the constant part of a near-ring N are subnear-rings of N, $\widetilde{H}(\rho)_0$ and $\widetilde{H}(\rho)_c$ are subnear-rings of $\widetilde{H}(\rho)$, $\forall\ \rho \in P$. Therefore $(\widetilde{H}, P)_0$ and $(\widetilde{H}, P)_c$ are SSNs of $(\widetilde{H}, P)$ over N. Hence for a given SN $(\widetilde{H}, P)$ over N we can obtain at least two SSNs $(\widetilde{H}, P)_0$ and $(\widetilde{H}, P)_c$ of $(\widetilde{H}, P)$.*

Remark 4.2. *Let $(\widetilde{H}_1, P_1)$ be a SZN over N and $(\widetilde{H}_2, P_2)$ is a non-null SS over N. Then $\widetilde{H}_1(\rho)\widetilde{H}_2(\rho) \subseteq \widetilde{H}_1(\rho){*}\widetilde{H}_2(\rho)$, $\forall\ \rho \in B_2$. That is $(\widetilde{H}_1, P_1)\widetilde{\circ}(\widetilde{H}, P_2) \subseteq (\widetilde{H}_1, P_1)\widetilde{*}(\widetilde{H}_2, P_2)$.*

Theorem 4.1. *The soft zero-symmetric part $(\widetilde{H}, P)_0$ of a SN $(\widetilde{H}, P)$ over N is a SBI of $(\widetilde{H}, P)$.*

Proof. *By Remark 4.1, $(\widetilde{H}, P)_0$ is a SSN of $(\widetilde{H}, P)$ and so $\widetilde{H}(\rho)_0$ is a subgroup of $\widetilde{H}(\rho)$, $\forall\ \rho \in P$. For all $\sigma, \sigma' \in \widetilde{H}(\rho)$ and $\sigma_0 \in \widetilde{H}(\rho)_0$, we have $(\sigma(\sigma' + \sigma_0) - \sigma\sigma')0 = \sigma(\sigma' + \sigma_0)0 - \sigma\sigma'0 = \sigma(\sigma'0 + \sigma_0 0) - \sigma\sigma'0 = 0$. Therefore $(\sigma(\sigma' + \sigma_0) - \sigma\sigma') \in \widetilde{H}(\rho)_0$. Consequently, $\widetilde{H}(\rho) * \widetilde{H}(\rho)_0 \subseteq \widetilde{H}(\rho)_0$, $\forall\ \rho \in P$.*

Now, $(\widetilde{H}(\rho)_0\widetilde{H}(\rho)\widetilde{H}(\rho)_0)\cap(\widetilde{H}(\rho)_0\widetilde{H}(\rho))*\widetilde{H}(\rho)_0 \subseteq (\widetilde{H}(\rho)_0\widetilde{H}(\rho)\widetilde{H}(\rho)_0)\cap\widetilde{H}(\rho)*\widetilde{H}(\rho)_0 \subseteq (\widetilde{H}(\rho)_0\widetilde{H}(\rho)\widetilde{H}(\rho)_0)\cap\widetilde{H}(\rho)_0 \subseteq \widetilde{H}(\rho)_0$, $\forall\ \rho \in P$. *Thus* $\widetilde{H}(\rho)_0 \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in P$. *Hence* $(\widetilde{H}, P)_0 \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$.

Theorem 4.2. *The soft constant part* $(\widetilde{H}, P)_c$ *of a SN* $(\widetilde{H}, P)$ *over* N *is a SBI of* $(\widetilde{H}, P)$.

Proof. *By Remark 4.1,* $(\widetilde{H}, P)_c$ *is a SSN of* $(\widetilde{H}, P)$. *So* $\widetilde{H}(\rho)_c$ *is a subgroup of* $\widetilde{H}(\rho)$, $\forall\ \rho \in P$. *For all* $\sigma \in \widetilde{H}(\rho)$ *and* $\sigma_c \in \widetilde{H}(\rho)_c$, *we have* $(\sigma\sigma_c)0 = \sigma(\sigma_c 0) = \sigma\sigma_c$. *Thus* $\sigma\sigma_c \in \widetilde{H}(\rho)_c$ *and so* $\widetilde{H}(\rho)\widetilde{H}(\rho)_c \subseteq \widetilde{H}(\rho)_c$, $\forall\ \rho \in P$. *Now,* $(\widetilde{H}(\rho)_c\widetilde{H}(\rho)\widetilde{H}(\rho)_c) \cap (\widetilde{H}(\rho)_c\widetilde{H}(\rho)) * \widetilde{H}(\rho)_c \subseteq (\widetilde{H}(\rho)_c\widetilde{H}(\rho)_c) \cap (\widetilde{H}(\rho)_c\widetilde{H}(\rho)) * \widetilde{H}(\rho)_c \subseteq \widetilde{H}(\rho)_c \cap (\widetilde{H}(\rho)_c\widetilde{H}(\rho)) * \widetilde{H}(\rho)_c \subseteq \widetilde{H}(\rho)_c$, $\forall\ \rho \in P$.

This implies that $\widetilde{H}(\rho)_c \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in P$. *Hence* $(\widetilde{H}, P)_c \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$.

Theorem 4.3. *Let* $(\widetilde{H}, P)$ *be a SZN over* N. *Then a SSG* $(\widetilde{K}, B)$ *of* $(\widetilde{H}, P)$ *is a SBI of* $(\widetilde{H}, P)$ *if and only if* $(\widetilde{K}, B)\tilde{\circ}(\widetilde{H}, P)\tilde{\circ}(\widetilde{K}, B) \subseteq (\widetilde{K}, B)$.

Proof. *Assume that* $(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$. *Then,* $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho))\cap(\widetilde{K}(\rho)\widetilde{H}(\rho))*\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. *Since* $(\widetilde{H}, P)$ *is a SZN over* N, *by Remark 4.2,* $\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{H}(\rho)*\widetilde{K}(\rho)$, $\forall\ \rho \in B$. *Now,* $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) = \widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)\cap \widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$. *Thus,* $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. *Hence* $(\widetilde{K}, B)\tilde{\circ}(\widetilde{H}, P)\tilde{\circ}(\widetilde{K}, B) \subseteq (\widetilde{K}, B)$.

Conversely, assume that $(\widetilde{K}, B)\tilde{\circ}(\widetilde{H}, P)\tilde{\circ}(\widetilde{K}, B) \subseteq (\widetilde{K}, B)$. *That is,* $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. *Then,* $(\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho) \cap (\widetilde{K}(\rho)\widetilde{H}(\rho)) * \widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. *Consequently,* $(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$.

Theorem 4.4. *Let* N *be a zero-symmetric near-ring. Then a SSG* $(\widetilde{K}, B)$ *over* N *is a SBI over* N *if and only if* $(\widetilde{K}, B)\tilde{\circ}A_N\tilde{\circ}(\widetilde{K}, B) \subseteq (\widetilde{K}, B)$.
Proof. *The proof is similar to that of Theorem 4.3.*

Theorem 4.5. *Let* $(\widetilde{K}, B)$ *be a SBI of a SZN* $(\widetilde{H}, P)$ *over* N. *Then* $m' \tilde{\circ}(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$ *and* $(\widetilde{K}, B)\tilde{\circ}\, m \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$ *where* $m, m' \in \widetilde{H}(\rho)$ *and* m' *is a distributive element in* $\widetilde{H}(\rho)$, $\forall\ \rho \in B$.
Proof. *Since* $(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$, $\widetilde{K}(\rho)$ *is a subgroup of* $\widetilde{H}(\rho)$ *and* $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$. *First we prove that* $m' \tilde{\circ}(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$. *Since* m' *is a distributive element in* $\widetilde{H}(\rho)$, $m'\widetilde{K}(\rho)$ *is a subgroup of* $\widetilde{H}(\rho)$, $\forall\ \rho \in B$. *Now,* $m'\,\widetilde{K}(\rho)\,\widetilde{H}(\rho)\,m'\,\widetilde{K}(\rho) \subseteq m'\,\widetilde{K}(\rho)\,\widetilde{H}(\rho)\,\widetilde{K}(\rho) \subseteq m'\,\widetilde{K}(\rho)$. *Thus,* $m'\,\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$. *Hence* $m' \tilde{\circ}(\widetilde{K}, B) \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$. *Clearly* $\widetilde{K}(\rho)m$ *is a*

subgroup of $\widetilde{H}(\rho)$, $\forall\, \rho \in B$. *Then,* $\widetilde{K}(\rho)m\widetilde{H}(\rho)\widetilde{K}(\rho)m \subseteq \widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho)m \subseteq \widetilde{K}(\rho)m$. *Thus,* $\widetilde{K}(\rho)m \lhd_b \widetilde{H}(\rho)$, $\forall\, \rho \in B$. *Hence* $(\widetilde{K}, B)\widetilde{\circ}\, m \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$.

Corollary 4.1. *Let* $(\widetilde{K}, B)$ *be a SBI of a SZN* $(\widetilde{H}, P)$ *over* N.
Then $m' \,\widetilde{\circ}(\widetilde{K}, B)\widetilde{\circ}\, m \,\widetilde{\lhd}_b\, (\widetilde{H}, P)$, *where* $m, m' \in \widetilde{H}(\rho)$ *and* m' *is a distributive element in* $\widetilde{H}(\rho)$, $\forall\, \rho \in B$.
Proof. The proof is straightforward using Theorem 4.5.

Theorem 4.6. *Let* $(\widetilde{H}, P)$ *be a SN over* N *and* $(\widetilde{K}, B)$ *be a SBI of* $(\widetilde{H}, P)$. *Then*

(i) $(\widetilde{K}, B)_0 = (\widetilde{K}, B)\,\widetilde{\cap}_R\,(\widetilde{H}, P)_0\,\widetilde{\lhd}_b\,(\widetilde{H}, P)$ *over* N *if it is non-null.*

(ii) $(\widetilde{K}, B)_c = (\widetilde{K}, B)\,\widetilde{\cap}_R\,(\widetilde{H}, P)_c\,\widetilde{\lhd}_b\,(\widetilde{H}, P)$ *over* N *if it is non-null.*

Proof. *The proof is straightforward using Corollary 3.1.*

Theorem 4.7. *Let* $(\widetilde{K}_1, I)$ *be a soft right-ideal and* $(\widetilde{K}_2, S)$ *be a SSN of a SZN* $(\widetilde{H}, P)$ *over* N. *Then* $(\widetilde{K}_1, I)\widetilde{\circ}(\widetilde{K}_2, S)\,\widetilde{\lhd}_b\,(\widetilde{H}, P)$.
Proof. *Since* $(\widetilde{K}_1, I)$ *is a soft right-ideal of* $(\widetilde{H}, P)$, $\widetilde{K}_1(\rho)\widetilde{H}(\rho) \subseteq \widetilde{K}_1(\rho)$ *and* $\widetilde{K}_1(\rho)\widetilde{K}_1(\rho) \subseteq \widetilde{K}_1(\rho)\widetilde{H}(\rho) \subseteq \widetilde{K}_1(\rho)$, $\forall\, \rho \in I$. *Let* $(\widetilde{J}, C) = (\widetilde{K}_1, I)\widetilde{\circ}(\widetilde{K}_2, S)$ *where* $C = I \cap S \neq \emptyset$ *and* $\widetilde{J}(\rho) = \widetilde{K}_1(\rho)\widetilde{K}_2(\rho)$, $\forall\, \rho \in C$. *Since* $\widetilde{K}_1(\rho)$ *is a normal subgroup of* $\widetilde{H}(\rho)$, $\widetilde{J}(\rho) = \widetilde{K}_1(\rho)\widetilde{K}_2(\rho)$ *is a subgroup of* $\widetilde{H}(\rho)$, $\forall$ $\rho \in C$. *Further,*

$$
\begin{aligned}
\widetilde{J}(\rho)\widetilde{H}(\rho)\widetilde{J}(\rho) &= (\widetilde{K}_1(\rho)\widetilde{K}_2(\rho))\widetilde{H}(\rho)(\widetilde{K}_1(\rho)\widetilde{K}_2(\rho)) \\
&= \widetilde{K}_1(\rho)(\widetilde{K}_2(\rho)\widetilde{H}(\rho))(\widetilde{K}_1(\rho)\widetilde{K}_2(\rho)) \\
&\subseteq \widetilde{K}_1(\rho)(\widetilde{H}(\rho)\widetilde{H}(\rho))(\widetilde{K}_1(\rho)\widetilde{K}_2(\rho)) \\
&\subseteq \widetilde{K}_1(\rho)\widetilde{H}(\rho)(\widetilde{K}_1(\rho)\widetilde{K}_2(\rho)) \\
&\subseteq \widetilde{K}_1(\rho)\widetilde{K}_1(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_1(\rho)\widetilde{K}_2(\rho) = \widetilde{J}(\rho).
\end{aligned}
$$

Thus, $\widetilde{J}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\, p \in B$. *Hence* $(\widetilde{K}_1, I)\widetilde{\circ}(\widetilde{K}_2, S)\,\widetilde{\lhd}_b\,(\widetilde{H}, P)$.

Theorem 4.8. *In a soft zero-symmetric commutative near-ring* $(\widetilde{H}, P)$ *over* N, *the product of any two SBIs of* $(\widetilde{H}, P)$ *is a SBI of* $(\widetilde{H}, P)$.
Proof. *Let* $(\widetilde{K}_1, B_1)$ *and* $(\widetilde{K}_2, B_2)$ *be any two SBIs of* $(\widetilde{H}, P)$. *Then,* $\widetilde{K}_1(\rho)$ *and* $\widetilde{K}_2(\rho)$ *are subgroups of* $\widetilde{H}(\rho)$, $\widetilde{K}_1(\rho)\widetilde{H}(\rho)\widetilde{K}_1(\rho) \subseteq \widetilde{K}_1(\rho)$, $\forall\, \rho \in B_1$ *and* $\widetilde{K}_2(\rho)\widetilde{H}(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_2(\rho)$, $\forall\, \rho \in B_2$.

Let $(\widetilde{K}, B) = (\widetilde{K}_1, B_1)\widetilde{\circ}(\widetilde{K}_2, B_2)$, *where* $B = B_1 \cap B_2 \neq \emptyset$ *and* $\widetilde{K}(\rho) = \widetilde{K}_1(\rho)\widetilde{K}_2(\rho)$, $\forall\ \rho \in B$. *Since* $(\widetilde{H}, P)$ *is a soft zero-symmetric commutative near-ring,* $\widetilde{K}(\rho) = \widetilde{K}_1(\rho)\widetilde{K}_2(\rho)$ *is a subgroup of* $\widetilde{H}(\rho)$, $\forall\ \rho \in B$. *Thus* $(\widetilde{K}, B)$ *is a soft subgroup of* $(\widetilde{H}, P)$.

Further, $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) = (\widetilde{K}_1(\rho)\widetilde{K}_2(\rho))\widetilde{H}(\rho)(\widetilde{K}_1(\rho)\widetilde{K}_2(\rho)) \subseteq \widetilde{K}_1(\rho)\widetilde{H}(\rho)\widetilde{H}(\rho)\ \widetilde{K}_1(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_1(\rho)\widetilde{H}(\rho)\ \widetilde{K}_1(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_1(\rho)\widetilde{K}_2(\rho) = \widetilde{K}(\rho)$. *Thus,* $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$. *Hence* $(\widetilde{K}, P_1)\widetilde{\circ}(\widetilde{K}, P_2)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$.

Theorem 4.9. *In a soft zero-symmetric commutative near-ring* $(\widetilde{H}, P)$ *over* N*, the product of any two SQIs of* $(\widetilde{H}, P)$ *is a SBI of* $(\widetilde{H}, P)$.
Proof. *The proof follows from Theorems 3.7 and 4.8.*

Remark 4.3. *For any two SBIs* $(\widetilde{K}_1, B_1)$ *and* $(\widetilde{K}_2, B_2)$ *of a SZN* $(\widetilde{H}, P)$ *with* $(\widetilde{K}_2, B_2) \subseteq (\widetilde{K}_1, B_1)$*, we have* $(\widetilde{K}_2, B_2)\ \widetilde{\lhd}_b\ (\widetilde{K}_1, B_1)$. *But if* $(\widetilde{K}_2, B_2)\ \widetilde{\lhd}_b\ (\widetilde{K}_1, B_1)$ *and* $(\widetilde{K}_1, B_1)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$*, then* $(\widetilde{K}_2, B_2)$ *is need not be a SBI of* $(\widetilde{H}, P)$ *over* N *in general.*

By considering the above remark in mind we have the following theorem.

Theorem 4.10. *Let* $(\widetilde{H}, P)$ *be a SZN over* N. *If* $(\widetilde{K}_1, B_1)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$ *and* $(\widetilde{K}_2, B_2)\ \widetilde{\lhd}_b\ (\widetilde{K}_1, B_1)$ *such that* $(\widetilde{K}_2, B_2)\widetilde{\circ}(\widetilde{K}_2, B_2) = (\widetilde{K}_2, B_2)$*, then* $(\widetilde{K}_2, B_2)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$.
Proof. *Being* $(\widetilde{K}_1, B_1)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$ *and* $(\widetilde{K}_2, B_2)\ \widetilde{\lhd}_b\ (\widetilde{K}_1, B_1)$*, we have* $\widetilde{K}_1(\rho)\widetilde{H}(\rho)\ \widetilde{K}_1(\rho) \subseteq \widetilde{K}_1(\rho)$, $\forall\ \rho \in B_1$ *and* $\widetilde{K}_2(\rho)\widetilde{K}_1(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_2(\rho)$, $\forall\ \rho \in B_2$. *Since* $(\widetilde{K}_2, B_2)$ *is a SSG of* $(\widetilde{K}_1, B_1)$, $(\widetilde{K}_2, B_2)$ *is also a SSG of* $(\widetilde{H}, P)$. *Now,* $\widetilde{K}_2(\rho)\widetilde{H}(\rho)\widetilde{K}_2(\rho) = \widetilde{K}_2(\rho)\widetilde{K}_2(\rho)\widetilde{H}(\rho)\widetilde{K}_2(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_2(\rho)\widetilde{K}_1(\rho)\widetilde{H}(\rho)\widetilde{K}_1(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_2(\rho)\widetilde{K}_1(\rho)\widetilde{K}_2(\rho) \subseteq \widetilde{K}_2(\rho)$. *This implies* $\widetilde{K}_2(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B_2$. *Hence* $(\widetilde{K}_2, B_2)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$.

Theorem 4.11. *If* $(\widetilde{K}, B)$ *is a soft right ideal of a soft left ideal of a SZN* $(\widetilde{H}, P)$ *over* N. *Then* $(\widetilde{K}, B)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$.
Proof. *Let* $(\widetilde{J}, I)$ *be soft left ideal of* $(\widetilde{H}, P)$ *and* $(\widetilde{K}, B)$ *be a soft right ideal* $(\widetilde{J}, I)$. *Then,* $(\widetilde{K}, B)\widetilde{\circ}(\widetilde{J}, I) \subseteq (\widetilde{K}, B)$ *and* $(\widetilde{H}, P)\widetilde{*}(\widetilde{J}, I) \subseteq (\widetilde{J}, I)$. *That is,* $\widetilde{K}(\rho)\widetilde{J}(\rho) \subseteq \widetilde{K}(\rho)$, $\forall\ \rho \in B$ *and* $\widetilde{H}(\rho) * \widetilde{J}(\rho) \subseteq \widetilde{J}(\rho)$, $\forall\ \rho \in I$. *Now,* $\widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{K}(\rho) \subseteq \widetilde{K}(\rho)\widetilde{H}(\rho)\widetilde{J}(\rho) = \widetilde{K}(\rho)(\widetilde{H}(\rho)\widetilde{J}(\rho)) \subseteq \widetilde{K}(\rho)(\widetilde{H}(\rho) * \widetilde{J}(\rho)) \subseteq \widetilde{K}(\rho)\widetilde{J}(\rho) \subseteq \widetilde{K}(\rho)$. *Thus* $\widetilde{K}(\rho) \lhd_b \widetilde{H}(\rho)$, $\forall\ \rho \in B$. *Hence* $(\widetilde{K}, B)\ \widetilde{\lhd}_b\ (\widetilde{H}, P)$.

5 Conclusion

In this paper, we have introduced the notions of soft bi-ideal of a soft near-ring and soft bi-ideal over a near-ring. We have explored the properties of these notions with illustrated examples. We have provided the condition for a soft subgroup of a soft near-ring to be a soft bi-ideal. We have showed that every soft left ideal (resp. soft right ideal, soft ideal, soft quasi-ideal, soft left N-subgroup, soft right N-subgroup, soft invariant subnear-ring) of a soft near-ring is a soft bi-ideal but the converses are not true in general. We have justified this by means of a counter example. Furthermore, we have obtained the characterization of soft bi-ideals of soft zero-symmetric near-rings. In the future, we will apply fuzzy soft sets to bi-ideals of near-rings and investigate the relations between soft bi-ideals and fuzzy soft bi-ideals.

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