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Mixed finite element method for flow of fluid in complex porous media with a new boundary condition

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Abstract

In this work, we approximate the steady Brinkman equations with a new boundary condition. We derive an adequate variational formulation to approach this problem by using MFE methods. We use a general block diagonal preconditioner to obtain a faster convergence. To control the error, we use two types of a posteriori error indicator equivalent to the true discretization error.

1 Introduction

This paper describes a numerical solution of steady Brinkman equations with a new boundary condition. This model describes the flow of fluid in complex porous media or two different situations porous/free flow media, this equation is presented and published by H.C. Brinkman in [14, 15]. We use the

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boundary condition generalizes the known conditions, especially the Dirichlet and the Neumann ones. We discretize this coupled partial differential equations by mixed finite element (MFE) method [23]. This lineair problem is a saddle point problem, a wealth of literature exists to treat much of it related events to particular applications. Perhaps the most popular work is [7], which considers assumptions under which the matrix formulation will be solvable and the authors use block diagonal preconditioners is focus on linear algebra. For the uniqueness of solution can be found in [7] or, in the substantial area of PDEs, in [10, 11]. Brinkman problems also arise in a natural way when the (unsteady), Stokes equations and Darcy law are coupled and simplified using classical operator splitting techniques [24]. Thus, the Stokes or Darcy equations can be obtained by suitable choices these parameters μ^* and $\mathbb{K}$ in (2.1) by defining them in vugular and rock matrix regions, respectively [19]. A posteriori error analysis in FEM received a lot of attention during the last decades. For the conforming case see [4, 5]. In the specific case of the Brinkman equations see [4] laid the basic foundation for the mathematical analysis of practical methods.

The plan of the paper is as follows. In the next section we present the model problem, and we present the discretization by FE method in section 3. A general block diagonal preconditioners for Brinkman problems is described in section 4. Section 5 shows the methods of a posteriori error estimator of the computed solution.

2 Governing equations

In this section, we define the governing equation where the unknown functions $\overrightarrow{u}$ is the fluid velocity, p is the pressure field, satisfying

$$\begin{cases} -\mu^*\nabla^2\overrightarrow{u} + \nabla p + \mu\mathbb{K}^{-1}\overrightarrow{u} = \overrightarrow{f} \text{ in } \Omega \\ \nabla\cdot\overrightarrow{u} = 0 \quad \text{in } \Omega \\ \mathbb{A}^{-1}\overrightarrow{u} + (\nabla\overrightarrow{u} - pI)\overrightarrow{n} = \overrightarrow{g} \ in \ \Gamma =: \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is an idealized, bounded, connected domain. The parameters μ^* is the effective viscosity (is only a parameter allows for matching the shear stress boundary conditions across free fluid or porous medium interface [16]) and μ is the physical dynamic viscosity that defines the fluid under consideration (e.g., water, oil, etc.). $\mathbb{K}$ (resp $\mathbb{A}$) is a permeability tensor, which is equal to the Darcy permeability in a porous media (resp the boundary). Distinguish two special cases for the Brinkman equation (2.1), by choosing

$\mu^* \simeq 0$ in a region or it is known that for moderately very small permeability and pore fractions i.e. $\mathbb{K}^{-1} >> 1$ this equation is reduced to Darcy's law, on the other hand by choosing $\mathbb{K}_{ij}$ or very large i.e. $\mathbb{K}_{ij} \longrightarrow \infty$ this equation is reduced to the Stokes equations and can be taken this equality $\mu^* = \mu$, see e.g. [19]. We propose study this problem with this new boundary condition where the vector $\overrightarrow{n}$ denote the outward pointing normal to the boundary. The functional $\overrightarrow{f}$ in the space $[L^2(\Omega)]^2$, the functional $\overrightarrow{g}$ in the space $[L^2(\Gamma)]^2$, the unknowns is the pressure function p defined in the space $L^2(\Omega)$ satisfy $\int_\Omega p\, dx = 0$ and the velocity vector $\overrightarrow{u}$. We Assume that there exist a positive constant numbers $a_1, a_2 > 0$ such that

$$a_1\zeta^t\zeta \leq \zeta^t\mathbb{K}^{-1}\zeta \leq a_2\zeta^t\zeta, \quad \zeta \in \mathbb{R}^2 \tag{2.2}$$

and $\mathbb{A}^{-1}$ is a nonzero continuous matrix such that $\exists c_1, c_2 > 0$ such that

$$c_1\zeta^t\zeta \leq \zeta^t\mathbb{A}^{-1}\zeta \leq c_2\zeta^t\zeta, \quad \zeta \in \mathbb{R}^2 \tag{2.3}$$

where ζ is understood as a column vector and ζ^t is the transpose of ζ. Before presenting the weak formulation of this problem we set these spaces $V = (H_0^1(\Omega))^2$ and $W = \{q \in L^2(\Omega) : \int_\Omega q(x)dx = 0\}$. To simplify this study we define $a : V \times V \longrightarrow \mathbb{R}$, $b : V \times W \longrightarrow \mathbb{R}$ and $d : W \times W \longrightarrow \mathbb{R}$.

$$a(\overrightarrow{u}, \overrightarrow{v}) = \int_\Omega \mu^*\nabla\overrightarrow{u}.\nabla\overrightarrow{v}dx + \int_\Omega \mu\mathbb{K}^{-1}\overrightarrow{u}.\overrightarrow{v}dx + \int_\Gamma \mathbb{A}^{-1}\overrightarrow{u}.\overrightarrow{v}, \tag{2.4}$$

$$b(\overrightarrow{v}, q) = -\int_\Omega (q\nabla.\overrightarrow{v})dx, \quad d(p, q) = \int_\Omega p\ q\ dx. \tag{2.5}$$

These inner products induce norms on the space V by $\|\cdot\|_V$ and for the space W denoted by $\|\cdot\|_W$ respectively.

$$\|\overrightarrow{u}\|_V = a(\overrightarrow{u}, \overrightarrow{u})^{\frac{1}{2}} \ \ \forall\overrightarrow{u} \in V, \ \ \|q\|_W = d(q, q)^{\frac{1}{2}} \ \ \forall q \in W. \tag{2.6}$$

It is simple to see that the norm $\|\cdot\|_V$ is both equivalent to $H^1(\Omega)$ norms and $(H^1(\Omega), \|\cdot\|_V)$ is a real Hilbert space. For the second member we given the continuous functional $l : V \longrightarrow \mathbb{R}$

$$l(\overrightarrow{v}) = \int_\Omega \overrightarrow{f}.\overrightarrow{v}\ dx + \int_{\partial\Omega} \overrightarrow{g}.\overrightarrow{v}dx. \tag{2.7}$$

By summing the all, the weak formulation of the Brinkman problem (2.1) is then, find $(\overrightarrow{u}, p) \in V \times W$ such that

$$\begin{cases} a(\overrightarrow{u}, \overrightarrow{v}) + b(\overrightarrow{v}, p) = l(\overrightarrow{v}) \\ b(\overrightarrow{u}, q) = 0, \end{cases} \tag{2.8}$$

for all $(\overrightarrow{v}, q) \in V \times W$.

It is well known that under the assumptions (2.2)-(2.3) the bilinear form $a(\cdot,\cdot)$ is positive continuous coercive. The bilinear form $b(\cdot,\cdot)$ is continuous and satisfies the $inf - sup$ condition [17] and the linear function $l(\cdot)$ is continuous. Then the problem (2.8) is well-posed and have only one solution [11].

3 Finite element method

The goal of this section is to introduce a mixed finite element algorithm to approximate our problem. Let P be a regular mesh in the sense as defined in [2] of our domain Ω in $2D$ into the union of N subdomains K verify that
• $N <\propto$,
• $\overline{\Omega} = \cup_{K\in P}\overline{K}$,
• $K \cap J$ is empty whenever $K \neq J$,
• each K is a convex Lipschitzian domain with piecewise smooth boundary ∂K.
The common boundary between subdomains K and J is denoted by: $\Gamma_{KJ} = \partial K \cap \partial J$. For any $K \in P$, ω_K is of rectangles sharing at least one edge with element K.
We let $\varepsilon_h = \cup_{K\in P}\ \varepsilon(K)$ denotes the set of all edges split into interior and boundary edges, $\varepsilon_h = \varepsilon_{h,\Omega} \cup \varepsilon_{h,\Gamma}$,
where $\varepsilon_{h,\Omega} = \{E \in \varepsilon_h : E \subset \Omega\}$ and $\varepsilon_{h,\Gamma} = \{E \in \varepsilon_h : E \subset \partial\Omega\}$.

We construct the usual manner finite element subspaces of these spaces V (resp W) noted by X^h (resp M^h) and so that the inclusion $X^h \times M^h \subset V \times W$ holds. Now, we define the MFE approximation to (2.8)
Find $(\overrightarrow{u}_h, p_h) \in X^h \times M^h$ such that

$$\begin{cases} a(\overrightarrow{u}_h, \overrightarrow{v}_h) + b(\overrightarrow{v}_h, p_h) = l(\overrightarrow{v}_h) \\ b(\overrightarrow{u}_h, q_h) = 0, \end{cases} \tag{3.9}$$

for all $(\overrightarrow{v}_h, q_h) \in X^h \times M^h$. The system (3.9) we can rewrite as a square matrix problem where the unknowns is these vectors U and P. to define these vectors we use a set of vector-valued basis functions $\{\overrightarrow{\varphi}_j\}$, so that

$$\overrightarrow{u}_h = \sum_{j=1}^{n_u} u_j \overrightarrow{\varphi}_j + \sum_{j=n_u+1}^{n_u+n_\partial} u_j \overrightarrow{\varphi}_j, \tag{3.10}$$

and we fix the coefficients $u_j : j = n_u+1, ..., n_u+n_\partial$, so that the second term interpolates the boundary data on $\partial\Omega_D$. For the pressure basis functions $\{\Psi_k\}$ and set

$$p_h = \sum_{k=1}^{n_p} p_k \Psi_k, \tag{3.11}$$

where n_u and n_p are the numbers of velocity and pressure basis functions, respectively. We obtain a system of linear equations that U and P have to satisfy

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}. \tag{3.12}$$

The matrix A is defined by

$$A = [a_{ij}],\ a_{ij} = \int_\Omega \mu^* \nabla \overrightarrow{\varphi}_i : \nabla \overrightarrow{\varphi}_j + \int_\Omega \mu \mathbb{K}^{-1} \overrightarrow{\varphi}_i . \overrightarrow{\varphi}_j + \int_{\partial\Omega} \mathbb{A}^{-1} \overrightarrow{\varphi}_i . \overrightarrow{\varphi}_j, \tag{3.13}$$

and the divergence matrix B is defined by

$$B = [b_{kj}],\ b_{kj} = -\int_\Omega \Psi_k \nabla . \overrightarrow{\varphi}_j, \tag{3.14}$$

for i and $j = 1, ..., n_u$ and $k = 1, ..., n_p$. The right-hand side vector F in (3.12) is

$$F = [F_i],\ F_i = \int_\Omega \overrightarrow{f} . \overrightarrow{\varphi}_i + \int_{\partial\Omega} \overrightarrow{g} . \overrightarrow{\varphi}_i, \tag{3.15}$$

for $i = 1, ..., n_u$ and $k = 1, ..., n_p$. In the next section, we use the fast iterative solution of stabilised Brinkman we propose method of Preconditioned Conjugate Residuals (PCR) for solving the big symmetric system (3.12). The convergence of Preconditioned Conjugate Residuals is independent of h [8].

4 General block diagonal preconditioners

To solving this large systems of linear equations we use iterative methods. All algorithms work in this way is referred to as Krylov subspace methods see [9], this method it is the most effective methods currently available in linear algebra system. We propose apply the preconditioned conjugate residuals (PCR) for the Brikman equation to have a fast and robust linear solvers for stabilized mixed approximations of this coupled system equations (2.8).

4.1 Preconditioned conjugate residuals

Is a simpel to see that the resulting discrete Brinkman system is a saddle-point system [6], can be expressed in this general form

$$\begin{pmatrix} A & B^t \\ B & -\beta C \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \tag{4.16}$$

where the unknowns are the vectors U, P there are a discretized representations of $\vec{u}$, p, with F taking into account the source term $\vec{f}$ and $\vec{g}$ as well as nonhomogeneous boundary conditions. The treatment of the problem (4.16) strongly depends on the properties of the system (2.1). In this part considers the more general class of block preconditioners [8], where the technique corresponds to partitioning into blocks of the velocity and pressure variables. The matrix A represents a block diagonal of discrete Laplacians, $-C$ present the stabilisation term omitted in the conventional (unstabilised) formulation where $\beta > 0$ is the stabilisation parameter and the matrix B is the block coupling between velocities and pressure. Let A is a positive definite $n \times n$ matrix, C is a positive semi definite $m \times m$ matrix and B definite $m \times n$. In all practical cases $n > m$ (see [6, Chapter 3]). Let

$$\bar{A} = \begin{pmatrix} A & B^t \\ B & -\beta C \end{pmatrix} \tag{4.17}$$

the Schur complement $\left(BA^{-l}B^t\right) + \beta C$ of $\bar{A}$ play a crucial role to to assess the compatibility or otherwise of the two spaces X^h and M^h. The general positive definite preconditioner of $\bar{A}$ defined by

$$M = \begin{pmatrix} M_A & 0 \\ 0 & M_C \end{pmatrix} \tag{4.18}$$

where M_A and M_C are two symmetric positive definite matrices. Note that, the convergence of this algorithm also determined by the values of the optimal minimax polynomials of increasing degree on the eigenvalues of the symmetrically preconditioned matrix

$$M^{-\frac{1}{2}} \bar{A} M^{-\frac{1}{2}} = \begin{pmatrix} \tilde{A} = M_A^{-\frac{1}{2}} A M_A^{-\frac{1}{2}} & \tilde{B}^t = M_A^{-\frac{1}{2}} B^t M_C^{-\frac{1}{2}} \\ \tilde{B} = M_C^{-\frac{1}{2}} B M_A^{-\frac{1}{2}} & -\beta \tilde{B}^t = -\beta M_C^{-\frac{1}{2}} C M_C^{-\frac{1}{2}} \end{pmatrix} = \tilde{\mathbb{A}} \tag{4.19}$$

The eigenvalues of $\bar{\mathbb{A}}$ is defined by

$$\mu_{-m} \leq \mu_{-m+1} \leq ... \leq \mu_{-1} \leq 0 \leq \mu_1 \leq ... \leq \mu_{n-1} \leq \mu_n \tag{4.20}$$

the singular values of $\tilde{B}$ as

$$\sigma_1 \leq ... \leq \sigma_{m-1} \leq \sigma_m \leq 0 \tag{4.21}$$

and the eigenvalues of the preconditioned $\tilde{A}$ as

$$0 \leq \lambda_1 \leq ... \leq \lambda_{n-1} \leq \lambda_n. \tag{4.22}$$

Note that, this is a slightly different approach from that used in [8]; there the preconditioner M explicitly involved the parameter β. Now we propose an important result is a convergence result see [8], Let r_k is the residual of the kth iterate, then

$$\frac{\|r_k\|}{\|r_0\|} \leq \min_{p\in\Pi_k^1} \max_i |p(\mu_i)| = e_k,$$

where Π_k^1 is the set of real polynomials of degree k satisfy $p(0) = 1$, and $\{\mu_i\}$ are the eigenvalues of the preconditioned Brinkman matrix. If the eigenvalues μ_i lie in inclusion intervals of the form

$$[-a, -b\alpha] \cup \left[c\alpha^2, d\right] \tag{4.23}$$

where $a,\ b,\ c$, and d are positive constants and α is an asymptotically small positive parameter, then the asymptotic PCR convergence rate [20]

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} = 1 - \alpha^{\frac{3}{2}} \sqrt{\frac{bc}{ad}} + 0\left(\alpha^{\frac{5}{2}}\right) \tag{4.24}$$

Proof. See [8] for the proof and further discussion of this estimate. □

The number of PCR iterations required to reduce the residual by a fixed factor is then $0\left(\alpha^{\frac{3}{2}}\right)$. The more "standard" estimate of PCR convergence is based on embedding a spectrum such as (4.23) in positive and negative intervals that are of equal length. Use of Lebedev's results [22] then gives $e_k^{\frac{1}{k}} \leq 2\left(\frac{1-\sqrt{\frac{\tilde{b}\tilde{c}}{\tilde{a}\tilde{d}}}}{1+\sqrt{\frac{\tilde{b}\tilde{c}}{\tilde{a}\tilde{d}}}}\right)^2$ where the spectrum is contained in $\left[-\tilde{a}, -\tilde{b}\right] \cup \left[\tilde{c}, \tilde{d}\right]$ and $\tilde{a} - \tilde{b} = \tilde{d} - \tilde{c} > 0$ (see Thmeorem. 3.2 in [25]). For a spectrum of the form (4.23) this approach gives an estimate of the form

$$\lim_{k\to\infty} e_k^{\frac{1}{k}} = 1 - 0\left(\alpha^2\right)$$

The remainder of the paper is set out as follows. In the next subsection we derive our eigenvalue bounds.

4.2 Eigenvalue estimates

For stable finite element discretizations of the Brinkman equation in particular, for an LBB stable element ($C = 0$) there exists an inf-sup constant γ such that

$$\gamma^2 \leq \frac{p^t \left(BA^{-l}B^t\right) p}{(p^t M_p p)} \text{ for all } p \tag{4.25}$$

where M_p is the pressure mass matrix, the boundedness of matrix B implies the existence a constant Λ such that

$$\frac{p^t \left(BA^{-l}B^t\right) p}{p^t M_p p} \leq \Lambda^2 \text{ for all } p. \tag{4.26}$$

Note that, only the upper bound (4.26) on the Schur complement holds for an unstable element. another element of instability is the LBB constant γ is zero (corresponding to "pure" spurious pressure modes) or dependent on the mesh size h. We assume that there exist a critical parameter value β_0 independent of h, such that for all parameter values $\beta > \beta_0$

$$\gamma^2 \leq \frac{p^t \left(BA^{-l}B^t + \beta C\right) p}{p^t M_p p} \text{ for all } p \tag{4.27}$$

with the constant γ independent of h and β, now we can derive h-independent eigenvalue bounds in the stabilised case. This condition is an obvious extension of (4.25) to the stabilized case. Note that combining (4.27) with the boundedness of C and B leads to the bound

$$\gamma^2 \leq \frac{p^t \left(BA^{-l}B^t + \beta C\right) p}{p^t M_p p} \leq \tilde{\Lambda}^2 \text{ for all } p \tag{4.28}$$

for all bounded values of $\beta > \beta_0$ and for some constant $\tilde{\Lambda}$ independent of h. In [8], (4.28) was referred to as the (uniform-) stabilisation condition. In this paper we aim to determine eigenvalue bounds explicitly in terms of β As shown in the next section, this can be more conveniently done using (4.27) and (4.26) directly in place of the combined bound (4.28). Also, as in [8], the authors assume the existence of constants θ, Φ independent of h such that

$$\theta^2 \leq \frac{p^t M_p p}{p^t M_C p} \leq \Phi^2 \text{ for all } p \tag{4.29}$$

i.e., the diagonal block M_c is required to be spectrally equivalent to the pressure mass matrix M_p. This condition ensures that the correct scaling is enforced between the velocity and pressure fields. For stable finite element discretizations $Q_2 - Q_1$ furthermore, (4.29) is satisfied with these constant $\theta = \frac{1}{2}$ and $\Phi = \sqrt{2}$.

We also make use of the boundedness of $\tilde{C}$, i.e.,

$$\max_p \frac{p^t \tilde{C} p}{p^t p} = \max_q \frac{q^t C q}{q^t M_C q} \leq \Delta \tag{4.30}$$

where Δ is a constant independent of h. In particular, $\Delta = 2$ for the two specific stabilised methods considered in the next section.

For the unstabilised case, $C = 0$, the following result of Rusten and Winther [18, Lemma 2.1] provides some useful eigenvalue bounds.

Lemma 4.1. *For the case $C = 0$,*

$$\frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\sigma_m^2}\right) \leq \mu_{-m}, \tag{4.31}$$

$$\mu_{-1} \leq \frac{1}{2}\left(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2}\right), \tag{4.32}$$

$$\lambda_1 \leq \mu_1, \tag{4.33}$$

$$\mu_n \leq \frac{1}{2}\left(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_m^2}\right). \tag{4.34}$$

In [18] a simple (3×3) example is given that shows that (4.31), (4.32), (4.33), and (4.34) are sharp. In the uniformly stabilised case, we modify the proof of [18] to obtain the following lemma (see also [21]).

Lemma 4.2. *For the case $C \neq 0$ where (4.30) holds, the eigenvalues of $\tilde{A}$ satisfy (4.32), (4.33), (4.34), and*

$$\frac{1}{2}\left(\lambda_1 - \beta\Delta - \sqrt{(\lambda_1 + \beta\Delta)^2 + 4\sigma_m^2}\right) \leq \mu_{-m}. \tag{4.35}$$

A tighter bound that holds for stable and uniformly stabilised formulations is provided by the following lemma.

Lemma 4.3.

$$\mu_{-1} \leq \frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\gamma^2\theta^2\lambda_1}\right). \tag{4.36}$$

We now estimate σ_m in terms of λ_n.

Lemma 4.4.

$$\sigma_m \leq \Lambda\Phi\sqrt{\mu_n}. \tag{4.37}$$

Theorem 4.5. *For a stable or stabilized discrete Stokes problem (1.2) on a quasiuniform sequence of grids, assuming (4.26) and uniform stability in the sense of (4.25) or (4.27), and also that (4.29) and (4.30) hold, the eigenvalues of the preconditioned matrix (4.19) lie in the union of intervals*

$$[a_1, b_1] \cup [c_1, d_1] \tag{4.38}$$

where $a_1 = \frac{1}{2}\left(\lambda_1 - \beta\Delta - \sqrt{(\lambda_1 + \beta\Delta)^2 + 4\Lambda^2\Phi^2\lambda_n}\right)$, $b_1 = \frac{1}{2}\left(\lambda_1 - \sqrt{\lambda_1^2 + 4\Lambda^2\Phi^2\lambda_1}\right)$, $c_1 = \lambda_1$, $d_1 = \frac{1}{2}\left(\lambda_n - \sqrt{\lambda_n^2 + 4\Lambda^2\Phi^2\lambda_n}\right)$ *($\Delta = 0$ in the stable case).*

To convert the eigenvalue bounds (4.38) into estimates in terms of the mesh size parameter h (which will approach zero under mesh refinement), we will assume that

$$g(h) \leq \frac{u^t A u}{u^t M_A u} \leq 1, \tag{4.39}$$

or equivalently, $g(h) < \lambda_1$ and $\lambda_n < 1$.

Theorem 4.6. *For a stable or stabilised discrete Stokes problem (1.2) on a quasiuniform sequence of grids, assuming (4.26) and uniform stability in the sense of (4.25) or (4.27), and also that (4.29), (4.30), and (4.39) hold with $g(h) \longrightarrow 0$ as $h \to 0$, then the eigenvalues of the preconditioned matrix (4.19) lie in the union of intervals*

$$[a_2, b_2] \cup [c_2, d_2] \tag{4.40}$$

where $a_2 = -\frac{\beta\Delta}{2} - \sqrt{\left(\frac{\beta\Delta}{2}\right)^2 + \Lambda^2\Phi^2\mu_n} + 0\left(g(h)\right)$, $b_2 = -\gamma\theta\sqrt{g(h)} + 0\left(g(h)\right)$, $c_2 = g(h)$ *and* $d_2 = \frac{1}{2} + \sqrt{\frac{1}{4} + \Lambda^2\Phi^2}$.

Theorem 4.7. *Given an unstabilized Stokes problem, that is (1.3) with $C = 0$, and using a preconditioner (4.18) with $M_A = A$ and M_C such that (4.29) holds, then assuming quasi-uniformity of the grid and LBB stability in the sense of (4.25) and (4.26), the eigenvalues of the preconditioned matrix (4.19) lie in the union of three intervals*

$$[-b, -a] \cup [1, 1] \cup [1 + a, 1 + b] \tag{4.41}$$

with constants

$$a = -\frac{1}{2} + \sqrt{1 + 4\gamma^2\theta^2}, \quad b = -\frac{1}{2} + \sqrt{1 + 4\Lambda^2\Phi^2}$$

independent of the grid parameter h.

Thus although uniform stabilization guarantees eigenvalue bounds that are independent of h, the symmetry of the stable clusters (4.41) is not possible if (4.25) is not satisfied.

5 A posteriori error estimators

To control the error of our problem, we propose two a posteriori error indicator, the first one is local Poisson problem estimator and the second one is residual error estimator which are equivalent to the global error estimates. By using the coercivity of the bilinear function $a(\cdot,\cdot)$ we have this lemma

Lemma 5.1.

$$\sup_{(\overrightarrow{v},q)\in V\times W} \frac{a(\overrightarrow{w},\overrightarrow{v})+d(s,q)}{\|\overrightarrow{v}\|_V+\|q\|_W} \geq \frac{1}{2}(\|\overrightarrow{w}\|_V+\|s\|_W), \tag{5.42}$$

for all $(\overrightarrow{w},s)\in V\times W$.

Proof. Let $(\overrightarrow{w},s)\in V\times W$, we have

$$\sup_{(\overrightarrow{v},q)\in V\times W} \frac{a(\overrightarrow{w},\overrightarrow{v})+d(s,q)}{\|\overrightarrow{v}\|_V+\|q\|_W} \geq \frac{a(\overrightarrow{w},\overrightarrow{w})+d(s,0)}{\|\overrightarrow{w}\|_V+\|0\|_W} = \|\overrightarrow{w}\|_V, \tag{5.43}$$

and we have

$$\sup_{(\overrightarrow{v},q)\in V\times W} \frac{a(\overrightarrow{w},\overrightarrow{v})+d(s,q)}{\|\overrightarrow{v}\|_V+\|q\|_W} \geq \frac{a(\overrightarrow{w},\overrightarrow{0})+d(s,s)}{\|\overrightarrow{0}\|_V+\|s\|_W} = \|s\|_W. \tag{5.44}$$

We gather (5.43) and (5.44) to get (5.42). □

Let $(\overrightarrow{e},E)\in V\times W$ be the errors in the finite element approximation where $\overrightarrow{e}=\overrightarrow{u}-\overrightarrow{u}_h$ and $E=p-p_h$. Our aim is to bound these values $\|\overrightarrow{e}\|_V$ and $\|e\|_W$ with respect to the energy norm for the velocity $\|\overrightarrow{u}\|_V=a(\overrightarrow{u},\overrightarrow{u})$ and the quotient norm for the pressure $\|p\|_W=\|p\|_{\Omega,0}$. Let the symmetric bilinear form

$$B\left[(\overrightarrow{u},p),(\overrightarrow{v},q)\right]=a(\overrightarrow{u},\overrightarrow{v})+b(\overrightarrow{v},p)+b(\overrightarrow{u},q)), \tag{5.45}$$

with the corresponding functional $F(\overrightarrow{v},q)=l(\overrightarrow{v})$. Let $\overrightarrow{u}$ (resp $\overrightarrow{u}_h$) be solution of (2.8) (resp (3.9)) the bilinear function satisfy

$$\begin{aligned} B\left[(\overrightarrow{e},E),(\overrightarrow{v},q)\right] &= B\left[(\overrightarrow{u}-\overrightarrow{u}_h,p-p_h),(\overrightarrow{v},q)\right] \\ &= B\left[(\overrightarrow{u},p),(\overrightarrow{v},q)\right]-B\left[(\overrightarrow{u}_h,p_h),(\overrightarrow{v},q)\right] \\ &= l(\overrightarrow{v})-a(\overrightarrow{u},\overrightarrow{v})-b(\overrightarrow{v},p)-b(\overrightarrow{u},q)), \end{aligned} \tag{5.46}$$

for all $(\overrightarrow{v}, q)$.

The stress jump across edge or face E adjoining elements T and S defined by

$$[[\nabla \overrightarrow{u}_h - p_h \overrightarrow{I}]] = ((\nabla \overrightarrow{u}_h - p_h \overrightarrow{I})|_T - (\nabla \overrightarrow{u}_h - p_h \overrightarrow{I})|_K) \overrightarrow{n}_{E,K},$$

where $\overrightarrow{n}_{E,K}$ is the outward pointing normal.
Now, we define these important equidistributed stress jump operators defined by

$$\overrightarrow{R}^*_E = \left\{ \begin{array}{c} \frac{1}{2}[[\nabla \vec{u}_h - p_h I]] \text{ if } E \in \varepsilon_{h,\Omega}, \\ \overrightarrow{g} - [\mathbb{A}^{-1}\overrightarrow{u} - (\nabla \overrightarrow{u} - pI)\overrightarrow{n})] \text{ if } E \in \varepsilon_{h,\Gamma}, \end{array} \right. \tag{5.47}$$

The interior residuals defined by

$$\overrightarrow{R}_K = \{\overrightarrow{f} + \mu^* \nabla^2 \overrightarrow{u}_h - \nabla p_h - \mu \mathbb{K}^{-1} \overrightarrow{u}_h\}|_K, \tag{5.48}$$

and

$$R_K = \{\nabla . \overrightarrow{u}_h\}|_K. \tag{5.49}$$

We define $(\overrightarrow{\phi}, \psi) \in V \times W$ to be the Ritz projection of the modified residuals

$$a(\overrightarrow{\phi}, \overrightarrow{v}) + d(\psi, q) = a(\overrightarrow{e}, \overrightarrow{v}) + b(\overrightarrow{v}, E) + b(\overrightarrow{e}, q), \tag{5.50}$$

for all $(\overrightarrow{v}, q) \in V \times W$.

Theorem 5.2. *There exist positive constants K_1 and K_2 such that*

$$K_1(\|\overrightarrow{\phi}\|^2_V + \|\psi\|^2_W) \leq \|\overrightarrow{u} - \overrightarrow{u}_h\|^2_V + \|p - p_h\|^2_W \leq K_2(\|\overrightarrow{\phi}\|^2_V + \|\psi\|^2_W) \tag{5.51}$$

Proof. See Ainsworth, M., and Oden, J. [4].□ □

The local velocity space on each subdomain $K \in P$ is

$$V_K = \{\overrightarrow{v} \in H^1(K) \times H^1(K) : \overrightarrow{v} = \overrightarrow{0} \text{ on } \partial\Omega \cap \partial K\}, \tag{5.52}$$

and the pressure space is $W_K = L^2(K)$. The bilinear forms $a_K : V_K \times V_K \longrightarrow \mathbb{R}$, $b_K : V_K \times W_K \longrightarrow \mathbb{R}$, and $d_K : W_K \times W_K \longrightarrow \mathbb{R}$ defined by

$$a_K(\overrightarrow{u}, \overrightarrow{v}) = \int_K \mu^* \nabla \overrightarrow{u} . \nabla \overrightarrow{v} + \mu \mathbb{K}^{-1} \overrightarrow{u} . \overrightarrow{v} \, dx + \int_{\Gamma \cap K} \mathbb{A}^{-1} \overrightarrow{u} . \overrightarrow{v} \, d\Gamma, \tag{5.53}$$

$$b_K(\overrightarrow{v}, q) = -\int_K (q \nabla . \overrightarrow{v}) dx, \quad d_K(p, q) = \int_K p \ q \ dx. \tag{5.54}$$

The continuous function $l_K : V_K \longrightarrow \mathbb{R}$ is defined by

$$l_K(\overrightarrow{v}) = \int_K \overrightarrow{f}.\overrightarrow{v}\ dx + \int_{\Gamma\cap K} \overrightarrow{g}.\overrightarrow{v} d\Gamma. \tag{5.55}$$

Hence for $\overrightarrow{v}, \overrightarrow{w} \in V$ and $q \in W$ we have

$$a(\overrightarrow{v}, \overrightarrow{w}) = \sum_{K\in P} a_K(\overrightarrow{v}_K, \overrightarrow{w}_K),\ b(\overrightarrow{v}, q) = \sum_{K\in P} b_K(\overrightarrow{v}_K, q_K). \tag{5.56}$$

The velocity space $V(P) = \prod_{K\in P} V_K$ and the broken pressure space $W(P) = \{q \in \prod_{K\in P} W_K : \int_\Omega q(x)dx = 0\}$. Examining the previous notations reveals that

$$W(P) = W. \tag{5.57}$$

We consider the space of continuous linear functional τ on $V(P) \times W(P)$ that vanish on the space $V \times W$.
Therefore, let $H(div, \Omega) = \{A \in L^2(\Omega)^{2\times 2} : div(A) \in L^2(\Omega)^2\}$, equipped with this norm $\|A\|_{H(div,\Omega)} = \{\|A\|^2_{L^2(\Omega)} + \|divA\|^2_{L^2(\Omega)}\}^{\frac{1}{2}}$.

Lemma 5.3. *A continuous linear functional τ on the space $V(P) \times W(P)$ vanishes on the space $V \times W$ if and only if there exists $A \in H(div, \Omega)$ such that*

$$\tau[(\overrightarrow{v}, q)] = \sum_{K\in P} \oint_{\partial K} \overrightarrow{n}_K.A.\overrightarrow{v}_K ds. \tag{5.58}$$

Proof. See Ainsworth, M., and Oden, J. [4]. □

It will be useful to introduce the stress like tensor $\sigma(\overrightarrow{v}, q)$ formally defined to be

$$\sigma_{ij}(\overrightarrow{v}, q) = \nu \frac{\partial v_i}{\partial x_j} - q\delta_{ij}, \tag{5.59}$$

Where δ_{ij} is the Kronecker symbol.
In order to define the value of the normal component of the stress on the interelement boundaries it is convenient to introduce notations for the jump on Γ_{KJ}:

$$[[\overrightarrow{v}.\sigma(\overrightarrow{v}_h, q_h)]] = \overrightarrow{n}_K.\sigma(\overrightarrow{v}_{h,K}, q_{h,K}) + \overrightarrow{n}_J.\sigma(\overrightarrow{v}_{h,J}, q_{h,J}). \tag{5.60}$$

For $\overrightarrow{v} \in V(P)$, we have

$$\sum_{K\in P} \oint_{\partial K} \overrightarrow{n}_K.\sigma(\overrightarrow{u}_h, p_h).\overrightarrow{v} ds = \sum_{\Gamma_{KJ}} \int_{\Gamma_{KJ}} \langle \overrightarrow{n}_K.\sigma(\overrightarrow{u}_h, p_h)\rangle.[[\overrightarrow{v}]]ds. \tag{5.61}$$

Lemma 5.4. *There exists $\widehat{\mu} \in H(div, \Omega)$ such that*

$$\widehat{\mu}[(\overrightarrow{w}, q)] = \sum_{\Gamma_{KJ}} \int_{\Gamma_{KJ}} \langle \overrightarrow{n}_K.\sigma(\overrightarrow{u}_h, q_h)\rangle.[[\overrightarrow{w}]]ds, \tag{5.62}$$

for all $(\overrightarrow{w}, q) \in V(P) \times W(P)$.

Proof. The right-hand side of equation (5.62) vanishes en $V \times W$. Applying Lemma 5.2, we obtain (5.62). □

We define the linear functional $R : V(P) \times W(P) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} R[(\overrightarrow{w}, q)] &= \sum_{K \in P} \{l_k(\overrightarrow{w}) - a_K(\overrightarrow{u}_h, \overrightarrow{w}) - b_K(\overrightarrow{w}, p_h) - b_K(\overrightarrow{u}_h, q)\} \\ &\quad + \oint_{\partial K} \overrightarrow{n}_K.\sigma(\overrightarrow{u}_h, p_h).\overrightarrow{w}_K ds - \widehat{\mu}[(\overrightarrow{w}, q)], \end{aligned} \tag{5.63}$$

for all $(\overrightarrow{w}, q) \in V(P) \times W(P)$.
For $(\overrightarrow{w}, q) \in V \times W$, we obtain

$$R[(\overrightarrow{w}, q)] = a(\overrightarrow{\phi}, \overrightarrow{w}) + d(\psi, q). \tag{5.64}$$

Let the lagrangian functional $L : V(P) \times W(P) \times H(div, \Omega) \longrightarrow \mathbb{R}$ such that

$$L[(\overrightarrow{w}, q), \mu] = \frac{1}{2}\{a(\overrightarrow{w}, \overrightarrow{w}) + d(q, q)\} - R[(\overrightarrow{w}, q)] - \mu[(\overrightarrow{w}, q)], \tag{5.65}$$

So that

$$Sup_{\mu \in H(div,\Omega)} L[(\overrightarrow{w}, q), q] = \begin{cases} \frac{1}{2}\{a(\overrightarrow{w}, \overrightarrow{w}) + d(q, q)\} - R[(\overrightarrow{w}, q)] \text{ if } (\overrightarrow{w}, q) \in V \times W, \\ = +\propto \ \textit{otherwise}, \end{cases} \tag{5.66}$$

and, by using the coercivity of the bilineaire function $a(\cdot, \cdot)$ for $(\overrightarrow{w}, q) \in V \times W$,

$$\begin{aligned} \frac{1}{2}\{a(\overrightarrow{w}, \overrightarrow{w}) + d(q, q)\} - R[(\overrightarrow{w}, q)] &= \frac{1}{2}\{a(\overrightarrow{w} - \overrightarrow{\phi}, \overrightarrow{w} - \overrightarrow{\phi}) + d(q - \psi, q - \psi) - \\ &\quad a(\overrightarrow{\phi}, \overrightarrow{\phi}) - d(\psi, \psi)\} \\ &\geq -\frac{1}{2}\{a(\overrightarrow{\phi}, \overrightarrow{\phi}) + d(\psi, \psi)\} = -\frac{1}{2}(\|\overrightarrow{\phi}\|_V^2 + \|\psi\|_W^2). \end{aligned}$$

Therefore,

$$\begin{aligned}
-\frac{1}{2}(\|\overrightarrow{\phi}\|_V^2+\|\psi\|_W^2) &= Inf_{(\overrightarrow{w},q)\in V(P)\times W(P)}Sup_{\mu\in H(div,\Omega)}L[(\overrightarrow{w},q),\mu] \\
&= Sup_{\mu\in H(div,\Omega)}Inf_{(\overrightarrow{w},q)\in V(P)\times W(P)}L[(\overrightarrow{w},q),\mu] \\
&\geq Inf_{(\overrightarrow{w},q)\in V(P)\times W(P)}L[(\overrightarrow{w},q),\mu] \qquad (5.67)\\
&= \sum_{K\in P} Inf_{\overrightarrow{w}_K\in V_K}\{\frac{1}{2}a(\overrightarrow{w}_K,\overrightarrow{w}_K)-l_k(\overrightarrow{w}_K)+a_K(\overrightarrow{u}_h,\overrightarrow{w}_K) \\
&\quad +b_K(\overrightarrow{w}_K,p_h)-\oint_{\partial K}\overrightarrow{n}_K.\sigma(\overrightarrow{u}_h,p_h).\overrightarrow{w}_K ds-\frac{1}{2}d_K(\nabla.\overrightarrow{u}_h,\nabla.\overrightarrow{u}_h)\}.
\end{aligned}$$

By using the inequality (5.67), we have:

Theorem 5.5. *Let $J_K : V_K \to \mathbb{R}$ be a quadratic functional*

$$\begin{aligned}
J_K(\overrightarrow{w}_K) &= \frac{1}{2}a(\overrightarrow{w}_K,\overrightarrow{w}_K)-l_k(\overrightarrow{w}_K)+a_K(\overrightarrow{u}_h,\overrightarrow{w}_K)+b_K(\overrightarrow{w}_K,p_h) \\
&\quad -\oint_{\partial K}\overrightarrow{n}_K.\sigma(\overrightarrow{u}_h,p_h).\overrightarrow{w}_K ds. \qquad (5.68)
\end{aligned}$$

Then

$$\|\overrightarrow{\phi}\|_V^2+\|\psi\|_W^2 \leq \sum_{K\in P}\{-2\ inf_{\overrightarrow{w}_K\in V_K}J_K(\overrightarrow{w}_K)+d_K(\nabla.\overrightarrow{u}_{h,K},\nabla.\overrightarrow{u}_{h,K})\} \qquad (5.69)$$

We have the problems on each subdomain $inf_{\overrightarrow{w}_K\in V_K}\ J_K(\overrightarrow{w}_K)$. Suppose that the minimum exists and characterized by finding $\overrightarrow{\phi}_K \in V_K$

Lemma 5.6. *Suppose that for each X_A the constants $\{\lambda^{(k)}_{KJ,A}\}$ satisfy*

$$-\sum_{J\in P}\lambda^{(k)}_{KJ,A}\rho^{(k)}_{KJ,A}=b^{(k)}_{K,A}, \qquad (5.70)$$

for k=1, 2 where

$$\begin{aligned}
b^{(k)}_{K,A} &= l_K(X_A\overrightarrow{\theta}_k)-a_K(\overrightarrow{u}_h,X_A\overrightarrow{\theta}_k)-b_K(X_A\overrightarrow{\theta}_k,p_h) \\
&\quad +\oint_{\partial K}X_A(s)\langle\overrightarrow{n}_K.\sigma(\overrightarrow{u}_h,p_h)\rangle.\overrightarrow{\theta}_k ds, \qquad (5.71)
\end{aligned}$$

and

$$\rho^{(k)}_{KJ,A}=\int_{\Gamma_{KJ}}[[\overrightarrow{n}.\sigma(\overrightarrow{u}_h,p_h)]].\overrightarrow{\theta}_k ds. \qquad (5.72)$$

Then

$$0 = l_K(\overrightarrow{\theta}) - a_K(\overrightarrow{u}_h, \overrightarrow{\theta}) - b_K(\overrightarrow{\theta}, p_h) + \oint_{\partial K} \langle \overrightarrow{n}_K.\sigma(\overrightarrow{u}_h, p_h)\rangle.\overrightarrow{\theta}\, ds, \quad (5.73)$$

for all $\overrightarrow{\theta} \in Ker[a, V_K]$.

Proof. see [13]. □

New we present the important results in this section. We define the global error estimator η_p by

$$\eta_p = (\sum_{K\in P} \eta_K^2)^{\frac{1}{2}}. \quad (5.74)$$

and we have this important Theorem.

Theorem 5.7. *There exists a constant* $C > 0$ *such that*

$$\|\overrightarrow{u} - \overrightarrow{u}_h\|_V^2 + \|p - p_h\|_W^2 \leq C \sum_{K\in P} \eta_K^2, \quad (5.75)$$

where

$$\eta_K = \{a_K(\overrightarrow{\phi}_K, \overrightarrow{\phi}_K) + d_K(\nabla.\overrightarrow{u}_h, \nabla.\overrightarrow{u}_h)\}^{\frac{1}{2}}. \quad (5.76)$$

The global residual error estimator η_r is given by

$$\eta_r = (\textstyle\sum_{K\in P} \ \eta_{r,K}^2)^{\frac{1}{2}}.$$

where $\eta_{r,K}$ is the element contribution of the residual error estimator defined by

$$\eta_{r,K}^2 = h_K^2 \|\overrightarrow{R}_K\|_{0,K}^2 + \|R_K\|_{0,K}^2 + \sum_{E\in\partial K} h_E \|\overrightarrow{R}_E^*\|_{0,E}^2, \quad (5.77)$$

and his elements defined in (5.47)-(5.49). In this theorem we can shown the equivalent between the estimators.

Theorem 5.8. *The estimators* $\eta_{r,K}$ *and* η_K *equivalent. There exist positive constants* c_1 *and* C_2 *such that*

$$c_1 \ \eta_K \leq \eta_{r,K} \leq C_2 \ \eta_K. \quad (5.78)$$

Proof. See Theorem 3.9 in [8]. □

Theorem 5.9. *There exist positive constant* C' *such that*

$$\|\overrightarrow{u} - \overrightarrow{u}_h\|_V^2 + \|p - p_h\|_W^2 \leq C' \sum_{K\in P} \eta_{r,K}^2. \quad (5.79)$$

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