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Applications of the Elzaki Transform Method for Solving Quadratic Riccati Differential Equations

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Abstract

In this paper, the Elzaki transform method is applied to solve the quadratic Riccati differential equation. The obtained results are expressed in terms of an infinite series that converges to closed form solutions. Some examples are presented to illustrate the efficiency of the method.

1 Introduction

Riccati differential equation plays a major role in the various fields of applied science [17]. This kind of equation is originally proposed by the Italian nobleman, Count Jacopo Francesco Riccati [2]. The applications of the equation can be found in random processes, optimal control, diffusion problem, stochastic realization theory, network synthesis, and financial mathematics [17]. In recent years, many researchers attempt to find its solution in both analytical and numerical form. The numerous iterative methods are suggested for solving the equation such as Adomian decomposition method

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(ADM) [3, 4], homotopy perturbation method (HPM)[1], variational iteration method (VIM)[10], differential transform method (DTM)[15] and the Bizier curves method [11]. Moreover, the numerical techniques based on using cubic B- spline scaling functions and Chebyshev cardinal functions were given in [13].

Elzaki transform method is a powerful device for constructing analytic approximate solution of scientific problems. It was initially introduced by Elzaki [5] in 2011 as a modification of the classical Sumudu transform. The authors then derived this transform for ordinary, partial or even fractional derivatives [5] [6] [7] [8] [9],[12].

In this work, we present the solution of the initial value problem for Riccati differential equation

$$\begin{cases} u'(t) = p(t) + q(t)u(t) + r(t)u^2(t), \quad t_0 \leq t \leq t_1 \\ u(0) = \alpha \end{cases} \tag{1.1}$$

where p, q, r are continuous functions and t_0, t_1, α are arbitrary constants. The existence of a solution of Cauchy problem can be found in the classical differential equation book. The purpose of this study is to apply the Elzaki transform to quadratic Riccati differential equation.

The manuscript is organized as follows. The basic definition, theorem, and properties of Elzaki transform are stated in Section 2 while the application of the Elzaki transformation method to Riccati differential equation is described in Section 3. Section 4 shows the illustrative examples and a conclusion is found in the last section.

2 Elzaki Transformation

In this section, we introduce some basic definitions and properties of Elzaki transform. As sufficient conditions for the existence of the Elzaki transform are that of exponential order, this implies that the ELzaki transform may or may not exist. Hence, to guarantee the existence let us consider the set $\mathcal{A}$ defined

$$\mathcal{A} = \left\{ f(t) \middle| \, \exists M, \, k_1, \, k_2 > 0 \text{ such that } \left| f(t) \right| \leq M e^{\frac{|t|}{k_j}} \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

Then the Elzaki transform of exponential ordered function can be stated as follows.

Definition 2.1. *[5] For any real function f defined for all $t \geq 0$, the Elzaki transform of $f(t)$ is defined by $E[f(t)]$:*

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt, \quad t > 0.$$

Theorem 2.2. *[5] Let $f(t)$ be in $\mathcal{A}$ and let $T_n(v)$ denote the Elzaki transform of nth derivative $f^{(n)}(t)$ of $f(t)$, then for $n \geq 1$*

$$T_n(v) = \frac{1}{v}T(v) - \sum_{k=0}^{n} v^{2-n+k}f^{(k)}(0). \tag{2.1}$$

By the above definition and theorem, the following results can be obtained

1. $E[f'(t)] = \frac{1}{v}T(v) - vf(0)$,
2. $E[f''(t)] = \frac{1}{v}T(v) - f(0) - vf'(0)$,
3. $E[tf(t)] = v^2\frac{d}{dv}[\frac{T(v)}{v}] - vT(v)$,
4. $E[t^2f(t)] = v^4\frac{d}{dv}T(v)$.

Moreover, Elzaki transform of some functions are shown in 1. The proof can be found in [5, 6]

3 Applications of Elzaki transform method to Quadratic Riccati Differential Equation.

Consider the initial value problem for quadratic Riccati differential equation,

$$u'(t) = Ru + Nu \tag{3.1}$$

subject to initial condition $u(0) = \alpha$, where Ru and Nu are linear and nonlinear terms respectively. Applying the Elzaki transform on both side of (3.1) and by theorem 2.2, one obtains

$$\frac{1}{v}E[u(t)] - vu(0) = E[Ru] + E[Nu]. \tag{3.2}$$

Table 1: Elzaki transformation of some functions

$f(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
t^n, $n = 1, 2, \ldots$	$n!v^{n+3}$
e^{at}	$\dfrac{v^2}{1-av}$
$\sin at$	$\dfrac{av^3}{1+a^2v^2}$
$\cos at$	$\dfrac{v^2}{1+a^2v^2}$
$\sinh at$	$\dfrac{av^3}{1-a^2v^2}$
$\cosh at$	$\dfrac{av^2}{1-a^2v^2}$

Because of initial condition $u(0) = \alpha$, it becomes

$$E[u(t)] = \alpha v^2 + vE[Ru] + vE[Nu]. \tag{3.3}$$

Taking inverse Elzaki transform on both side of (3.3), we have

$$u(t) = E^{-1}[\alpha v^2] + E^{-1}[vE[Ru]] + E^{-1}[vE[Nu]] \tag{3.4}$$

Here, the Elzaki transform method represents solution $u(t)$ of (3.1) as an infinite series of components in the form

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \tag{3.5}$$

and the non-linear term Nu can be represented by infinite series

$$Nu = \sum_{n=0}^{\infty} A_n \tag{3.6}$$

where A_n are Adomian polynomials of $u_0, u_1, \ldots, u_n$ which can be calculated by the formula

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\lambda^n} N\Big(\sum_{n=0}^{\infty} \lambda^n u_n\Big)\Big]_{\lambda=0}. \tag{3.7}$$

Substituting (3.5)-(3.7) into (3.4), one gets

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(t) &= E^{-1}[\alpha v^2] + E^{-1}[vE[R\Big(\sum_{n=0}^{\infty} u_n(t)\Big)]] + E^{-1}[vE[\sum_{n=0}^{\infty} A_n(t)]] \\ &= E^{-1}[\alpha v^2] + E^{-1}[vE[p(t) + q(t)\Big(\sum_{n=0}^{\infty} u_n(t)\Big)]] + E^{-1}[vE[\sum_{n=0}^{\infty} A_n(t)]]\end{aligned}$$

The recursive relation defined by

$$\begin{cases} u_0(t) = E^{-1}[\alpha v^2] + E^{-1}\Big[vE[p(t)]\Big] = H(t) \\ u_{n+1}(t) = E^{-1}\Big[vE[q(t)u_n(t)]\Big] + E^{-1}\Big[vE[A_n(t)]\Big], \quad n = 1, 2, \dots \end{cases} \tag{3.8}$$

The function $H(t)$ represents the term arising from source term and prescribe initial condition. The modified Elzaki transform method is considered that the function $H(t)$ defined above in (3.8) be decomposed into two parts, namely $H_0(t)$ and $H_1(t)$. Such that

$$H(t) = H_0(t) + H_1(t)$$

Under this assumption, we propose a slight variation only in the components u_0, u_1. The variation we propose is that only the part H_0 be assigned to the u_0, whereas the remaining part $H_1(t)$ be combined with the other terms to define u_1. In view of these suggestion, the initial solution is important, and the choice of Eq. (3.8) as the initial solution always leads to noise oscillation during the iteration procedure. Instead of the iteration procedure, Eqs.(3.8) we suggest the following modification

$$\begin{cases} u_0(t) = H_0(t), \\ u_1(t) = H_1(t) + E^{-1}\big[vE[q(t)u_0(t)]\big] + E^{-1}\big[vE[A_0(t)]\big] \\ u_{n+1}(t) = E^{-1}[vE[q(t)u_n(t)]] + E^{-1}\big[vE[A_n(t)]\big], \; n = 1, 2, \dots \end{cases} \tag{3.9}$$

The solution through the modified Elzaki transform method highly depends upon the choice of and We will show how to suitably choose and by example.

4 Numerical Examples

Example 4.1. Consider non-linear Riccati differential equation

$$u'(t) = 1 - u^2(t) \tag{4.1}$$

subject to initial condition $u(0) = 0$. The exact solution of this problem is $u(t) = \tanh t$. By applying the Elzaki transform method, here $p(t) = 1, q(t) = 0$ and $r(t) = -1$, the recursive relation is given by

$$\begin{aligned} u_0(t) &= t, \\ u_{n+1}(t) &= -E^{-1}\Big[vE[A_n]\Big], \quad n = 0, 1, 2, \dots \end{aligned}$$

So, the other components of the solution can be easily calculated

$$\begin{aligned} u_1(t) &= -E^{-1}\Big[vE[A_0(t)]\Big] = -\frac{1}{3}t^3, \\ u_2(t) &= -E^{-1}\Big[vE[A_1(t)]\Big] = \frac{2}{5}t^5, \\ u_3(t) &= -E^{-1}\Big[vE[A_2(t)]\Big] = -\frac{17}{315}t^7, \\ u_4(t) &= -E^{-1}\Big[vE[A_3(t)]\Big] = \frac{62}{2835}t^9, \\ &\vdots \end{aligned}$$

Hence, the series solution is given by

$$u(t) = t - \frac{1}{3}t^3 + \frac{2}{5}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 + \dots$$

which its closed form solution is $u(t) = \tanh t$.

Example 4.2. Now, consider non-linear Riccati differential equation

$$u'(t) = 2t - t^2u(t) + u^2(t) \tag{4.2}$$

subject to initial condition $u(0) = 0$. The exact solution of this problem is $u(t) = t^2$. By the recursive relation (3.8), one gets

$$\begin{aligned} u_0(t) &= t^2 \\ u_{n+1}(t) &= -E^{-1}\Big[v^5\frac{d^2}{dv^2}E[u_n(t)]\Big] + E^{-1}\Big[vE[A_n]\Big], \quad n = 0, 1, 2, \dots \end{aligned}$$

which generates other components of solution, $u_n(t) = 0$, $n = 1, 2, \dots$. Hence, the series solution is $u(t) = t^2$ which is the closed form.

Example 4.3. Let us consider non-linear Riccati differential equation

$$u'(t) = 1 - t^2 + u^2(t) \tag{4.3}$$

subject to initial condition $u(0) = 0$. The exact solution of this problem is $u(t) = t$. By utilizing the classical Elzaki Transform method, we obtain the recursive iteration

$$\begin{aligned} u_0(t) &= t - \frac{1}{3}t^3 \\ u_{n+1}(t) &= E^{-1}\Big[vE[A_n(t)]\Big], \quad n = 0, 1, 2, \ldots \end{aligned}$$

The other components of solution are

$$\begin{aligned} u_1(t) &= -E^{-1}\Big[vE[A_0(t)]\Big] = -\frac{1}{3}t^3 - \frac{2}{15}t^5 + \frac{1}{63}t^7 \\ u_2(t) &= -E^{-1}\Big[vE[A_1(t)]\Big] = \frac{2}{5}t^5 - \frac{22}{315}t^7 + \frac{38}{2835}t^9 - \frac{2}{2079}t^{11} \\ u_3(t) &= -E^{-1}\Big[vE[A_2(t)]\Big] = -\frac{17}{315}t^7 - \frac{20}{567}t^9 + \frac{206}{22275}t^{11} - \frac{1412}{121215}t^{13} + \frac{13}{218325}t^{15} \\ &\vdots \end{aligned}$$

The series solution is

$$u(t) = t - \frac{62}{2835}t^9 + \frac{1292}{155925}t^{11} - \frac{1412}{121215}t^{13} + \frac{13}{218325}t^{15} + \ldots$$

Obviously, the series slowly converge. By improving recursive relation,

$$\begin{aligned} u_0(t) &= t \\ u_1(t) &= -\frac{1}{3}t^3 + E^{-1}\Big[vE[A_0(t)]\Big] \\ u_{n+1}(t) &= E^{-1}\Big[vE[A_n(t)]\Big], \quad n = 1, 2, \ldots, \end{aligned}$$

One can find that the other components $u_n(t) = 0$, $n = 1, 2, \ldots$. Therefore, the series solution is $u(t) = t$.

Example 4.4. Consider non-linear Riccati differential equation

$$u'(t) + u(t) + 2u^2(t) + 3 = 0 \tag{4.4}$$

subject to initial condition $u(0) = 1$. By taking the classical Elzaki transform method, the recursive relation is

$$\begin{aligned} u_0(t) &= 1 - 3t \\ u_{n+1}(t) &= -E^{-1}\Big[vE[u_n(t)]\Big] - E^{-1}\Big[2vE[A_n(t)]\Big], \quad n = 0, 1, 2, \ldots \end{aligned}$$

One obtains

$$\begin{aligned} u_1(t) &= -3t + \frac{15}{2}t^2 - 6t^3, \\ u_2(t) &= \frac{15}{2}t^2 - \frac{49}{2}t^3 + 30t^4 - \frac{72}{5}t^5, \\ u_3(t) &= -\frac{27}{2}t^3 + \frac{477}{8}t^4 - \frac{1017}{10}t^5 + \frac{486}{5}t^6 - \frac{1224}{35}t^7 \\ &\vdots \end{aligned}$$

So, the series solution is

$$u(t) = 1 - 6t - 15t^2 - 44t^3 + \frac{315}{8}t^4 - \frac{1161}{10}t^5 + \frac{486}{5}t^6 - \frac{1224}{35}t^7 + \ldots$$

For improving the Elzaki transform method, the recursive relation is

$$\begin{aligned} u_0(t) &= 1 \\ u_1(t) &= -3t - E^{-1}\Big[vE[u_0(t)]\Big] - E^{-1}\Big[2vE[A_0(t)]\Big] \\ u_{n+1}(t) &= -E^{-1}\Big[vE[u_n(t)]\Big] - E^{-1}\Big[2vE[A_n(t)]\Big], \quad n = 1, 2, \ldots \end{aligned}$$

The other component of solution are

$$\begin{aligned} u_1(t) &= -6t, \\ u_2(t) &= 15t^2 \\ u_3(t) &= -49t^3 \\ &\vdots \end{aligned}$$

Thus, the series solution is

$$u(t) = 1 - 6t + 15t^2 - 49t^3 + \ldots$$

which satisfies the result in [17].

5 Conclusion

In this work, we successfully solve the quadratic Riccati differential equation by the Elzaki transform method. The results have shown the efficiency of the proposed method. The solution is expressed in term of an infinite series which converges to exact solution as in Example 4.1, 4.2 4.3 whereas the solution of Example 4.4 is identical to the result in [17]. The advantage is that linearization and perturbation are not required in the process.

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