Poset matrix and recognition of series-parallel posets

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Abstract

We introduce the notion of poset matrix for representing finite partially ordered sets (posets). We study the interpretations of different forms of poset matrix. We show that every poset matrix can be re-labeled to upper (equivalently lower) triangular form representing a unique poset up to isomorphism. We also show that the direct sum and ordinal sum of poset matrices represent the direct sum and ordinal sum of posets respectively. We define the properties of block of 0s, block of 1s and complete blocks of 1s on poset matrix and give an application of poset matrix in recognizing series-parallel posets.

1 Introduction

The class of series-parallel posets is one of the most important classes of posets specially due to their computationally tractability property for which they provide the basic structures of a number of applied and theoretical problems in several fields of computer science, electrical engineering and operation research [9, 10, 11, 12]. This class of posets is introduced firstly by Lawler [10] and studied further by numerous authors [1, 3, 6, 13, 20]. Kaerkes [7] gave a recognition of series-parallel posets in terms of forbidden substructure.

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and showed that a poset is series-parallel if and only if it does not contain an induced subposet isomorphic to the \( N \)-shaped poset (4-element zigzag poset).

Due to many computational aspects of incidence matrices, they have classical applications in recognition of different posets and graphs [2, 5, 8, 14, 18, 19]. Rhee [14] described a doubly-stochastic matrix \( D_P \), an incidence matrix representation of a poset \( P \) and showed that if \( P \) is a series-parallel poset except a chain then \( D_P \) is singular, that is, \( |D_P| = 0 \). Butler [2] described a particular solution to the problem of counting the number of posets with \( n \) elements by interpreting a partial order relation on \( n \)-element set as a non-singular idempotent Boolean relation matrix of order \( n \). This intuition gives us the idea of defining the properties of reflexivity, antisymmetry and transitivity on a square \((0,1)\)-matrix which we call a poset matrix.

Fulkerson and Gross [5] characterized interval graphs as graphs whose dominant clique-vertex incidence matrix has the property of perfect 1s for columns. Roberts [16, 17] characterized proper interval graphs as graphs whose augmented adjacency matrix has perfect 1s property for columns. Tucker [19] characterized circular-arc graphs and proper circular-arc graphs by using the properties of perfect 0s, circular 1s and circularly compatible 1s defined on the augmented adjacency matrix. These classical results give us the idea of defining the properties of block of 0s, block of 1s and complete blocks of 1s on poset matrix.

In Section 2 of this paper, we recall some important definitions and common notations related to series-parallel posets. In Section 3, we define poset matrix and describe some interpretations regarding matrix transpose and relabeling (interchanging rows and columns simultaneously) of poset matrix. We show that every poset matrix can be relabeled to upper (or lower) triangular form with 1s in the main diagonal by a finite times of relabeling. In Section 4, we show the constructions of the block matrices for representing direct sum and ordinal sum of posets. In Section 5, we define the properties of block of 0s, block of 1s and complete blocks of 1s on poset matrix. We then use these properties of poset matrix for recognition of \( P \)-graphs, \( P \)-series and series-parallel posets.
Figure 1: Hasse diagrams of $D_n, Z_n, P_{m,n}$ and $T_{t,n}$.

2 Preliminaries

A partially ordered set or poset is a structure $P = \langle P, \leq \rangle$ consisting of the nonempty set $P$ with the order relation $\leq$ on $P$, that is, $\leq$ is reflexive, antisymmetric and transitive. The set $P$ is called the underlying set or ground set of the poset $P$. For $x, y \in P$, we call $y$ covers $x$ if $x \leq y$ and for any $z \in P$, $x \leq z \leq y$ implies either $z = x$ or $z = y$. We write $x \prec y$ whenever $y$ covers $x$ or $x$ is covered by $y$. If for $x, y \in P$ neither $x \leq y$ in $P$ nor $y \leq x$ in $P$ then we say that $x$ and $y$ are incomparable in $P$ and we write $x \nmid y$. For details on posets, readers are referred to the classical book by Davey and Priestley [4].

We use the notations $1$ for the singleton poset, $C_n(n \geq 1)$ for $n$-element chain poset, $I_n(n \geq 1)$ for $n$-element antichain poset, $D_n(n \geq 4)$ for $n$-element diamond poset, $Z_n(n \geq 4)$ for $n$-element zigzag poset, $B_{m,n}(m \geq 1, n \geq 1)$ for complete bipartite poset with $m$ minimal elements and $n$ maximal elements, $P_{m,n}(m \geq 1, n \geq 1)$ for polygonal poset with maximal chains of lengths $m+2$ and $n+2$ and $T_{t,n}(n > t \geq 1)$ for $n$-element $t$-ary rooted tree as poset. The Hasse diagrams of $D_n, Z_n, P_{m,n}$ and $T_{t,n}$ (height-balanced) are shown in Figure 1. Through this paper, we assume every poset is finite and nonempty.

Let $P = \langle P, \leq_P \rangle$ and $Q = \langle Q, \leq_Q \rangle$ be two posets. A bijective map $\phi : P \to Q$ is called an order isomorphism if for all $x, y \in P$, $x \leq_P y$ if and only if $\phi(x) \leq_Q \phi(y)$. We write $P \cong Q$ whenever $P$ and $Q$ are order isomorphic. For example, $C_1 \cong I_1 \cong 1$, $P_{1,1} \cong D_4$, $C_2 \cong B_{1,1} \cong T_{1,2}$, $C_n \cong T_{1,n}(n \geq 2)$ and $B_{1,n} \cong T_{n,n+1}(n \geq 2)$.

Let $P = \langle P, \leq \rangle$ be a poset. The dual order of $\leq$ on $P$, denoted by $\leq^D$, is
defined as for all \( x, y \in P \), \( x \leq y \) if and only if \( y \leq x \) in \( P \). The *dual poset* of \( P \), denoted by \( P^\circ \), is defined as the poset with the dual order \( \leq^\circ \) on \( P \), that is, \( P^\circ = \langle P, \leq^\circ \rangle \). Obviously, \( C_n^\circ \cong C_n \) and \( D_n^\circ \cong D_n \). Also \( B_{m,n}^\circ \cong B_{n,m} \).

Let \( P = \langle P, \leq_P \rangle \) and \( Q = \langle Q, \leq_Q \rangle \) be posets such that \( P \) and \( Q \) are disjoint.

1. The *direct sum* or *disjoint sum* or *free sum* of \( P \) and \( Q \), denoted by \( P + Q \), is defined as the poset \( \langle P \cup Q, \leq_+ \rangle \) such that for all \( x, y \in P \cup Q \), \( x \leq_+ y \) if and only if either (i) \( x \leq_P y \) or (ii) \( x \leq_Q y \).

2. The *ordinal sum* of \( P \) and \( Q \), denoted by \( P \oplus Q \), is defined as the poset \( \langle P \cup Q, \leq_\oplus \rangle \) such that for all \( x, y \in P \cup Q \), \( x \leq_\oplus y \) if and only if either (i) \( x \leq_P y \) or (ii) \( x \leq_Q y \) or (iii) \( x \in P \) and \( y \in Q \).

Here \( P \) and \( Q \) are called the *direct terms* of \( P + Q \) and the *ordinal terms* of \( P \oplus Q \).

A poset \( P \) is called a *P-graph* if it can be expressed as the ordinal sum of singleton posets or antichain posets, that is, there exist the singleton posets or antichain posets \( P_i, 1 \leq i \leq n \) such that \( P = P_1 \oplus P_2 \oplus \cdots \oplus P_n \). The posets \( 1 \) and \( I_n \) are trivial examples of P-graphs. The posets \( C_n, B_{m,n} \) and \( D_n \) are some common examples of P-graphs because \( C_n = 1 \oplus 1 \oplus \cdots \oplus 1, B_{m,n} = I_m \oplus I_n \) and \( D_n = 1 \oplus I_{n-2} \oplus 1 \).

A poset \( P \) is called a *P-series* if it can be expressed as the direct sum of P-graphs, that is, there exist P-graphs \( P_i, 1 \leq i \leq n \) such that \( P = P_1 + P_2 + \cdots + P_n \). Every P-graph is trivially a \( P \)-series. The poset \( C_m + C_n \) is a \( P \)-series which is not a \( P \)-graph if either \( m \geq 2 \) or \( n \geq 2 \). However, the posets \( Z_m, P_{m,n} \) \( (m \geq 2 \text{ or } n \geq 2) \) and \( T_{t,n} \) \( (n \geq t + 2) \) are neither \( P \)-graphs nor \( P \)-series.

A poset \( P \) is called *series-parallel* if it can be expressed as the sum of singleton posets using direct sum or ordinal sum. In other words, \( P \) is series-parallel if there exist \( P \)-graphs or \( P \)-series \( P_i, 1 \leq i \leq n \) such that \( P = P_1 \ast P_2 \ast \cdots \ast P_n \) where \( \ast \) is either direct sum or ordinal sum. Every \( P \)-graph and \( P \)-series is trivially series-parallel. Also \( P_{m,n} \) and \( T_{t,n} \) are non-trivial series-parallel posets that can be shown particularly as follows.

(i) \( P_{m,n} = 1 \oplus (C_m + C_n) \oplus 1 \) where \( C_r = 1 \oplus 1 \oplus \cdots \oplus 1 \) (ordinal sum of \( r \) singletons). This shows that \( P_{m,n} \) can be expressed as the sum of
singleton posets using direct sum or ordinal sum.

(ii) \( T_{t,n} = r_0 \oplus (T_{t,n_0} + T_{t,n_0} + \cdots + T_{t,n_0}) \), where \( r_0 \) is the root (singleton poset) of \( T_{t,n} \) and \( T_{t,n_0} \) is either a leaf (singleton poset) or a subtree with number of elements \( n_0 = \frac{n-1}{t} \), particularly if the tree \( T_{t,n} \) is full. This continues recursively expressing \( T_{t,n} \) as the sum of singleton posets using direct sum or ordinal sum.

It is easy to check that every term (direct or ordinal) of a series-parallel poset is also series-parallel. However, the poset \( Z_n \) is not a series-parallel poset.

### 3 Poset matrix and some interpretations

From now on we use the notations \( M_{m,n} \) for an \( m \times n \) matrix and \( M_m \) for a square matrix of order \( m \). In particular, we use \( I_n \) for the identity matrix of order \( n \) and \( C_n \) for the matrix \( [c_{ij}], 1 \leq i, j \leq n \) defined as \( c_{ij} = 1 \) for all \( i \leq j \) and \( c_{ij} = 0 \) otherwise. Obviously \( C_1 = I_1 = 1 \).

**Definition 3.1.** A square \((0,1)\)-matrix \( M = [a_{ij}], 1 \leq i, j \leq n \) is called a poset matrix if the following conditions hold.

1. \( a_{ii} = 1 \) for all \( 1 \leq i \leq n \) i.e. \( M \) is reflexive;
2. \( a_{ij} = 1 \) and \( a_{ji} = 1 \) imply \( i = j \) i.e. \( M \) is antisymmetric;
3. \( a_{ij} = 1 \) and \( a_{jk} = 1 \) imply \( a_{ik} = 1 \) i.e. \( M \) is transitive.

**Example 3.1.**

\[
L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Here \( L \) and \( M \) are poset matrices and \( N \) is not a poset matrix. All the matrices \( L \), \( M \) and \( N \) are both reflexive and antisymmetric because they have all entries 1s in the main diagonal and unequal entries, except both are 0s, in opposite positions with respect to the main diagonal. Also \( L \) and \( M \) are trivially transitive. On the other hand, \( N = [a_{ij}], 1 \leq i, j \leq 3 \) is not transitive because \( a_{12} = 1 \) and \( a_{23} = 1 \) but \( a_{13} = 0 \).
An upper (or lower) triangular \((0, 1)\)-matrix with entries 1s in the main diagonal is clearly reflexive and antisymmetric. Therefore, an upper (or lower) triangular \((0, 1)\)-matrix with entries 1s in the main diagonal is a poset matrix if it is transitive. For example, both \(I_n\) and \(C_n\) are poset matrices because these are upper triangular and clearly transitive.

To each poset matrix we can associate a poset. Let \(M = [a_{ij}]\) be a poset matrix and \(P = \{x_1, x_2, \ldots, x_m\}\) where \(x_i\) corresponds the \(i\)-th row (or column) of \(M\). We define a relation \(\leq\) on \(P\) such that for all \(1 \leq i, j \leq m\), \(x_i \leq x_j\) if and only if \(a_{ij} = 1\). Since \(M\) is a poset matrix, clearly \(\leq\) is an order relation on \(P\). Thus \(P = (P, \leq)\) is a poset. We say that the poset matrix \(M\) represents the poset \(P\) and vice versa. The poset matrices \(L\) and \(M\), as in the Example 3.1, represent the posets \(1 + C_2\) and \(B_{2,1}\) (Figure 2) respectively. Obviously, the poset matrices \(I_n\) and \(C_n\) represent the posets \(I_n\), where \(x_i \parallel x_j\) for all \(1 \leq i, j \leq n\), and \(C_n\), where \(x_1 \prec x_2 \prec \cdots \prec x_n\), with the ground set \(\{x_1, x_2, \ldots, x_n\}\) respectively. In particular, both \(I_1\) and \(C_1\) represent the singleton poset \(1\).

We observe that the matrix transpose \(M^t\) of the poset matrix \(M\) (Example 3.1) represents the poset \(B_{1,2}\) which is dual to the poset \(B_{2,1}\) (Figure 2). We establish this result in general as follows.

**Theorem 3.2.** The matrix transpose \(M^t\) of a poset matrix \(M\) is a poset matrix and it represents the poset dual to the poset represented by \(M\).

**Proof.** Let \(M = [a_{ij}], 1 \leq i, j \leq n\). Then \(M^t = [a_{ji}], 1 \leq i, j \leq n\). Since \(a_{ii} = 1\) for all \(1 \leq i \leq n\) in both \(M\) and \(M^t\), clearly \(M^t\) is reflexive. Also \(a_{ij} = 1\) and \(a_{ji} = 1\) imply \(i = j\) in both \(M\) and \(M^t\). Thus \(M^t\) is antisymmetric. Let \(a_{ij} = 1\) and \(a_{jk} = 1\) in \(M^t\). Then \(a_{ji} = 1\) and \(a_{kj} = 1\) in \(M\). Since \(M\) is transitive, \(a_{ki} = 1\) in \(M\). This implies \(a_{ik} = 1\) in \(M^t\). Thus
\[ M^t \] is transitive. Therefore, \( M^t \) is a poset matrix.

To show that \( M^t \) represents the poset dual to the poset represented by \( M \), let \( M \) represents the poset \( P = \langle X, \leq_P \rangle \) and \( M^t \) represents the poset \( Q = \langle X, \leq_Q \rangle \) with the same underlying set \( X = \{ x_1, x_2, \ldots, x_n \} \). Let \( x_i \leq_P x_j \) for some \( i \) and \( j \). Then \( a_{ij} = 1 \) in \( M \) implies \( a_{ji} = 1 \) in \( M^t \). Thus \( x_j \leq_Q x_i \).

This shows that the order \( \leq_Q \) is dual to \( \leq_P \) on \( X \), that is, \( Q \cong P^\varnothing \).

**Definition 3.3.** Let \( M_m \) be a poset matrix. Then for some \( 1 \leq i, j \leq m \), interchanges of \( i \)-th and \( j \)-th rows along with interchanges of \( i \)-th and \( j \)-th columns in \( M_m \) is called \((i,j)\)-relabeling of \( M_m \).

**Example 3.2.**

\[
M = \begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{(1,2)\text{-relabeling}}
\begin{bmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\xrightarrow{(2,3)\text{-relabeling}}
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{(1,2)\text{-relabeling}}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
= M'
\]

We observe that the matrix \( M' \) obtained by some relabeling of the poset matrix \( M \) equals the poset matrix \( C_4 \). Moreover, \( M \) and \( M' \) represent the same poset \( C_4 \) with the ground set \( \{ x_1, x_2, x_3, x_4 \} \) such that \( x_3 \prec x_2 \prec x_1 \prec x_4 \) on \( M \) and \( x_1 \prec x_2 \prec x_3 \prec x_4 \) on \( M' \). Thus relabeling of a poset matrix \( M \) does not make any change to the order relation on \( M \), it provides just a renaming of the elements in the poset represented by \( M \). We establish these facts as follows.

**Theorem 3.4.** Any relabeling of a poset matrix is a poset matrix and it represents the same poset up to isomorphism.

**Proof.** Let \( M_m = [a_{ij}] \) be a poset matrix and \( N_m = [b_{ij}] \) be the matrix obtained by \((k,l)\)-relabeling of \( M_m \). Then we have the following equalities.

1. \( b_{kk} = a_{ll} \) and \( b_{ll} = a_{kk} \)
2. \( b_{kl} = a_{lk} \) and \( b_{lk} = a_{kl} \)
3. \( b_{ik} = a_{il} \) and \( b_{il} = a_{ik} \) for all \( 1 \leq i \leq m, i \neq k, i \neq l \)
4. $b_{kj} = a_{ij}$ and $b_{lj} = a_{kj}$ for all $1 \leq j \leq m, j \neq k, j \neq l$

5. $b_{ij} = a_{ij}$ for all $1 \leq i, j \leq m, i \neq k, i \neq l, j \neq k, j \neq l$

Since $M_m$ is a poset matrix, the above equalities show that $N_m$ is a poset matrix.

To show that $M_m = [a_{ij}]$ and $N_m = [b_{ij}]$ represent the same poset up to isomorphism, let they represent $P = \langle P, \leq_P \rangle$ and $Q = \langle Q, \leq_Q \rangle$ respectively where $P = \{x_1, x_2, \ldots, x_m\}$ and $Q = \{y_1, y_2, \ldots, y_m\}$. Define $\phi : P \to Q$ as follows.

$$\phi(x_t) = \begin{cases} 
y_l & \text{if } t = k, 
y_k & \text{if } t = l, 
y_t & \text{otherwise.} \end{cases}$$

Since $|P| = m = |Q|$, clearly $\phi$ is bijective. Let $x_i \leq_P x_j$. Then we have the following cases.

1. $i = k$.
   
   (a) $j = k$. Then $1 = a_{ij} = a_{kk} = b_{ll}$ (equality 1) implies $y_l \leq_Q y_t$.
   
   (b) $j = l$. Then $1 = a_{ij} = a_{lk} = b_{lk}$ (equality 2) implies $y_l \leq_Q y_k$.
   
   (c) $j \neq k, j \neq l$. Then $1 = a_{ij} = a_{kj} = b_{lj}$ (equality 4) implies $y_l \leq_Q y_j$.

2. $i = l$.
   
   (a) $j = k$. Then $1 = a_{ij} = a_{lk} = b_{kl}$ (equality 2) implies $y_k \leq_Q y_l$.
   
   (b) $j = l$. Then $1 = a_{ij} = a_{il} = b_{kk}$ (equality 1) implies $y_k \leq_Q y_k$.
   
   (c) $j \neq k, j \neq l$. Then $1 = a_{ij} = a_{lj} = b_{kj}$ (equality 4) implies $y_k \leq_Q y_j$.

3. $i \neq k, i \neq l$.
   
   (a) $j = k$. Then $1 = a_{ij} = a_{ik} = b_{il}$ (equality 3) implies $y_i \leq_Q y_l$.
   
   (b) $j = l$. Then $1 = a_{ij} = a_{il} = b_{ik}$ (equality 3) implies $y_i \leq_Q y_k$.
   
   (c) $j \neq k, j \neq l$. Then $1 = a_{ij} = b_{ij}$ (equality 5) implies $y_i \leq_Q y_j$.

Thus $\phi(x_i) \leq_Q \phi(x_j)$ and $\phi : P \to Q$ is an order isomorphism. 

We observe that the poset matrix $M'$ (Example 3.2) obtained by some relabeling of the non-triangular poset matrix $M$ is in upper triangular form. Below we establish this result in general.
Theorem 3.5. Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite time of relabeling.

Proof. Let \( M = [a_{ij}], 1 \leq i, j \leq n \) be a poset matrix. Suppose \( a_{ij} = 1 \) for some \( i > j \) such that for all \( 1 \leq k \leq j - 1, a_{ik} = 0 \) and the partition \( [a_{uv}], 1 \leq u, v \leq i - 1 \) of \( M \) is upper triangular. Then \( a_{(i-1)i} = 0 \). Otherwise, since \( a_{ij} = 1 \), we have the following cases.

1. \( i = j + 1 \). Then \( a_{ji} = a_{(i-1)i} \) contradicts that \( M \) is antisymmetric.
2. \( i > j + 1 \). Then transitivity of \( M \) implies \( a_{(i-1)j} = 1 \) which contradicts that \( P \) is upper triangular.

Then by \((i, j)\)-relabeling of \( M \), we have \( a_{(i-1)j} = 1 \) and \( a_{ik} = 0 \) for all \( 1 \leq k \leq i - 1 \). Similarly, the matrix \( M' \) obtained by \((k, k-1)\)-relabeling of \( M \) for all \( i - 1 \geq k \geq j + 1 \) with \( i > j + 1 \) is a matrix having partition \( [a_{uv}], 1 \leq u, v \leq i \) of \( M' \) in upper triangular form. Since \( n \) is finite, continuing the same process for all \( 1 \leq j < i \leq n \) with \( a_{ij} = 1 \), we have \( M' \) in upper triangular form. By Theorem 3.4, \( M' \) is a poset matrix and hence it has 1s in the main diagonal. We show similarly the lower triangular case. 

Corollary 3.6. Let \( M \) be any square \((0,1)\)-matrix. Then \( M \) is a poset matrix if and only if \( M \) is transitive and upper (or lower) triangular with 1s in the main diagonal.

4 Direct sum and ordinal sum of poset matrices

From now on by a poset matrix we mean a poset matrix in upper triangular form. Also, we use the notations \( Z_{m,n} \) for an \( m \)-by-\( n \) matrix having entries 0s only and \( O_{m,n} \) for an \( m \)-by-\( n \) matrix having entries 1s only.

Definition 4.1. The direct sum of the matrices \( M_{m,p} \) and \( N_{n,q} \), denoted by \( M_{m,p} + N_{n,q} \), is defined as an \((m+n)\)-by-\((p+q)\) block matrix such that

\[
M_{m,p} + N_{n,q} = \begin{bmatrix}
M_{m,p} & Z_{m,q} \\
--- & ---
\end{bmatrix}
\begin{bmatrix}
--- & ---
Z_{n,p} & N_{n,q}
\end{bmatrix}
\]

In this case, we call \( M_{m,p} \) and \( N_{n,q} \) as the direct terms of \( M_{m,p} + N_{n,q} \).
Theorem 4.2. Let $M_m$ represents the poset $A$ and $N_n$ represents the poset $B$. Then the matrix $M_m + N_n$ is a poset matrix and it represents the poset $A + B$.

Proof. Let $M_m = [a_{ij}]$, $N_n = [b_{ij}]$ and $M_m + N_n = S_{m+n} = [s_{ij}]$ with block representation $[S_{ij}], 1 \leq i, j \leq 2$. Since $S_{m+n}$ is upper triangular with elements 1s in the main diagonal, because $S_{21} = Z_{n,m}$, and $M_m$ and $N_n$ are poset matrices, $S_{m+n}$ is clearly reflexive and antisymmetric. For transitivity of $S_{m+n}$, let $s_{ij} = s_{jk} = 1$ for some $i \leq j \leq k$. Then we have the following cases.

1. $k \leq m$. Then $i \leq j \leq k \leq m$ implies $s_{ij}, s_{jk}, s_{ik} \in M_m$. Since $M_m$ is transitive, $s_{ik}$ = 1.

2. $k > m$.
   
   (a) $j \leq m$. Then $j \leq m < k$ implies $s_{jk} \in Z_{m,n}$ which contradicts that $s_{jk}$ = 1.
   
   (b) $j > m$.
   
   i. $i \leq m$. Then $i \leq m < j$ implies $s_{ij} \in Z_{m,n}$ which contradicts that $s_{ij}$ = 1.
   
   ii. $i > m$. Then $m < i \leq j \leq k$ implies $s_{ij}, s_{jk}, s_{ik} \in N_n$. Since $N_n$ is transitive, $s_{ik}$ = 1.

Thus $S_{m+n}$ is transitive and hence a poset matrix.

Let $A = \langle A; \leq_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $B = \langle B; \leq_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We show that $S_{m+n}$ represents $A + B = \langle A \cup B; \leq_{+} \rangle$, where $A \cup B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $s_{ij} = 1$ in $S_{m+n}$ for some $1 \leq i, j \leq m + n$. Then either $s_{ij} \in M_m$ or $s_{ij} \in N_n$. Since $M_m$ and $N_n$ represent $A$ and $B$ respectively, either $x_i \leq_A x_j$ or $x_i \leq_B x_j$. By definition, $x_i \leq_{+} x_j$ in $A \cup B$. Hence $S_{m+n}$ represents the poset $A + B$.

The following theorem gives a generalization of the Theorem 4.2.
Theorem 4.3. Let $M_{m_i}, 1 \leq i \leq n$ be the poset matrices representing the posets $P_i, 1 \leq i \leq n$ respectively. Then the matrix $S_s = [S_{ij}], 1 \leq i, j \leq n$, defined as

$$S_{ij} = \begin{cases} M_{m_i} & \text{if } i = j, \\ Z_{m_i,m_j} & \text{if } i < j, \\ Z_{m_j,m_i} & \text{otherwise.} \end{cases}$$

where $s = \sum_{i=1}^{n} m_i$, is a poset matrix representing the poset $P_1+P_2+\cdots+P_n$.

Proof. The proof follows inductively by the Theorem 4.2. For instance, the blocks of the matrix $S_s = M_1 + M_2 + \cdots + M_n$ can be viewed as follows.

$$S_s = \begin{bmatrix} M_{m_1} & Z_{m_1,m_2} & \cdots & Z_{m_1,m_n} \\ Z_{m_2,m_1} & M_{m_2} & \cdots & Z_{m_2,m_n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{m_n,m_1} & Z_{m_n,m_2} & \cdots & M_{m_n} \end{bmatrix}$$

Definition 4.4. The ordinal sum of the matrices $M_{m,p}$ and $N_{n,q}$, denoted by $M_{m,p} \oplus N_{n,q}$, is defined as an $(m+n)$-by-$(p+q)$ block matrix such that

$$M_{m,p} \oplus N_{n,q} = \begin{bmatrix} M_{m,p} & O_{m,q} \\ Z_{n,p} & N_{n,q} \end{bmatrix}$$

In this case, we call $M_{m,p}$ and $N_{n,q}$ as the ordinal terms of $M_{m,p} \oplus N_{n,q}$.

Example 4.2.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 | 1 & 1 & 1 \\ 0 & 1 | 1 & 1 & 1 \\ - & - | - & - & - \\ 0 & 0 | 1 & 0 & 1 \\ 0 & 0 | 0 & 1 & 1 \\ 0 & 0 | 0 & 0 & 1 \end{bmatrix}$$

Theorem 4.5. Let $M_m$ represents the poset $A$ and $N_n$ represents the poset $B$. Then the matrix $M_m \oplus N_n$ is a poset matrix and it represents the poset $A \oplus B$. 
Proof. Let $M = [a_{ij}]$, $N = [b_{ij}]$ and $M + N = T = [t_{ij}]$ with block representation $[T_{ij}], 1 \leq i, j \leq 2$. Since $T$ is upper triangular with elements 1s in the main diagonal, because $T_{21} = Z_{n,m}$, and $M$ and $N$ are poset matrices, $T$ is clearly reflexive and antisymmetric. For transitivity of $T$, let $t_{ij} = t_{jk} = 1$ for some $i \leq j \leq k$. Then we have the following cases.

1. $k \leq m$. Then $i \leq j \leq k \leq m$ implies $t_{ij}, t_{jk}, t_{ik} \in M$. Since $M$ is transitive, $t_{ik} = 1$.

2. $k > m$.

   (a) $j \leq m$. Then $i \leq j \leq m < k$ implies $t_{ik} \in O_{m,n}$ and hence $t_{ik} = 1$.

   (b) $j > m$.

      i. $i \leq m$. Then $i \leq m < j \leq k$ implies $t_{ik} \in O_{m,n}$ and hence $t_{ik} = 1$.

      ii. $m < i$. Then $m < i \leq j \leq k$ implies $t_{ij}, t_{jk}, t_{ik} \in N$ and since $N$ is transitive, $t_{ik} = 1$.

Thus $T$ is transitive and hence a poset matrix.

Let $A = \langle A; \leq_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $B = \langle B; \leq_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We show that $T$ represents $A \oplus B = \langle A \cup B; \leq_{A \cup B} \rangle$, where $A \cup B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $t_{ij} = 1$ in $T$ for some $1 \leq i, j \leq m + n$. Then either $t_{ij} \in M$ or $t_{ij} \in N$ or $t_{ij} \in Z_{m,n}$. Therefore, either $x_i \leq_A x_j$ (because $M$ represents $A$) or $x_i \leq_B x_j$ (because $N$ represents $B$) or $x_i \in A$ and $x_j \in B$. By definition, $x_i \leq_{A \cup B} x_j$ in $A \cup B$. Hence $T$ represents $A \oplus B$.

The following theorem gives a generalization of the Theorem 4.5.

**Theorem 4.6.** Let $M_i, 1 \leq i \leq n$ be the poset matrices representing the posets $P_i, 1 \leq i \leq n$ respectively. Then the matrix $T = [T_{ij}], 1 \leq i, j \leq n$, defined as

$$T_{ij} = \begin{cases} M_i & \text{if } i = j, \\ O_{m_i,m_j} & \text{if } i < j, \\ Z_{m_j,m_i} & \text{otherwise.} \end{cases}$$

where $t = \sum_{i=1}^n m_i$, is a poset matrix representing the poset $P_1 \oplus P_2 \oplus \cdots \oplus P_n$. 

Proof. The proof follows inductively by the Theorem 4.5. For instance, the blocks of the matrix $T_t = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ can be viewed as follows.

$$T_t = \begin{bmatrix}
M_{m_1} & O_{m_1,m_2} & \cdots & O_{m_1,m_n} \\
Z_{m_2,m_1} & M_{m_2} & \cdots & O_{m_2,m_n} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{m_n,m_1} & Z_{m_n,m_2} & \cdots & M_{m_n}
\end{bmatrix}$$

5 Recognition of series-parallel posets

Definition 5.1. Let $M_m = [a_{ij}]$ be a poset matrix. We say that $M_m$ has the property of block of 0s (block of 1s) of length $r$, $1 \leq r < m$ if and only if $a_{ij} = 0$ ($a_{ij} = 1$) for all $1 \leq i \leq r$ and $r + 1 \leq j \leq m$.

Example 5.1.

$$A = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

Here the poset matrices $A$ and $B$ satisfy the property of block of 0s of length 2 and the property of block of 1s of length 3 respectively. Obviously, for every $n \geq 2$, the poset matrices $I_n$ and $C_n$ satisfy the property of block of 0s and the property block of 1s respectively of lengths 1, 2, \ldots, $n-2$ and $n-1$.

Note that the poset matrices $A$ and $B$ (Example 5.1) represent the posets $C_2 + B_{1,2}$ and $(1 + C_2) \oplus I_2$ (Figure 3) respectively. On the other hand, the poset matrix $C$ (Example 5.1) satisfies neither the property of block of 0s nor the property of block of 1s for any labeling and it represents the poset $Z_4$ (Figure 3).

We observe that a block of 0s (block of 1s) of length $r$ in a poset matrix $M_m$ can be viewed as $Z_{r,m-r}$ ($O_{r,m-r}$) embedded on the top-right corner of $M_m$. Thus if a poset matrix $M_m$ satisfies the property of block of 0s (block of 1s) of length $r$ then $M_m$ must have the direct (ordinal) terms $M_r$ and $M_{m-r}$, that is, $M_m = M_r + M_{m-r}$ ($M_m = M_r \oplus M_{m-r}$). We generalize this result as follows.
Theorem 5.2. (a) A poset matrix $M_m$ satisfies the property of block of 0s if and only if $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$ for some $m_i, 1 \leq i \leq n$.

(b) A poset matrix $M_m$ satisfies the property of block of 1s if and only if $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ for some $m_i, 1 \leq i \leq n$.

Proof. (a) Let $M_m = [a_{uv}], 1 \leq u, v \leq m$ satisfies the property of block of 0s of lengths $r_1, r_2, \ldots, r_{n-2}$ and $r_{n-1}$. Let $m_0 = 0, m_1 = r_1, m_{i+1} = r_{i+1} - r_i, (1 \leq i \leq n - 2)$ and $m_n = m - r_{n-1}$. Since $a_{uv} = 0$ for all $1 \leq u \leq r_i, r_i + 1 \leq v \leq m$, for every $1 \leq i \leq n - 1$ the block $[a_{uv}], m_{i-1} + 1 \leq u \leq m_i, r_i + 1 \leq v \leq m$ can be considered as an $m_i$-by-$(m - r_i)$ matrix of entries 0s only, that is, $[a_{uv}] = Z_{m_i,m-r_i}$. Then for every $1 \leq i \leq n - 1$, we have $Z_{m_i,m_{i+j}}$, $1 \leq j \leq n - i$ as the horizontal partitions of the augmented matrix $Z_{m_i,m_{i+j}}$. Then by the construction shown in the Theorem 4.3, we have $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$.

Conversely, let $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$. There exist matrices $Z_{m_i,m_{i+j}}, 1 \leq i \leq n - 1, 1 \leq j \leq n - i$ giving the construction of the direct sum shown in the Theorem 4.3. Then for every $1 \leq i \leq n - 1$, we have the augmented matrices $Z_{m_i,c_i}$, where $c_i = \sum_{j=1}^{i-1} m_{i+j}$, having $Z_{m_i,m_{i+j}}, 1 \leq j \leq n - i$ as the horizontal partitions. Then for every $1 \leq i \leq n - 1$, the blocks $[a_{uv}], 1 \leq u \leq r_i, r_i + 1 \leq v \leq m$, where $r_i = \sum_{j=1}^{i} m_j$, can be considered as the block of 0s. This shows that $M_m$ satisfies the property of block of 0s of lengths $r_1, r_2, \ldots, r_{n-2}$ and $r_{n-1}$.

(b) Follows similarly by Theorem 4.6.

Definition 5.3. Let $M_m = [a_{ij}]$ be a poset matrix. Then we say that $M_m$ has the property of complete blocks of 1s of length $\{r_1, r_2, \ldots, r_n\}$, where $0 \leq r_1 < r_2 < \cdots < r_n < m$, if and only if for all $1 \leq i < j \leq m$
Poset matrix and recognition of series-parallel posets

Figure 4: Hasse diagram of $I_3 \oplus I_2 \oplus 1$.

$$a_{ij} = \begin{cases} 
1 & \text{if } 1 \leq i \leq r_k \text{ and } r_k + 1 \leq j \leq m \ (1 \leq k \leq n), \\
0 & \text{otherwise}.
\end{cases}$$

Example 5.2.

$$D = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Here the poset matrix $D$ satisfies the complete blocks of 1s property of length $\{3, 5\}$. Obviously, for every $n \geq 1$, the poset matrix $I_n$ satisfies the property of complete blocks of 1s of length $\{0\}$ and $C_n$ satisfies the property of complete blocks of 1s of length $\{1, 2, \ldots, n-1\}$.

Note that the matrix $D$ (Example 5.2) represents the $P$-graph $I_3 \oplus I_2 \oplus 1$ (Figure 4).

We observe that although the poset matrix $B$ (Example 5.1) satisfies the property of block of 1s of length 3, it does not satisfy the complete blocks of 1s property of length $\{3\}$. On the other hand, if a poset matrix $M_m \neq I_m$ satisfies the property of complete blocks of 1s of length $\{r_1, r_2, \ldots, r_n\}$ then $M_m$ must satisfy the property of block of 1s of lengths $r_1, r_2, \ldots, r_{n-1}$ and $r_n$. We establish this result as follows.
Theorem 5.4. A poset matrix $M_m \neq I_m$ satisfies the property of complete blocks of 1s if and only if for some $m_i, 1 \leq i \leq n$, $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$.

Proof. The proof follows by the Theorem 5.2 and the definition of the property of complete blocks of 1s.

The following theorem gives a recognition of $P$-graphs.

Theorem 5.5. Let the poset matrix $M_m$ represents the poset $P$. Then $P$ is a $P$-graph if and only if $M_m$ satisfies the property of complete blocks of 1s.

Proof. Let $P$ be a $P$-graph. Then there exists $n \leq m$ such that $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$, where for every $1 \leq i \leq n$, $P_i$ is either the singleton poset or an antichain poset. Let for every $1 \leq i \leq n$, $M_{m_i}$ represents the poset $P_i$, where $m_i = |P_i|$. Then we have $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ (Theorem 4.6). Since for every $1 \leq i \leq n$, $P_i$ is either the singleton poset or an antichain poset, $M_{m_i} = I_{m_i}$ and hence $M_m$ satisfies the property of complete blocks of 1s (Theorem 5.4).

For the converse, let $M_m$ satisfies the property of complete blocks of 1s of length $\{0\}$, that is, $M_m = I_m$. Then $M_m$ is trivially a $P$-graph. Otherwise, let $M_m \neq I_m$ satisfies the property of complete blocks of 1s. Then for some $m_i, 1 \leq i \leq n$, we have $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$ (Theorem 5.4). Let for every $1 \leq i \leq n$, $M_{m_i}$ represents the poset $P_i$. Then $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ (Theorem 4.6). Since $M_{m_i} = I_{m_i}$, for every $1 \leq i \leq n$, $P_i$ is either the singleton poset or an antichain poset. Hence, by definition, $P$ is a $P$-graph.

The following theorem gives a recognition of $P$-series.

Theorem 5.6. Let the poset matrix $M_m$ represents the poset $P$. Then $P$ is a $P$-series if and only if $M_m$ can be relabeled in such a form that either $M_m$ satisfies the property of complete blocks of 1s or $M_m$ satisfies the property of block of 0s and every direct term of $M_m$ satisfies the property of complete blocks of 1s.

Proof. Let $P$ be a $P$-series. If $P$ is a $P$-graph then $M_m$ satisfies the property of complete blocks of 1s (Theorem 5.5). If $P$ is not a $P$-graph then there exists $n \leq m$ such that $P = P_1 + P_2 + \cdots + P_n$, where for every $1 \leq i \leq n$, $P_i$ is a $P$-graph. Let for every $1 \leq i \leq n$, $M_{m_i}$ ($m_i = |P_i|$) represents the poset $P_i$. Then $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$ (Theorem 4.3) and hence $M_m$ satisfies the property of block of 0s (Theorem 5.2). Moreover,
for every $1 \leq i \leq n$, since $M_{m_i}$ represents the $P$-graph $P_i$, $M_{m_i}$ satisfies the property of complete blocks of 1s (Theorem 5.5).

For the converse, let $M_m$ can be relabeled so that $M_m$ satisfies the property of complete blocks of 1s. Then clearly $P$ is a $P$-graph (Theorem 5.5) and hence trivially a $P$-series. Otherwise, let $M_m$ can be relabeled so that $M_m$ satisfies the property of block of 0s and every direct term of $M_m$ satisfies the property of complete blocks of 1s. Then $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$ (Theorem 5.1), where for every $1 \leq i \leq n$, $M_{m_i}$ satisfies the property of complete blocks of 1s. Clearly, for every $1 \leq i \leq n$, $M_{m_i}$ represents the $P$-graph $P_i$ (Theorem 5.5) and $P = P_1 + P_2 + \cdots + P_n$ (Theorem 4.3). This shows that $P$ can be expressed as the direct sum of $P$-graphs and hence, by definition, $P$ is a $P$-series.

The following theorem gives a recognition of series-parallel posets.

**Theorem 5.7.** Let the poset matrix $M_m$ represent the poset $P \not\cong 1$. Then $P$ is series-parallel if and only if $M_m$ can be relabeled in such a form that $M_m$ satisfies either the property of block of 0s or the property of block of 1s and every term (direct or ordinal) until 1 satisfies either the property of block of 0s or the property of block of 1s.

**Proof.** Let $P \not\cong 1$ be a series-parallel poset. Then there exist at least the posets $P_1$ and $P_2$ with $|P_1| + |P_2| = |P|$ such that either $P = P_1 + P_2$ or $P = P_1 \oplus P_2$. Let $M_{m_{1i}}$ and $M_{m_{12}}$, where $m_{1i} = |P_i|, 1 \leq i \leq 2$, represent the posets $P_1$ and $P_2$ respectively. Then either $M_m = M_{m_{11}} + M_{m_{12}}$ or $M_m = M_{m_{11}} \oplus M_{m_{12}}$. There exists either the matrix $Z_{m_{11}, m_{12}}$ (as a block of 0s of length $m_{11}$) or the matrix $O_{m_{11}, m_{12}}$ (as a block of 1s of length $m_{11}$) as in the construction of direct sum and ordinal sum (Theorem 4.2 and Theorem 4.5). This shows that $M_m$ satisfies either the property of block of 0s of length $m_{11}$ or the property of block of 1s of length $m_{11}$. Since every term (direct or ordinal) of a series-parallel poset is series-parallel, if $P_i \not\cong 1, 1 \leq i \leq 2$, we show similarly that for every $1 \leq i \leq 2, M_{m_{1i}}$ satisfy either the property of block of 0s of length $m_{2i}$ or the property of block of 1s of length $m_{2i}$. Continuing the above process we show that every term (direct or ordinal) until 1 satisfies either the block of 0s property or the block of 1s property.

Conversely, let $M_m$ can be relabeled in such a form that it satisfies either the property of block 0s or the property of block 1s of length $m_1 < m$. Then there exist $M_{m_1}$ and $M_{m_2}$, where $m_2 = m - m_1$, such that either $M_m = M_{m_1} + M_{m_2}$ or $M_m = M_{m_1} \oplus M_{m_2}$. Then either $P = P_{01} + P_{02}$ or $P = P_{01} \oplus P_{02}$, where $M_{m_1}$ and $M_{m_2}$ represent the posets $P_{01}$ and $P_{02}$.
respectively (Theorem 4.2 and Theorem 4.5). Since every term (direct or ordinal) $M_{m_i}, 1 \leq i \leq 2$ until 1 satisfies either the block of 0s property or the block of 1s property, we show similarly that there exist the posets $P_{i1}$ and $P_{i2}$ $(1 \leq i \leq 2)$ such that $P_{0i} = P_{i1} + P_{i2}$ or $P_{0i} = P_{i1} \oplus P_{i2}$. Continuing the above process, we show that the poset $P$ can be expressed as the sum of singleton posets using direct sum or ordinal sum. Hence $P$ is a series-parallel poset.

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