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Geometric Path Decomposition of graphs

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Abstract

Let $G = (V,\, E)$ be a simple connected graph with p vertices and q edges. A decomposition $(G_a, G_{ar}, G_{ar^2}, G_{ar^3}, \ldots, G_{ar^{n-1}})$ of G is said to be a Geometric Decomposition(GD) if each $G_{ar^{i-1}}$ is connected and $|E(G_{ar^{i-1}})| = ar^{i-1}$ for every i = 1, 2, ..., n and $a, r \in N$. A GD in which each $G_{ar^{i-1}}$ is a path is said to be a Geometric Path Decomposition(GPD). In this paper, we investigate Geometric Path Decompositions of some graphs.

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1 Introduction

All graphs considered here are finite, undirected simple connected (without loops or multiple edges). Let $G = (V, E)$ be a simple connected graph with p vertices and q edges. If $G_1, G_2, \ldots, G_n$ are connected edge disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then
$(G_1, G_2, \ldots, G_n)$ is said to be a decomposition of G.
A decomposition $(G_1, G_2, \ldots, G_n)$ of G is said to be an arithmetic decomposition if each G_i is connected and $|E(G_i)| = a + (i-1)d$ for every $i = 1, 2, \ldots, n$ and $a, d \in N$. In [1] Alavi et. al. introduced Ascending subgraph decomposition (ASD). In [3] Gnanadhas and Paulraj Joseph introduced Continuous Monotonic Decomposition of graphs. In [2] E. Ebin Raja Merly and D. Subitha introduced Geometric Decomposition of spider tree.
A semi total block - cut vertex graph $T_{bc}(G)$ of a graph G is a graph with vertex set $B(G) \cup C(G)$ and any two vertices in $T_{bc}(G)$ are adjacent if and only if the corresponding cut vertices are adjacent or the corresponding members are incident. The number of edges in $T_{bc}(G)$ is $\frac{1}{2}[(m + n - 1) + \sum_{h \in B(G)} [d_c(h)]^2]$, where m is the number of blocks, n is the number of cut vertices and $d_c(h)$ is the number of cut vertices incident on h, $h \in B(G)$.
The Total block - cut vertex graph $T_{BC}(G)$ of a graph G is a graph with vertex set $B(G) \cup C(G)$ and any two vertices in $T_{BC}(G)$ are adjacent if and only if the corresponding members are adjacent or incident. The number of edges in $T_{BC}(G)$ is $\frac{1}{2}[\sum_{h \in B(G)} [d_c(h)]^2] + \sum_{c \in C(G))} [d_{vb}(c)]^2]$ where $d_{vb}(c)$ is the number of vertices or blocks incident on c , $c \in C(G)$. The detailed study has been done by surekha [7].
A semi total block graph $T_b(G)$ of a graph G is a graph with vertex set $B(G) \cup V(G)$ and two vertices in $T_b(G)$ are adjacent if and only if the corresponding vertices are adjacent in G. The number of edges in $T_b(G)$ is $q(G) + \sum_{i=1}^{p} b_i$, where b_i's are blocks to which each point v_i belongs to G. The detailed study has been done by V. R. Kulli [5]. The point block graph $P_b(G)$ of a graph G is the graph whose point set is the set of points and blocks of G in which two points are adjacent if the corresponding blocks are adjacent or the corresponding members are incident. The number of edges in $P_b(G)$ is $\frac{1}{2}\sum_{i=1}^{p} b_i(b_i + 1)$, where b_i is the number of blocks to which point v_i belongs to G. A detailed study has been done by V. R. Kulli [4].
In this paper, we investigate Geometric Path Decompositions of some graphs.

2 Geometric path decomposition of graphs

Definition 2.1. Let $G = (V, E)$ be a simple connected graph with p vertices and q edges. A decomposition $(G_a, G_{ar}, G_{ar^2}, G_{ar^3}, \ldots, G_{ar^{n-1}})$ of G is said to be a Geometric Decomposition(GD) if each $G_{ar^{i-1}}$ is connected and $|E(G_{ar^{i-1}})| = ar^{i-1}$ for every $i = 1, 2, ..., n$ and $a, r \in N$ or (a, r, n) - Decomposable. Clearly $q = \frac{a(r^n-1)}{r-1}$.

Definition 2.2. A decomposition $(G_a, G_{ar}, G_{ar^2}, G_{ar^3}, ..., G_{ar^{n-1}})$ of G is said to be a Geometric Path Decomposition(GPD) or (a, r, n) - Path Decomposable if i) G admits GD. ii) Each $G_{ar^{i-1}}$ is a path for every $i = 1, 2, \ldots, n$ and $a, r \in N$

Theorem 2.3. *Let e_1 and e_2 be pendant edges of $T_{bc}(P_k)$. Then there is an integer m such that*
(i) $T_{bc}(P_k)$ is $(2, 2, 2m + 1)$ - Path Decomposable iff $k = \frac{2^{2m+2}+2}{3}$.
(ii) $T_{bc}(P_k)$ is $(2, 3, m + 1)$ - Path Decomposable iff $k = 3^m + 1$.
(iii) $T_{bc}(P_k) - \{e_1, e_2\}$ is $(a, 2, 2m)$ - Path Decomposable iff $k = \frac{a(2^{2m}-1)+6}{3}$.

Proof. Assume that P_k is a path of length k. Then P_k has k blocks and $k - 1$ cut vertices. Let B and C be the blocks and cut vertices of P_k such that $B = \{b_1, b_2, \ldots, b_k\}$ and $C = \{c_1, c_2, \ldots, c_{k-1}\}$. Then $B(P_k) \cup C(P_k)$ is a vertex set of $T_{bc}(P_k)$.

Let R_1, R_2 be the maximum length in between b_1 and b_k, c_1 and c_{k-1} respectively and also let R_3 be the minimum length in between c_1 and c_{k-1}.
Now, $q[T_{bc}(P_k)] = 3k - 4 \ldots\ldots (1)$
Therefore, $q[T_{bc}(P_k)]$ - $\{e_1, e_2\} = 3k - 6 \ldots\ldots (2)$
(i) Suppose $T_{bc}(P_k)$ is $(2, 2, 2m + 1)$ - Path Decomposable. Then $q[T_{bc}(P_k)] = 2(2^{2m+1} - 1)$; i.e., $k = \frac{2^{2m+2}+2}{3}$
Conversely, assume that the path P_k with length k, $k = \frac{2^{2m+2}+2}{3}$. By using (1), $q[T_{bc}(P_k)] = 2(2^{2m+1} - 1)$. Therefore, B and C are the blocks and cut vertices of $P_{\frac{2^{2m+2}+2}{3}}$ such that $B = \{b_1, b_2, \ldots, b_{\frac{2^{2m+2}+2}{3}}\}$ and $C = \{c_1, c_2, \ldots, c_{\frac{2^{2m+2}-1}{3}}\}$. We know that $B(P_{\frac{2^{2m+2}+2}{3}}) \cup C(P_{\frac{2^{2m+2}+2}{3}})$ is a vertex set of $T_{bc}(P_{\frac{2^{2m+2}+2}{3}})$. Now, R_1 is the maximum length in between b_1 and $b_{\frac{2^{2m+2}+2}{3}}$. Then the length of $R_1 = \frac{2^{2m+3}-2}{3}$. Thus R_1 can be decomposed into P_2, P_8, $\ldots$, $P_{2^{2m+1}}$. Also, R_3 is the minimum length in between c_1 and $c_{\frac{2^{2m+2}-1}{3}}$. Then the length of $R_3 = \frac{2^{2m+2}-2^2}{3}$. Thus R_3 can be decomposed into P_4, P_{16}, $\ldots$, $P_{2^{2m}}$. Therefore, $T_{bc}(P_k)$ is decomposed into P_2, P_4, $\ldots$, $P_{2^{2m}}$, $P_{2^{2m+1}}$.

Hence, $T_{bc}(P_k)$ is $(2, 2, 2m + 1)$ - Path Decomposable.
ii) Suppose $T_{bc}(P_k)$ is $(2, 3, m + 1)$ - Path Decomposable.
Then $q[T_{bc}(P_k)] = 3^{m+1} - 1$ Therefore, $k = 3^m + 1$.
Conversely, assume that $P_k$is a path with length k, $k = 3^m + 1$. By using (1), $q[T_{bc}(P_k)] = 3^{m+1} - 1$. Therefore, B and C are the blocks and cut vertices of P_{3^m+1} such that $B = \{b_1, b_2, \ldots, b_{3^m+1}\}$ and $C = \{c_1, c_2, \ldots, c_{3^m}\}$. We know that $B(P_{3^m+1}) \cup C(P_{3^m+1})$ is a vertex set of $T_{bc}(P_{3^m+1})$. Now, R_1 is the maximum length between b_1 and b_{3^m+1}. Then the length of $R_1 = 2 \times 3^m$. Thus R_1 can be decomposed into $P_{2\times 3^m}$. Also, R_3 is the minimum length in between c_1 and c_{3^m}. Then the length of $R_3 = 3^m - 1$. Thus R_3 can be decomposed into P_2, P_6, P_{18}, $\ldots$, $P_{2\times 3^{m-1}}$. Therefore, $T_{bc}(P_k)$ is decomposed into P_2, $P_{2\times 3}$, $\ldots$, $P_{2\times 3^{m-1}}$, $P_{2\times 3^m}$. Hence, $T_{bc}(P_k)$ is $(2, 3, m + 1)$ - Path Decomposable.

iii) Suppose $T_{bc}(P_k) - \{e_1, e_2\}$ is $(a, 2, 2m)$-Path Decomposable. Then $q[T_{bc}(P_k)]$ - $\{e_1, e_2\} = a(2^{2m} - 1)$. Therefore, $k = \frac{a(2^{2m}-1)+6}{3}$.
Conversely, assume that P_k is a path with length $k, k = \frac{a(2^{2m}-1)+6}{3}$. By using (2), $q[T_{bc}(P_k)] - \{e_1, e_2\} = a(2^{2m} - 1)$. Therefore, B and C are the blocks and cut vertices of $P_{\frac{a(2^{2m}-1)+6}{3}}$ such that $B = \{b_1, b_2, \ldots, b_{\frac{a(2^{2m}-1)+6}{3}}\}$ and $C = \{c_1, c_2, \ldots, c_{\frac{a(2^{2m}-1)+3}{3}}\}$. We know that $B(P_{\frac{a(2^{2m}-1)+6}{3}}) \cup C(P_{\frac{a(2^{2m}-1)+6}{3}})$ is a vertex set of $T_{bc}(P_{\frac{a(2^{2m}-1)+6}{3}}$. Now, R_2 is the maximum length in between c_1 and $c_{\frac{a(2^{2m}-1)+3}{3}}$. Then the length of $R_2 = \frac{2a(2^{2m}-1)}{3}$. Thus R_2 can be decomposed into $P_{2a}, P_{8a}, P_{32a}, \ldots, P_{a2^{2m-1}}$. Also, R_3 is the minimum length in between c_1 and $c_{\frac{a(2^{2m}-1)+3}{3}}$. Then the length of $R_3 = \frac{a(2^{2m}-1)}{3}$. Thus R_3 can be decomposed into $P_a, P_{4a}, P_{16a}, \ldots, P_{a2^{2m-2}}$. Therefore, $q[T_{bc}(P_k)] - \{e_1, e_2\}$ is decomposed into $P_a, P_{2a}, P_{4a}, \ldots, P_{a2^{2m-2}}, P_{a2^{2m-1}}$. Hence, $q[T_{bc}(P_k)] - \{e_1, e_2\}$ is $(a, 2, 2m)$-Path Decomposable. □

Theorem 2.4. *Let e_1 and e_2 be pendant edges of $P_b(P_k)$. Then there is an integer m such that*
(i) $P_b(P_k)$ is (2, 2, 2m + 1) - Path Decomposable iff $k = \frac{2^{2m+2}-1}{3}$.
(ii) $P_b(P_k)$ is $(2, 3, m + 1)$ - Path Decomposable iff $k = 3^m$.
(iii) $P_b(P_k)$ - $\{e_1, e_2\}$ is $(a, 2, 2m)$ - Path Decomposable iff $k = \frac{2^{2m}+2}{3}$.

Proof. The proof is similar to that of theorem (2.3) □

Theorem 2.5. *Let P_k be a path of length k. Then there is an integer a and m such that $T_b(P_k)$ is $(a, 2, 2m)$ - Path Decomposable iff $k = \frac{a(2^{2m}-1)}{3}$.*

Proof. Assume that P_k be the path of length k. Therefore, P_k has k blocks and $k+1$ vertices. Let B and V be the blocks and vertices of P_k such that $B = \{ b_1, b_2, \ldots, b_k \}$ and $V = \{ v_1, v_2, \ldots, v_{k+1}\}$. Therefore, $B(P_k) \cup V(P_k)$ is a vertex set of $T_b(P_k)$. Now, $q[T_b(P_k)] = 3k \ldots\ldots (3)$
Suppose $T_b(P_k)$ is $(a, 2, 2m)$-Path Decomposable. Then $q[T_b(P_k)] = a(2^{2m}-1)$. Therefore, $k = \frac{a(2^{2m}-1)}{3}$.
Conversely, assume that P_k is a path with length k, $k = \frac{a(2^{2m}-1)}{3}$. By using (3), $q[T_b(P_k)] = a(2^m - 1)$. Therefore, B and V are the blocks and vertices of $P_{\frac{a(2^{2m}-1)}{3}}$ such that $B = \{b_1, b_2, \ldots, b_{\frac{a(2^{2m}-1)}{3}}\}$ and $V = \{v_1, v_2, \ldots, v_{\frac{a(2^{2m}-1)+3}{3}}\}$. We know that $B(P_{\frac{a(2^{2m}-1)}{3}}) \cup V(P_{\frac{a(2^{2m}-1)}{3}})$ is a vertex set of $T_b(P_{\frac{a(2^{2m}-1)}{3}})$. Now, R_1 is the maximum length in between v_1 and $v_{\frac{a(2^{2m}-1)+3}{3}}$. Then the length of $R_1 = \frac{2a(2^{2m}-1)}{3}$. Thus R_1 can be decomposed into $P_{2a}, P_{2^3a}, P_{2^5a} \ldots, P_{2^{2m-1}a}$. Also, R_2 is the minimum length between v_1 and $v_{\frac{a(2^{2m}-1)+3}{3}}$. Then the length of $R_2 = \frac{a(2^{2m}-1)}{3}$. Thus R_2 can be decomposed into $P_a, P_{2^2a}, P_{2^4a} \ldots, P_{2^{2m-2}a}$. Therefore, $T_b(P_k)$ decomposed into $P_a, P_{2a}, P_{2^2a}, P_{2^4a}, \ldots, P_{2^{2m-1}a}$. Hence, $T_b(P_k)$ is $(a, 2, 2m)$ - Path Decomposable.

□

Theorem 2.6. *Let P_k be the path of length k. Then there is an integer m such that $T_{BC}(P_k)$ is $(1, 2, m+2)$ Path Decomposable iff $k = 2^m + 1$.*

Proof. Assume that P_k is the path of length k. Then P_k has k blocks and $k-1$ cut vertices. Let B and C be the blocks and cut vertices of P_k such that $B = \{b_1, b_2, \ldots, b_k\}$ and $C = \{c_1, c_2, \ldots, c_{k-1}\}$. Therefore,
$B(P_k) \cup C(P_k)$ is a vertex set of $T_{BC}(P_k)$. Now,
$q[T_{BC}(P_k)] = 4k - 5 \ldots\ldots$ (4)
Suppose $T_{BC}(P_k)$ is $(1, 2, m+2)$-Path Decomposable.
Then $q[T_{BC}(P_k)] = 2^{m+2} - 1$. Therefore, $k = 2^m + 1$.
Conversely, assume that the path P_k with length k, $k = 2^m + 1$. By using (4), $q[T_{BC}(P_k)] = 2^{m+2} - 1$. Therefore, B and C are the blocks and cut vertices of P_{2^m+1} such that $B = \{b_1, b_2, \ldots, b_{2^m+1}\}$ and $C = \{c_1, c_2, \ldots, c_{2^m}\}$. We know that $B(P_{2^m+1}) \cup C(P_{2^m+1})$ is a vertex set of $T_{BC}(P_{2^m+1})$. Now, R_1 is the maximum length in between b_1 and b_{2^m+1}. Then the length of
$R_1 = 2^{m+1}$. Thus R_1 can be decomposed into $P_{2^{m+1}}$. Also, R_2 is the minimum length in between b_1 and b_{2^m+1}. Then the length of $R_2 = 2^m$. Thus R_2 can be decomposed into P_{2^m}. Also, let R_3 be the minimum length in between c_1 and c_{2^m}. Then the length of $R_3 = 2^m - 1$. Thus R_3 can be de-

composed into $P_1, P_2, P_4, \ldots, P_{2^{m-1}}$. Therefore, $T_{BC}(P_k)$ is decomposed into $P_1, P_2, P_4, \ldots, P_{2^{m-1}}, P_{2^m}, P_{2^{m+1}}$. Hence, $T_{BC}(P_k)$ is $(1, 2, m+2)$ - Path Decomposable. □

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