International Journal of Mathematics and
Computer Science, **15**(2020), no. 1, 107–125

Poset matrix and recognition of series-parallel posets

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(Received July 18, 2019, Accepted October 3, 2019)

Abstract

We introduce the notion of poset matrix for representing finite partially ordered sets (posets). We study the interpretations of different forms of poset matrix. We show that every poset matrix can be relabeled to upper (equivalently lower) triangular form representing a unique poset up to isomorphism. We also show that the direct sum and ordinal sum of poset matrices represent the direct sum and ordinal sum of posets respectively. We define the properties of block of 0s, block of 1s and complete blocks of 1s on poset matrix and give an application of poset matrix in recognizing series-parallel posets.

1 Introduction

The class of series-parallel posets is one of the most important classes of posets specially due to their computationally tractability property for which they provide the basic structures of a number of applied and theoretical problems in several fields of computer science, electrical engineering and operation research [9, 10, 11, 12]. This class of posets is introduced firstly by Lawler [10] and studied further by numerous authors [1, 3, 6, 13, 20]. Kaerkes [7] gave a recognition of series-parallel posets in terms of forbidden substructure

Key words and phrases: Partially ordered set, Series-parallel poset, Poset recognition, Matrix representation.
AMS (MOS) Subject Classifications: 06A06, 06A11, 15B36.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

and showed that a poset is series-parallel if and only if it does not contain an induced subposet isomorphic to the N-shaped poset (4-element zigzag poset).

Due to many computational aspects of incidence matrices, they have classical applications in recognition of different posets and graphs [2, 5, 8, 14, 18, 19]. Rhee [14] described a doubly-stochastic matrix D_P, an incidence matrix representation of a poset $\mathbf{P}$ and showed that if $\mathbf{P}$ is a series-parallel poset except a chain then D_P is singular, that is, $|D_P| = 0$. Butler [2] described a particular solution to the problem of counting the number of posets with n elements by interpreting a partial order relation on n-element set as a non-singular idempotent Boolean relation matrix of order n. This intuition gives us the idea of defining the properties of reflexivity, antisymmetry and transitivity on a square $(0, 1)$-matrix which we call a poset matrix.

Fulkerson and Gross [5] characterized interval graphs as graphs whose dominant clique-vertex incidence matrix has the property of perfect 1s for columns. Roberts [16, 17] characterized proper interval graphs as graphs whose augmented adjacency matrix has perfect 1s property for columns. Tucker [19] characterized circular-arc graphs and proper circular-arc graphs by using the properties of perfect 0s, circular 1s and circularly compatible 1s defined on the augmented adjacency matrix. These classical results give us the idea of defining the properties of block of 0s, block of 1s and complete blocks of 1s on poset matrix.

In Section 2 of this paper, we recall some important definitions and common notations related to series-parallel posets. In Section 3, we define poset matrix and describe some interpretations regarding matrix transpose and relabeling (interchanging rows and columns simultaneously) of poset matrix. We show that every poset matrix can be relabeled to upper (or lower) triangular form with 1s in the main diagonal by a finite times of relabeling. In Section 4, we show the constructions of the block matrices for representing direct sum and ordinal sum of posets. In Section 5, we define the properties of block of 0s, block of 1s and complete blocks of 1s on poset matrix. We then use these properties of poset matrix for recognition of P-graphs, P-series and series-parallel posets.

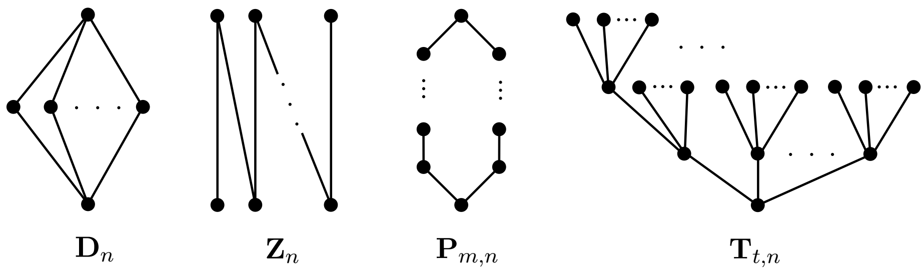


Figure 1: Hasse diagrams of $\mathbf{D}_n, \mathbf{Z}_n, \mathbf{P}_{m,n}$ and $\mathbf{T}_{t,n}$.

2 Preliminaries

A *partially ordered set* or *poset* is a structure $\mathbf{P} = \langle P, \leqslant\rangle$ consisting of the nonempty set P with the order relation $\leqslant$ on P, that is, $\leqslant$ is reflexive, antisymmetric and transitive. The set P is called the *underlying set* or *ground set* of the poset $\mathbf{P}$. For $x, y \in P$, we call y *covers* x if $x \leqslant y$ and for any $z \in P$, $x \leqslant z \leqslant y$ implies either $z = x$ or $z = y$. We write $x \prec y$ whenever y covers x or x is covered by y. If for $x, y \in P$ neither $x \leqslant y$ in P nor $y \leqslant x$ in P then we say that x and y are *incomparable* in $\mathbf{P}$ and we write $x||y$. For details on posets, readers are referred to the classical book by Davey and Priestley [4].

We use the notations $\mathbf{1}$ for the singleton poset, $\mathbf{C}_n(n \geq 1)$ for n-element chain poset, $\mathbf{I}_n(n \geq 1)$ for n-element antichain poset, $\mathbf{D}_n(n \geq 4)$ for n-element diamond poset, $\mathbf{Z}_n(n \geq 4)$ for n-element zigzag poset, $\mathbf{B}_{m,n}(m \geq 1, n \geq 1)$ for complete bipartite poset with m minimal elements and n maximal elements, $\mathbf{P}_{m,n}(m \geq 1, n \geq 1)$ for polygonal poset with maximal chains of lengths $m+2$ and $n+2$ and $\mathbf{T}_{t,n}(n > t \geq 1)$ for n-element t-ary rooted tree as poset. The Hasse diagrams of $\mathbf{D}_n$, $\mathbf{Z}_n$, $\mathbf{P}_{m,n}$ and $\mathbf{T}_{t,n}$ (height-balanced) are shown in Figure 1. Through this paper, we assume every poset is finite and nonempty.

Let $\mathbf{P} = \langle P, \leqslant_P\rangle$ and $\mathbf{Q} = \langle Q, \leqslant_Q\rangle$ be two posets. A bijective map $\phi : P \to Q$ is called an *order isomorphism* if for all $x, y \in P$, $x \leqslant_P y$ if and only if $\phi(x) \leqslant_Q \phi(y)$. We write $\mathbf{P} \cong \mathbf{Q}$ whenever $\mathbf{P}$ and $\mathbf{Q}$ are order isomorphic. For example, $\mathbf{C}_1 \cong \mathbf{I}_1 \cong \mathbf{1}$, $\mathbf{P}_{1,1} \cong \mathbf{D}_4$, $\mathbf{C}_2 \cong \mathbf{B}_{1,1} \cong \mathbf{T}_{1,2}$, $\mathbf{C}_n \cong \mathbf{T}_{1,n}(n \geq 2)$ and $\mathbf{B}_{1,n} \cong \mathbf{T}_{n,n+1}(n \geq 2)$.

Let $\mathbf{P} = \langle P, \leqslant\rangle$ be a poset. The *dual order* of $\leqslant$ on P, denoted by $\leqslant^\partial$, is defined as for all $x, y \in P$, $x \leqslant^{\partial} y$ if and only if $y \leqslant x$ in P. The *dual poset* of $\mathbf{P}$, denoted by $\mathbf{P}^{\partial}$, is defined as the poset with the dual order $\leqslant^{\partial}$ on P, that is, $\mathbf{P}^{\partial} = \langle P, \leqslant^{\partial} \rangle$. Obviously, $\mathbf{C}_n^{\partial} \cong \mathbf{C}_n$ and $\mathbf{D}_n^{\partial} \cong \mathbf{D}_n$. Also $\mathbf{B}_{m,n}^{\partial} \cong \mathbf{B}_{n,m}$.

Let $\mathbf{P} = \langle P, \leqslant_P \rangle$ and $\mathbf{Q} = \langle Q, \leqslant_Q \rangle$ be posets such that P and Q are disjoint.

1. The *direct sum* or *disjoint sum* or *free sum* of $\mathbf{P}$ and $\mathbf{Q}$, denoted by $\mathbf{P}+\mathbf{Q}$, is defined as the poset $\langle P \cup Q, \leqslant_+ \rangle$ such that for all $x, y \in P \cup Q$, $x \leqslant_+ y$ if and only if either (i) $x \leqslant_P y$ or (ii) $x \leqslant_Q y$.

2. The *ordinal sum* of $\mathbf{P}$ and $\mathbf{Q}$, denoted by $\mathbf{P} \oplus \mathbf{Q}$, is defined as the poset $\langle P \cup Q, \leqslant_{\oplus} \rangle$ such that for all $x, y \in P \cup Q$, $x \leqslant_{\oplus} y$ if and only if either (i) $x \leqslant_P y$ or (ii) $x \leqslant_Q y$ or (iii) $x \in P$ and $y \in Q$.

Here $\mathbf{P}$ and $\mathbf{Q}$ are called the *direct terms* of $\mathbf{P}+\mathbf{Q}$ and the *ordinal terms* of $\mathbf{P} \oplus \mathbf{Q}$.

A poset $\mathbf{P}$ is called a *P-graph* if it can be expressed as the ordinal sum of singleton posets or antichain posets, that is, there exist the singleton posets or antichain posets $\mathbf{P}_i, 1 \leq i \leq n$ such that $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \cdots \oplus \mathbf{P}_n$. The posets $\mathbf{1}$ and $\mathbf{I}_n$ are trivial examples of P-graphs. The posets $\mathbf{C}_n$, $\mathbf{B}_{m,n}$ and $\mathbf{D}_n$ are some common examples of P-graphs because $\mathbf{C}_n = \mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$, $\mathbf{B}_{m,n} = \mathbf{I}_m \oplus \mathbf{I}_n$ and $\mathbf{D}_n = \mathbf{1} \oplus \mathbf{I}_{n-2} \oplus \mathbf{1}$.

A poset $\mathbf{P}$ is called a *P-series* if it can be expressed as the direct sum of P-graphs, that is, there exist P-graphs $\mathbf{P}_i, 1 \leq i \leq n$ such that $\mathbf{P} = \mathbf{P}_1+\mathbf{P}_2+\cdots+\mathbf{P}_n$. Every P-graph is trivially a P-series. The poset $\mathbf{C}_m+\mathbf{C}_n$ is a P-series which is not a P-graph if either $m \geq 2$ or $n \geq 2$. However, the posets $\mathbf{Z}_n$, $\mathbf{P}_{m,n}$ ($m \geq 2$ or $n \geq 2$) and $\mathbf{T}_{t,n}$ ($n \geq t+2$) are neither P-graphs nor P-series.

A poset $\mathbf{P}$ is called *series-parallel* if it can be expressed as the sum of singleton posets using direct sum or ordinal sum. In other words, $\mathbf{P}$ is series-parallel if there exist P-graphs or P-series $\mathbf{P}_i, 1 \leq i \leq n$ such that $\mathbf{P} = \mathbf{P}_1 * \mathbf{P}_2 * \cdots * \mathbf{P}_n$ where $*$ is either direct sum or ordinal sum. Every P-graph and P-series is trivially series-parallel. Also $\mathbf{P}_{m,n}$ and $\mathbf{T}_{t,n}$ are non-trivial series-parallel posets that can be shown particularly as follows.

(i) $\mathbf{P}_{m,n} = \mathbf{1} \oplus (\mathbf{C}_m + \mathbf{C}_n) \oplus \mathbf{1}$ where $\mathbf{C}_r = \mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$ (ordinal sum of r singletons). This shows that $\mathbf{P}_{m,n}$ can be expressed as the sum of singleton posets using direct sum or ordinal sum.

(ii) $\mathbf{T}_{t,n} = r_0 \oplus (\mathbf{T}_{t,n_0} + \mathbf{T}_{t,n_0} + \cdots + \mathbf{T}_{t,n_0})$, where r_0 is the root (singleton poset) of $\mathbf{T}_{t,n}$ and $\mathbf{T}_{t,n_0}$ is either a leaf (singleton poset) or a subtree with number of elements $n_0 = \frac{n-1}{t}$, particularly if the tree $\mathbf{T}_{t,n}$ is full. This continues recursively expressing $\mathbf{T}_{t,n}$ as the sum of singleton posets using direct sum or ordinal sum.

It is easy to check that every term (direct or ordinal) of a series-parallel poset is also series-parallel. However, the poset $\mathbf{Z}_n$ is not a series-parallel poset.

3 Poset matrix and some interpretations

From now on we use the notations $M_{m,n}$ for an m-by-n matrix and M_m for a square matrix of order m. In particular, we use I_n for the identity matrix of order n and C_n for the matrix $[c_{ij}], 1 \leq i, j \leq n$ defined as $c_{ij} = 1$ for all $i \leq j$ and $c_{ij} = 0$ otherwise. Obviously $C_1 = I_1 = 1$.

Definition 3.1. *A square* $(0,1)$*-matrix* $M = [a_{ij}], 1 \leq i, j \leq n$ *is called a* poset matrix *if the following conditions hold.*

1. $a_{ii} = 1$ *for all* $1 \leq i \leq n$ *i.e.* M *is* reflexive*;*
2. $a_{ij} = 1$ *and* $a_{ji} = 1$ *imply* $i = j$ *i.e.* M *is* antisymmetric*;*
3. $a_{ij} = 1$ *and* $a_{jk} = 1$ *imply* $a_{ik} = 1$ *i.e.* M *is* transitive.

Example 3.1.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Here L and M are poset matrices and N is not a poset matrix. All the matrices L, M and N are both reflexive and antisymmetric because they have all entries 1s in the main diagonal and unequal entries, except both are 0s, in opposite positions with respect to the main diagonal. Also L and M are trivially transitive. On the other hand, $N = [a_{ij}], 1 \leq i, j \leq 3$ is not transitive because $a_{12} = 1$ and $a_{23} = 1$ but $a_{13} = 0$.

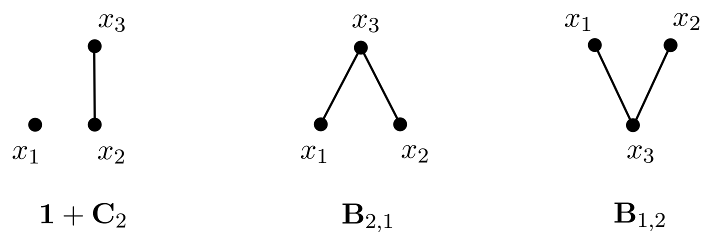


Figure 2: Hasse diagrams of $\mathbf{1}+\mathbf{C}_2, \mathbf{B}_{2,1}$ and $\mathbf{B}_{1,2}$.

An upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is clearly reflexive and antisymmetric. Therefore, an upper (or lower) triangular $(0,1)$-matrix with entries 1s in the main diagonal is a poset matrix if it is transitive. For example, both I_n and C_n are poset matrices because these are upper triangular and clearly transitive.

To each poset matrix we can associate a poset. Let $M_m = [a_{ij}]$ be a poset matrix and $P = \{x_1, x_2, \ldots, x_m\}$ where x_i corresponds the i-th row (or column) of M_m. We define a relation $\leqslant$ on P such that for all $1 \leq i, j \leq m$, $x_i \leqslant x_j$ if and only if $a_{ij} = 1$. Since M_m is a poset matrix, clearly $\leqslant$ is an order relation on P. Thus $\mathbf{P} = \langle P, \leqslant \rangle$ is a poset. We say that the poset matrix M_m *represents* the poset $\mathbf{P}$ and vice versa. The poset matrices L and M, as in the Example 3.1, represent the posets $\mathbf{1}+\mathbf{C}_2$ and $\mathbf{B}_{2,1}$ (Figure 2) respectively. Obviously, the poset matrices I_n and C_n represent the posets $\mathbf{I}_n$, where $x_i || x_j$ for all $1 \leq i, j \leq n$, and $\mathbf{C}_n$, where $x_1 \prec x_2 \prec \cdots \prec x_n$, with the ground set $\{x_1, x_2, \ldots, x_n\}$ respectively. In particular, both I_1 and C_1 represent the singleton poset $\mathbf{1}$.

We observe that the matrix transpose M^t of the poset matrix M (Example 3.1) represents the poset $\mathbf{B}_{1,2}$ which is dual to the poset $\mathbf{B}_{2,1}$ (Figure 2). We establish this result in general as follows.

Theorem 3.2. *The matrix transpose M^t of a poset matrix M is a poset matrix and it represents the poset dual to the poset represented by M.*

Proof. Let $M = [a_{ij}], 1 \leq i, j \leq n$. Then $M^t = [a_{ji}], 1 \leq i, j \leq n$. Since $a_{ii} = 1$ for all $1 \leq i \leq n$ in both M and M^t, clearly M^t is reflexive. Also $a_{ij} = 1$ and $a_{ji} = 1$ imply $i = j$ in both M and M^t. Thus M^t is antisymmetric. Let $a_{ij} = 1$ and $a_{jk} = 1$ in M^t. Then $a_{ji} = 1$ and $a_{kj} = 1$ in M. Since M is transitive, $a_{ki} = 1$ in M. This implies $a_{ik} = 1$ in M^t. Thus

M^t is transitive. Therefore, M^t is a poset matrix.

To show that M^t represents the poset dual to the poset represented by M, let M represents the poset $\mathbf{P} = \langle X, \leqslant_P \rangle$ and M^t represents the poset $\mathbf{Q} = \langle X, \leqslant_Q \rangle$ with the same underlying set $X = \{x_1, x_2, \ldots, x_n\}$. Let $x_i \leqslant_P x_j$ for some i and j. Then $a_{ij} = 1$ in M implies $a_{ji} = 1$ in M^t. Thus $x_j \leqslant_Q x_i$. This shows that the order $\leqslant_Q$ is dual to $\leqslant_P$ on X, that is, $\mathbf{Q} \cong \mathbf{P}^\partial$. ∎

Definition 3.3. *Let M_m be a poset matrix. Then for some $1 \leq i, j \leq m$, interchanges of i-th and j-th rows along with interchanges of i-th and j-th columns in M_m is called* (i,j)-relabeling *of M_m.*

Example 3.2.

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\textit{(1,2)-relabeling}} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\textit{(2,3)-relabeling}} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\textit{(1,2)-relabeling}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M'$$

We observe that the matrix M' obtained by some relabeling of the poset matrix M equals the poset matrix C_4. Moreover, M and M' represent the same poset $\mathbf{C}_4$ with the ground set $\{x_1, x_2, x_3, x_4\}$ such that $x_3 \prec x_2 \prec x_1 \prec x_4$ on M and $x_1 \prec x_2 \prec x_3 \prec x_4$ on M'. Thus relabeling of a poset matrix M does not make any change to the order relation on M, it provides just a renaming of the elements in the poset represented by M. We establish these facts as follows.

Theorem 3.4. *Any relabeling of a poset matrix is a poset matrix and it represents the same poset up to isomorphism.*

Proof. Let $M_m = [a_{ij}]$ be a poset matrix and $N_m = [b_{ij}]$ be the matrix obtained by (k, l)-relabeling of M_m. Then we have the following equalities.

1. $b_{kk} = a_{ll}$ and $b_{ll} = a_{kk}$
2. $b_{kl} = a_{lk}$ and $b_{lk} = a_{kl}$
3. $b_{ik} = a_{il}$ and $b_{il} = a_{ik}$ for all $1 \leq i \leq m, i \neq k, i \neq l$

4. $b_{kj} = a_{lj}$ and $b_{lj} = a_{kj}$ for all $1 \leq j \leq m, j \neq k, j \neq l$
5. $b_{ij} = a_{ij}$ for all $1 \leq i, j \leq m, i \neq k, i \neq l, j \neq k, j \neq l$

Since M_m is a poset matrix, the above equalities show that N_m is a poset matrix.

To show that $M_m = [a_{ij}]$ and $N_m = [b_{ij}]$ represent the same poset up to isomorphism, let they represent $\mathbf{P} = \langle P, \leqslant_P \rangle$ and $\mathbf{Q} = \langle Q, \leqslant_Q \rangle$ respectively where $P = \{x_1, x_2, \ldots, x_m\}$ and $Q = \{y_1, y_2, \ldots, y_m\}$. Define $\phi : P \to Q$ as follows.

$$\phi(x_t) = \begin{cases} y_l \text{ if } t = k, \\ y_k \text{ if } t = l, \\ y_t \text{ otherwise.} \end{cases}$$

Since $|P| = m = |Q|$, clearly ϕ is bijective. Let $x_i \leqslant_P x_j$. Then we have the following cases.

1. $i = k$.
   (a) $j = k$. Then $1 = a_{ij} = a_{kk} = b_{ll}$ (equality 1) implies $y_l \leqslant_Q y_l$.
   (b) $j = l$. Then $1 = a_{ij} = a_{kl} = b_{lk}$ (equality 2) implies $y_l \leqslant_Q y_k$.
   (c) $j \neq k, j \neq l$. Then $1 = a_{ij} = a_{kj} = b_{lj}$ (equality 4) implies $y_l \leqslant_Q y_j$.
2. $i = l$.
   (a) $j = k$. Then $1 = a_{ij} = a_{lk} = b_{kl}$ (equality 2) implies $y_k \leqslant_Q y_l$.
   (b) $j = l$. Then $1 = a_{ij} = a_{ll} = b_{kk}$ (equality 1) implies $y_k \leqslant_Q y_k$.
   (c) $j \neq k, j \neq l$. Then $1 = a_{ij} = a_{lj} = b_{kj}$ (equality 4) implies $y_k \leqslant_Q y_j$.
3. $i \neq k, i \neq l$.
   (a) $j = k$. Then $1 = a_{ij} = a_{ik} = b_{il}$ (equality 3) implies $y_i \leqslant_Q y_l$.
   (b) $j = l$. Then $1 = a_{ij} = a_{il} = b_{ik}$ (equality 3) implies $y_i \leqslant_Q y_k$.
   (c) $j \neq k, j \neq l$. Then $1 = a_{ij} = b_{ij}$ (equality 5) implies $y_i \leqslant_Q y_j$.

Thus $\phi(x_i) \leqslant_Q \phi(x_j)$ and $\phi : \mathbf{P} \to \mathbf{Q}$ is an order isomorphism. ∎

We observe that the poset matrix M' (Example 3.2) obtained by some relabeling of the non-triangular poset matrix M is in upper triangular form. Below we establish this result in general.

Theorem 3.5. *Every poset matrix can be relabeled to an upper (or lower) triangular matrix with 1s in the main diagonal by a finite times of relabeling.*

Proof. Let $M = [a_{ij}], 1 \leq i,j \leq n$ be a poset matrix. Suppose $a_{ij} = 1$ for some $i > j$ such that for all $1 \leq k \leq j-1$, $a_{ik} = 0$ and the partition $[a_{uv}], 1 \leq u,v \leq i-1$ of M is upper triangular. Then $a_{(i-1)i} = 0$. Otherwise, since $a_{ij} = 1$, we have the following cases.

1. $i = j+1$. Then $a_{ji} = a_{(i-1)i}$ contradicts that M is antisymmetric.
2. $i > j+1$. Then transitivity of M implies $a_{(i-1)j} = 1$ which contradicts that P is upper triangular.

Then by (i,j)-relabeling of M, we have $a_{(i-1)j} = 1$ and $a_{ik} = 0$ for all $1 \leq k \leq i-1$. Similarly, the matrix M' obtained by $(k, k-1)$-relabeling of M for all $i-1 \geq k \geq j+1$ with $i > j+1$ is a matrix having partition $[a_{uv}], 1 \leq u,v \leq i$ of M' in upper triangular form. Since n is finite, continuing the same process for all $1 \leq j < i \leq n$ with $a_{ij} = 1$, we have M' in upper triangular form. By Theorem 3.4, M' is a poset matrix and hence it has 1s in the main diagonal. We show similarly the lower triangular case. ∎

Corollary 3.6. *Let M be any square $(0,1)$-matrix. Then M is a poset matrix if and only if M is transitive and upper (or lower) triangular with 1s in the main diagonal.*

4 Direct sum and ordinal sum of poset matrices

From now on by a poset matrix we mean a poset matrix in upper triangular form. Also, we use the notations $Z_{m,n}$ for an m-by-n matrix having entries 0s only and $O_{m,n}$ for an m-by-n matrix having entries 1s only.

Definition 4.1. *The* direct sum *of the matrices $M_{m,p}$ and $N_{n,q}$, denoted by $M_{m,p} + N_{n,q}$, is defined as an $(m+n)$-by-$(p+q)$ block matrix such that*

$$M_{m,p} + N_{n,q} = \begin{bmatrix} M_{m,p} & | & Z_{m,q} \\ -- & . & -- \\ Z_{n,p} & | & N_{n,q} \end{bmatrix}$$

In this case, we call $M_{m,p}$ and $N_{n,q}$ as the direct terms *of $M_{m,p} + N_{n,q}$.*

Example 4.1.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & | & 0 & 0 & 0 \\ 0 & 1 & | & 0 & 0 & 0 \\ - & - & . & - & - & - \\ 0 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Theorem 4.2. *Let M_m represents the poset $\mathbf{A}$ and N_n represents the poset $\mathbf{B}$. Then the matrix $M_m + N_n$ is a poset matrix and it represents the poset $\mathbf{A} + \mathbf{B}$.*

Proof. Let $M_m = [a_{ij}]$, $N_n = [b_{ij}]$ and $M_m + N_n = S_{m+n} = [s_{ij}]$ with block representation $[S_{ij}], 1 \leq i, j \leq 2$. Since S_{m+n} is upper triangular with elements 1s in the main diagonal, because $S_{21} = Z_{n,m}$, and M_m and N_n are poset matrices, S_{m+n} is clearly reflexive and antisymmetric. For transitivity of S_{m+n}, let $s_{ij} = s_{jk} = 1$ for some $i \leq j \leq k$. Then we have the following cases.

1. $k \leq m$. Then $i \leq j \leq k \leq m$ implies $s_{ij}, s_{jk}, s_{ik} \in M_m$. Since M_m is transitive, s_{ik}=1.
2. $k > m$.
   (a) $j \leq m$. Then $j \leq m < k$ implies $s_{jk} \in Z_{m,n}$ which contradicts that $s_{jk} = 1$.
   (b) $j > m$.
      i. $i \leq m$. Then $i \leq m < j$ implies $s_{ij} \in Z_{m,n}$ which contradicts that $s_{ij} = 1$.
      ii. $i > m$. Then $m < i \leq j \leq k$ implies $s_{ij}, s_{jk}, s_{ik} \in N_n$. Since N_n is transitive, s_{ik}=1.

Thus S_{m+n} is transitive and hence a poset matrix.

Let $\mathbf{A} = \langle A; \leqslant_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We show that S_{m+n} represents $\mathbf{A} + \mathbf{B} = \langle A \cup B; \leqslant_+ \rangle$, where $A \cup B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $s_{ij} = 1$ in S_{m+n} for some $1 \leq i, j \leq m + n$. Then either $s_{ij} \in M_m$ or $s_{ij} \in N_n$. Since M_m and N_n represent $\mathbf{A}$ and $\mathbf{B}$ respectively, either $x_i \leqslant_A x_j$ or $x_i \leqslant_B x_j$. By definition, $x_i \leqslant_+ x_j$ in $A \cup B$. Hence S_{m+n} represents the poset $\mathbf{A} + \mathbf{B}$. ∎

The following theorem gives a generalization of the Theorem 4.2.

Theorem 4.3. *Let $M_{m_i}, 1 \leq i \leq n$ be the poset matrices representing the posets $\mathbf{P}_i, 1 \leq i \leq n$ respectively. Then the matrix $S_s = [S_{ij}], 1 \leq i, j \leq n$, defined as*

$$S_{ij} = \begin{cases} M_{m_i} \text{ if } i = j, \\ Z_{m_i,m_j} \text{ if } i < j, \\ Z_{m_j,m_i} \text{ otherwise.} \end{cases}$$

where $s = \sum_{i=1}^{n} m_i$, is a poset matrix representing the poset $\mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n$.

Proof. The proof follows inductively by the Theorem 4.2. For instance, the blocks of the matrix $S_s = M_1 + M_2 + \cdots + M_n$ can be viewed as follows.

$$S_s = \begin{bmatrix} M_{m_1} & Z_{m_1,m_2} & \cdots & Z_{m_1,m_n} \\ Z_{m_2,m_1} & M_{m_2} & \cdots & Z_{m_2,m_n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{m_n,m_1} & Z_{m_n,m_2} & \cdots & M_{m_n} \end{bmatrix}$$

∎

Definition 4.4. *The* ordinal sum *of the matrices $M_{m,p}$ and $N_{n,q}$, denoted by $M_{m,p} \oplus N_{n,q}$, is defined as an $(m+n)$-by-$(p+q)$ block matrix such that*

$$M_{m,p} \oplus N_{n,q} = \begin{bmatrix} M_{m,p} & | & O_{m,q} \\ -- & . & -- \\ Z_{n,p} & | & N_{n,q} \end{bmatrix}$$

In this case, we call $M_{m,p}$ and $N_{n,q}$ as the ordinal terms *of $M_{m,p} \oplus N_{n,q}$.*

Example 4.2.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & | & 1 & 1 & 1 \\ 0 & 1 & | & 1 & 1 & 1 \\ - & - & . & - & - & - \\ 0 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

Theorem 4.5. *Let M_m represents the poset $\mathbf{A}$ and N_n represents the poset $\mathbf{B}$. Then the matrix $M_m \oplus N_n$ is a poset matrix and it represents the poset $\mathbf{A} \oplus \mathbf{B}$.*

Proof. Let $M_m = [a_{ij}]$, $N_n = [b_{ij}]$ and $M_m + N_n = T_{m+n} = [t_{ij}]$ with block representation $[T_{ij}], 1 \leq i, j \leq 2$. Since T_{m+n} is upper triangular with elements 1s in the main diagonal, because $T_{21} = Z_{n,m}$, and M_m and N_n are poset matrices, T_{m+n} is clearly reflexive and antisymmetric. For transitivity of T_{m+n}, let $t_{ij} = t_{jk} = 1$ for some $i \leq j \leq k$. Then we have the following cases.

1. $k \leq m$. Then $i \leq j \leq k \leq m$ implies $t_{ij}, t_{jk}, t_{ik} \in M_m$. Since M_m is transitive, t_{ik}=1.

2. $k > m$.

   (a) $j \leq m$. Then $i \leq j \leq m < k$ implies $t_{ik} \in O_{m,n}$ and hence t_{ik}=1.

   (b) $j > m$.

      i. $i \leq m$. Then $i \leq m < j \leq k$ implies $t_{ik} \in O_{m,n}$ and hence t_{ik}=1.

      ii. $m < i$. Then $m < i \leq j \leq k$ implies $t_{ij}, t_{jk}, t_{ik} \in N_n$ and since N_n is transitive, t_{ik}=1.

Thus T_{m+n} is transitive and hence a poset matrix.

Let $\mathbf{A} = \langle A; \leqslant_A \rangle$, where $A = \{x_1, x_2, \ldots, x_m\}$ and $\mathbf{B} = \langle B; \leqslant_B \rangle$, where $B = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. We show that T_{m+n} represents $\mathbf{A} \oplus \mathbf{B} = \langle A \cup B; \leqslant_\oplus \rangle$, where $A \cup B = \{x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $t_{ij} = 1$ in T_{m+n} for some $1 \leq i, j \leq m+n$. Then either $t_{ij} \in M_m$ or $t_{ij} \in N_n$ or $t_{ij} \in Z_{m,n}$. Therefore, either $x_i \leqslant_A x_j$ (because M_m represents $\mathbf{A}$) or $x_i \leqslant_B x_j$ (because N_n represents $\mathbf{B}$) or $x_i \in A$ and $x_j \in B$. By definition, $x_i \leqslant_\oplus x_j$ in $A \cup B$. Hence T_{m+n} represents $\mathbf{A} \oplus \mathbf{B}$. ∎

The following theorem gives a generalization of the Theorem 4.5.

Theorem 4.6. *Let $M_{m_i}, 1 \leq i \leq n$ be the poset matrices representing the posets $\mathbf{P}_i, 1 \leq i \leq n$ respectively. Then the matrix $T_t = [T_{ij}], 1 \leq i, j \leq n$, defined as*

$$T_{ij} = \begin{cases} M_{m_i} \ \textit{if } i = j, \\ O_{m_i,m_j} \ \textit{if } i < j, \\ Z_{m_j,m_i} \ \textit{otherwise.} \end{cases}$$

where $t = \sum_{i=1}^{n} m_i$, is a poset matrix representing the poset $\mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \cdots \oplus \mathbf{P}_n$.

Proof. The proof follows inductively by the Theorem 4.5. For instance, the blocks of the matrix $T_t = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ can be viewed as follows.

$$T_t = \begin{bmatrix} M_{m_1} & O_{m_1,m_2} & \cdots & O_{m_1,m_n} \\ Z_{m_2,m_1} & M_{m_2} & \cdots & O_{m_2,m_n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{m_n,m_1} & Z_{m_n,m_2} & \cdots & M_{m_n} \end{bmatrix}$$

∎

5 Recognition of series-parallel posets

Definition 5.1. *Let $M_m = [a_{ij}]$ be a poset matrix. We say that M_m has the property of* block of 0s *(*block of 1s*) of length r, $1 \leq r < m$ if and only if $a_{ij} = 0$ ($a_{ij} = 1$) for all $1 \leq i \leq r$ and $r+1 \leq j \leq m$.*

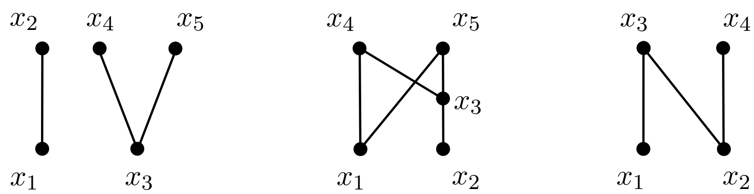
Example 5.1.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here the poset matrices A and B satisfy the property of block of 0s of length 2 and the property of block of 1s of length 3 respectively. Obviously, for every $n \geq 2$, the poset matrices I_n and C_n satisfy the property of block of 0s and the property block of 1s respectively of lengths $1, 2, \ldots, n-2$ and $n-1$.

Note that the poset matrices A and B (Example 5.1) represent the posets $\mathbf{C}_2 + \mathbf{B}_{1,2}$ and $(\mathbf{1} + \mathbf{C}_2) \oplus \mathbf{I}_2$ (Figure 3) respectively. On the other hand, the poset matrix C (Example 5.1) satisfies neither the property of block of 0s nor the property of block of 1s for any labeling and it represents the poset $\mathbf{Z}_4$ (Figure 3).

We observe that a block of 0s (block of 1s) of length r in a poset matrix M_m can be viewed as $Z_{r,m-r}$ ($O_{r,m-r}$) embedded on the top-right corner of M_m. Thus if a poset matrix M_m satisfies the property of block of 0s (block of 1s) of length r then M_m must have the direct (ordinal) terms M_r and M_{m-r}, that is, $M_m = M_r + M_{m-r}$ ($M_m = M_r \oplus M_{m-r}$). We generalize this result as follows.



$\mathbf{C}_2 + \mathbf{B}_{1,2}$ $\qquad$ $(\mathbf{1} + \mathbf{C}_2) \oplus \mathbf{I}_2$ $\qquad$ $\mathbf{Z}_4$

Figure 3: Hasse diagrams of $\mathbf{C}_2 + \mathbf{B}_{1,2}, (\mathbf{1} + \mathbf{C}_2) \oplus \mathbf{I}_2$ and $\mathbf{Z}_4$.

Theorem 5.2. *(a) A poset matrix M_m satisfies the property of block of 0s if and only if $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$ for some $m_i, 1 \leq i \leq n$.*
(b) A poset matrix M_m satisfies the property of block of 1s if and only if $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ for some $m_i, 1 \leq i \leq n$.

Proof. (a) Let $M_m = [a_{uv}], 1 \leq u, v \leq m$ satisfies the property of block of 0s of lengths $r_1, r_2, \ldots, r_{n-2}$ and r_{n-1}. Let $m_0 = 0$, $m_1 = r_1$, $m_{i+1} = r_{i+1} - r_i$ $(1 \leq i \leq n-2)$ and $m_n = m - r_{n-1}$. Since $a_{uv} = 0$ for all $1 \leq u \leq r_i, r_i + 1 \leq v \leq m$, for every $1 \leq i \leq n-1$ the block $[a_{uv}], m_{i-1} + 1 \leq u \leq m_i, r_i + 1 \leq v \leq m$ can be considered as an m_i-by-$(m - r_i)$ matrix of entries 0s only, that is, $[a_{uv}] = Z_{m_i, m-r_i}$. Then for every $1 \leq i \leq n-1$, we have $Z_{m_i, m_{i+j}}$, $1 \leq j \leq n-i$ as the horizontal partitions of the augmented matrix $Z_{m_i, m-r_i}$. Then by the construction shown in the Theorem 4.3, we have $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$.

Conversely, let $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$. There exist matrices $Z_{m_i, m_{i+j}}$, $1 \leq i \leq n-1, 1 \leq j \leq n-i$ giving the construction of the direct sum shown in the Theorem 4.3. Then for every $1 \leq i \leq n-1$, we have the augmented matrices Z_{m_i, c_i}, where $c_i = \sum_{j=1}^{n-i} m_{i+j}$, having $Z_{m_i, m_{i+j}}$, $1 \leq j \leq n-i$ as the horizontal partitions. Then for every $1 \leq i \leq n-1$, the blocks $[a_{uv}], 1 \leq u \leq r_i, r_i + 1 \leq v \leq m$, where $r_i = \sum_{j=1}^{i} m_j$, can be considered as the block of 0s. This shows that M_m satisfies the property of block of 0s of lengths $r_1, r_2, \ldots, r_{n-2}$ and r_{n-1}.

(b) Follows similarly by Theorem 4.6. ∎

Definition 5.3. *Let $M_m = [a_{ij}]$ be a poset matrix. Then we say that M_m has the property of* complete blocks of 1s *of length $\{r_1, r_2, \ldots, r_n\}$, where $0 \leq r_1 < r_2 < \cdots < r_n < m$, if and only if for all $1 \leq i < j \leq m$*

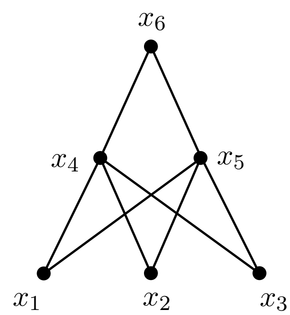


Figure 4: Hasse diagram of $\mathbf{I}_3 \oplus \mathbf{I}_2 \oplus \mathbf{1}$.

$$a_{ij} = \begin{cases} 1 \textit{ if } 1 \leq i \leq r_k \textit{ and } r_k + 1 \leq j \leq m \ (1 \leq k \leq n), \\ 0 \textit{ otherwise.} \end{cases}$$

Example 5.2.

$$D = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here the poset matrix D satisfies the complete blocks of 1s property of length $\{3, 5\}$. Obviously, for every $n \geq 1$, the poset matrix I_n satisfies the property of complete blocks of 1s of length $\{0\}$ and C_n satisfies the property of complete blocks of 1s of length $\{1, 2, \ldots, n-1\}$.

Note that the matrix D (Example 5.2) represents the P-graph $\mathbf{I}_3 \oplus \mathbf{I}_2 \oplus \mathbf{1}$ (Figure 4).

We observe that although the poset matrix B (Example 5.1) satisfies the property of block of 1s of length 3, it does not satisfy the complete blocks of 1s property of length $\{3\}$. On the other hand, if a poset matrix $M_m \neq I_m$ satisfies the property of complete blocks of 1s of length $\{r_1, r_2, \ldots, r_n\}$ then M_m must satisfy the property of block of 1s of lengths $r_1, r_2, \ldots, r_{n-1}$ and r_n. We establish this result as follows.

Theorem 5.4. *A poset matrix $M_m \neq I_m$ satisfies the property of complete blocks of 1s if and only if for some $m_i, 1 \leq i \leq n$, $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$.*

Proof. The proof follows by the Theorem 5.2 and the definition of the property of complete blocks of 1s. ∎

The following theorem gives a recognition of P-graphs.

Theorem 5.5. *Let the poset matrix M_m represents the poset $\mathbf{P}$. Then $\mathbf{P}$ is a P-graph if and only if M_m satisfies the property of complete blocks of 1s.*

Proof. Let $\mathbf{P}$ be a P-graph. Then there exists $n \leq m$ such that $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \cdots \oplus \mathbf{P}_n$, where for every $1 \leq i \leq n$, $\mathbf{P}_i$ is either the singleton poset or an antichain poset. Let for every $1 \leq i \leq n$, M_{m_i} represents the poset $\mathbf{P}_i$, where $m_i = |P_i|$. Then we have $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ (Theorem 4.6). Since for every $1 \leq i \leq n$, $\mathbf{P}_i$ is either the singleton poset or an antichain poset, $M_{m_i} = I_{m_i}$ and hence M_m satisfies the property of complete blocks of 1s (Theorem 5.4).

For the converse, let M_m satisfies the property of complete blocks of 1s of length $\{0\}$, that is, $M_m = I_m$. Then M_m is trivially a P-graph. Otherwise, let $M_m \neq I_m$ satisfies the property of complete blocks of 1s. Then for some $m_i, 1 \leq i \leq n$, we have $M_m = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_n}$ such that $M_{m_i} = I_{m_i}$ (Theorem 5.4). Let for every $1 \leq i \leq n$, M_{m_i} represents the poset $\mathbf{P}_i$. Then $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \cdots \oplus \mathbf{P}_n$ (Theorem 4.6). Since $M_{m_i} = I_{m_i}$, for every $1 \leq i \leq n$, $\mathbf{P}_i$ is either the singleton poset or an antichain poset. Hence, by definition, $\mathbf{P}$ is a P-graph. ∎

The following theorem gives a recognition of P-series.

Theorem 5.6. *Let the poset matrix M_m represents the poset $\mathbf{P}$. Then $\mathbf{P}$ is a P-series if and only if M_m can be relabeled in such a form that either M_m satisfies the property of complete blocks of 1s or M_m satisfies the property of block of 0s and every direct term of M_m satisfies the property of complete blocks of 1s.*

Proof. Let $\mathbf{P}$ be a P-series. If $\mathbf{P}$ is a P-graph then M_m satisfies the property of complete blocks of 1s (Theorem 5.5). If $\mathbf{P}$ is not a P-graph then there exists $n \leq m$ such that $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n$, where for every $1 \leq i \leq n$, $\mathbf{P}_i$ is a P-graph. Let for every $1 \leq i \leq n$, M_{m_i} $(m_i = |P_i|)$ represents the poset $\mathbf{P}_i$. Then $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$ (Theorem 4.3) and hence M_m satisfies the property of block of 0s (Theorem 5.2). Moreover, for every $1 \leq i \leq n$, since M_{m_i} represents the P-graph $\mathbf{P}_i$, M_{m_i} satisfies the property of complete blocks of 1s (Theorem 5.5).

For the converse, let M_m can be relabeled so that M_m satisfies the property of complete blocks of 1s. Then clearly $\mathbf{P}$ is a P-graph (Theorem 5.5) and hence trivially a P-series. Otherwise, let M_m can be relabeled so that M_m satisfies the property of block of 0s and every direct term of M_m satisfies the property of complete blocks of 1s. Then $M_m = M_{m_1} + M_{m_2} + \cdots + M_{m_n}$ (Theorem 5.1), where for every $1 \leq i \leq n$, M_{m_i} satisfies the property of complete blocks of 1s. Clearly, for every $1 \leq i \leq n$, M_{m_i} represents the P-graph $\mathbf{P}_i$ (Theorem 5.5) and $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_n$ (Theorem 4.3). This shows that $\mathbf{P}$ can be expressed as the direct sum of P-graphs and hence, by definition, $\mathbf{P}$ is a P-series. ∎

The following theorem gives a recognition of series-parallel posets.

Theorem 5.7. *Let the poset matrix M_m represent the poset $\mathbf{P} \ncong \mathbf{1}$. Then $\mathbf{P}$ is series-parallel if and only if M_m can be relabeled in such a form that M_m satisfies either the property of block of 0s or the property of block of 1s and every term (direct or ordinal) until 1 satisfies either the property of block of 0s or the property of block of 1s.*

Proof. Let $\mathbf{P} \ncong \mathbf{1}$ be a series-parallel poset. Then there exist at least the posets $\mathbf{P}_1$ and $\mathbf{P}_2$ with $|P_1| + |P_2| = |P|$ such that either $\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2$ or $\mathbf{P} = \mathbf{P_1} \oplus \mathbf{P}_2$. Let $M_{m_{11}}$ and $M_{m_{12}}$, where $m_{1i} = |P_i|, 1 \leq i \leq 2$, represent the posets $\mathbf{P}_1$ and $\mathbf{P}_2$ respectively. Then either $M_m = M_{m_{11}} + M_{m_{12}}$ or $M_m = M_{m_{11}} \oplus M_{m_{12}}$. There exists either the matrix $Z_{m_{11},m_{12}}$ (as a block of 0s of length m_{11}) or the matrix $O_{m_{11},m_{12}}$ (as a block of 1s of length m_{11}) as in the construction of direct sum and ordinal sum (Theorem 4.2 and Theorem 4.5). This shows that M_m satisfies either the property of block of 0s of length m_{11} or the property of block of 1s of length m_{11}. Since every term (direct or ordinal) of a series-parallel poset is series-parallel, if $P_i \ncong \mathbf{1}, 1 \leq i \leq 2$, we show similarly that for every $1 \leq i \leq 2$, $M_{m_{1i}}$ satisfy either the property of block of 0s of length m_{2i} or the property of block of 1s of length m_{2i}. Continuing the above process we show that every term (direct or ordinal) until 1 satisfies either the block of 0s property or the block of 1s property.

Conversely, let M_m can be relabeled in such a form that it satisfies either the property of block 0s or the property of block of 1s of length $m_1 < m$. Then there exist M_{m_1} and M_{m_2}, where $m_2 = m - m_1$, such that either $M_m = M_{m_1} + M_{m_2}$ or $M_m = M_{m_1} \oplus M_{m_2}$. Then either $\mathbf{P} = \mathbf{P}_{01} + \mathbf{P}_{02}$ or $\mathbf{P} = \mathbf{P}_{01} \oplus \mathbf{P}_{02}$, where M_{m_1} and M_{m_2} represent the posets $\mathbf{P}_{01}$ and $\mathbf{P}_{02}$

respectively (Theorem 4.2 and Theorem 4.5). Since every term (direct or ordinal) $M_{m_i}, 1 \leq i \leq 2$ until 1 satisfies either the block of 0s property or the block of 1s property, we show similarly that there exist the posets $\mathbf{P}_{i1}$ and $\mathbf{P}_{i2}$ $(1 \leq i \leq 2)$ such that $\mathbf{P}_{0i} = \mathbf{P}_{i1} + \mathbf{P}_{i2}$ or $\mathbf{P}_{0i} = \mathbf{P}_{i1} \oplus \mathbf{P}_{i2}$. Continuing the above process, we show that the poset $\mathbf{P}$ can be expressed as the sum of singleton posets using direct sum or ordinal sum. Hence $\mathbf{P}$ is a series-parallel poset. ∎

Acknowledgements

All our appreciation to Dr. Shamsun Naher Begum for various discussions and many helpful remarks that contributed to the presentation of the paper. We would like to thank the authority of Ministry of Science and Technology, Bangladesh for the financial support through the NST Fellowship 2017-18 [Grant no. 39.00.0000.012.02.009.17-474].

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