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Localization Functor, Homological Functor and Derivation Functor in the Category A-Alg

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Abstract

The purpose of this paper is to study the relation between the functor localization $S^{-1}()$ and the homological functor when it acts on the differential algebra $\Omega_{B/A}$ in the non-commuative case. Given a A-algebra B and S a central multiplicatively closed subset of A, we prove that the derivation functor $Der_A(S^{-1}B,\text{-})$ is a representable monofunctor of representation the pair $(\Omega_{S^{-1}B/A}, d_{S^{-1}B/A})$. In addition, we show the following isomorphism of left $S^{-1}B\text{-}algebra$

$$S^{-1}B \otimes_B \widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}).$$

1 Introduction

In this paper, unless otherwise stated, all rings are assumed to be unitary, associative, and non-commutative. In our paper [19] we defined the functor $\widehat{Ext}^n_B(\text{-}, \mathscr{B})$ so that its action on an object of the category of algebras $A\text{-}Alg$ gives an object of $A\text{-}Alg$. In this paper we study the relationship between the functor localization $S^{-1}()$ and the homological functor $\widehat{Ext}^n_B(\text{-}, \mathscr{B})$ when it acts on the differential algebra $\Omega_{B/A}$ in the non-commuative case. This paper is divided into three parts. In the first part

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entitled "preliminary results" we recall some basic results. In the second section we study the relation between the localization functor $S^{-1}()$ and the homological functor $\widehat{Ext}_B^n(\text{-}, \mathscr{B})$ and we show that the derivation functor $Der_A(S^{-1}B, \text{-}) : S^{-1}B\text{-}Alg \to Ens$ is a representable monofunctor. In the third section we study the relation between the functor localization $S^{-1}()$ and the homological functor $\widehat{Ext}_B^n(\text{-}, \mathscr{B})$ when it acts on the differential algebra $\Omega_{B/A}$ and we show the following isomorphism of left B-algebras, $S^{-1}B \otimes_B \widehat{Ext}_B^n(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}_{S^{-1}B}^n(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B})$ (see theorem 4.1).

2 Preliminaries Results

Theorem 2.1. *Let A be a ring and S a multiplicatively closed subset of A satisfying the left Ore conditions. The binary relation defined in $S \times M$ by*

$$(s, m)\mathcal{R}(s', m') \Longleftrightarrow \exists x, y \in S : \begin{cases} xm = ym' \\ xs = ys' \end{cases}$$

is an equivalence relation.

Proof. See [16]. □

Theorem 2.2. *Let A be a ring and S a multiplicatively closed subset of A satisfying the left condition of Ore. Then $S^{-1}A$ under the following operations is a ring:*

- $\frac{a}{t} + \frac{b}{s} = \frac{xa+yb}{ys}$ *where $x, y \in S : xt = ys$*
- $\frac{a}{t} \times \frac{b}{s} = \frac{zb}{wt}$ *where $(w, z) \in S \times A : wa = zs$.*

Proof. see [16]. □

Definition 2.3. *Let A be a ring, B an A-algebra of structural homomorphism φ and M a left B-module. A derivation of B in M is a map $d : B \longrightarrow M$ satisfying the following conditions:*

1. $d(x + y) = d(x) + d(y)$, $\forall\ x, y \in B$
2. $d(xy) = d(x)y + xd(y)$, $\forall\ x, y \in B$
3. $d(\varphi(A)) = 0$; *i.e.*, $d(\varphi(a)) = 0$, $\forall\ a \in A$.

We also say that d is an A-derivation of B in M.

Notation: We denote by $Der_A(B, M)$ the set of A-derivations of B in M.

Proposition 2.4. *Let A be a ring, B a left A-algebra and $\mathscr{B}$ a left B-algebra, then $Der_A(Z(B), \mathscr{B})$ has a structure of left B-module under the following operation:*

$$(b \cdot d)(b') = bd(b'), \forall b, b' \in Z(B), d \in Der_A(Z(B), \mathscr{B}).$$

Lemma 2.5. *Let A be a ring, B an A-algebra, $\mathscr{B}$ and $\mathscr{B}'$ two left B-algebras, f a map B-linear from $\mathscr{B}$ into $\mathscr{B}'$. If $d \in Der_A(B, \mathscr{B})$, then $f \circ d \in Der_A(B, \mathscr{B}')$.*

Proof. See [10]. □

Lemma 2.6. *Let B be a left A-algebra, $\mathscr{B}$ a left B-algebra and $d \in Der_A(B, \mathscr{B})$. Then*

1. *$Kerd$ is a subalgebra of B.*

2. *d is A-linear if and only if $A.1_B \subseteq Kerd$.*

Proposition 2.7. *Let B be an A-algebra. Then $B^{(B)}$ has a structure of left B-module under the following operation:*

$$\begin{aligned} \bullet : B \times B^{(B)} &\longrightarrow B^{(B)} \\ (b, \varphi) &\longmapsto b \bullet \varphi : B \longrightarrow B \\ &\qquad\qquad b' \longmapsto (b \bullet \varphi) = b\varphi(b') \end{aligned}$$

Proof. Easy. □

Proposition 2.8. *Let B be an A-algebra. Then the left B-module $B^{(B)}$ is free.*

Proof. Let $b \in B$. Denote by $e_b : B \longrightarrow B$ the characteristic function at the point b. Now

$$b' \longmapsto \begin{cases} 1 \text{ if } b = b' \\ 0 \text{ if } b \neq b' \end{cases}$$

It is not difficult to verify that $B^{(B)} = Vect(e_b)_{b \in B}$ and that $\{e_b\}_{b \in B}$ is a linearly independent set. □

Remark 2.9. *An element φ of $B^{(B)}$ must take the form $\varphi = \sum_{b \in B} \alpha_b e_b$.*

Theorem 2.10. *Let B be a left A-algebra. Then $B^{(B)}$ has a structure of left B-algebra under the following operations:*

$$(\sum_{b\in B}\alpha_b e_b)(\sum_{b'\in B}\mu_{b'}e'_b) = \sum \alpha_b\mu_{b'}e_{bb'}$$
$$[b\cdot\sum_{b'\in B}\alpha_{b'}e_{b'}](b'') = b\sum_{b'\in B}\alpha_{b'}e_{b'}(b'')$$

Proof. From before, $B^{(B)}$ is a left B-module. Te proof is complete because it is easy to verify that the two laws are compatible. □

Definition 2.11. *Let B be an A-algebra. We denote by $\Omega_{B/A}$ the quotient B-algebra of $B^{(B)}$ and the ideal I generated by the elements of one of the forms:*

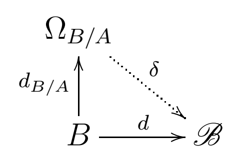
$$e_{xx'} - xe_{x'} - x'e_x, (x, x' \in B)$$

$$e_{\varphi(a)}, (a \in A).$$

$\Omega_{B/A}$ is called the differential algebra of the A-algebra B.

Proposition 2.12. *Let B be a finitely (resp. finitely presented) A-algebra. Then $\Omega_{B/A}$ is a finitely (resp. finitely presented).*

Theorem 2.13. *Let B be an A-algebra. Then there exists an B-algebra $\Omega_{B/A}$ and an A-derivation $d_{B/A}$ of B in $\Omega_{B/A}$ such that for any B-algebra $\mathscr{B}$ and for any derivation d of B in $\mathscr{B}$, there exists an unique map B-linear δ from $\Omega_{B/A}$ into $\mathscr{B}$ such that the following diagram is commutative*



Proposition 2.14. *Let B be an A-algebra. If B is finitely generated, then so is $\Omega_{B/A}$.*

Corollary 2.15. *Let B be an A-algebra and $\mathscr{B}$. Then there are isomorphisms of left B-modules,*

$$Der_A(B,\mathscr{B}) \cong Hom_B(\Omega_{B/A},\mathscr{B}).$$

Proof. Consider the following correspondence:

$$\psi : Hom_B(\Omega_{B/A}, \mathscr{B}) \longrightarrow Der_A(B, \mathscr{B})$$
$$f \longmapsto f \circ d_{B/A}$$

$*$ ψ makes sense because according to lemma 2.5, $f \circ d_{B/A} \in Der_A(B, \mathscr{B})$ is well defined.
$*$ Let $f, g \in Hom_B(\Omega_{B/A}, \mathscr{B})$, $b \in B$.
We have

$$\psi(f+g) = (f+g) \circ d_{B/A} = f \circ d_{B/A} + g \circ d_{B/A} = \psi(f) + \psi(g).$$

In addition

$$\psi(bf) = (bf) \circ d_{B/A} = b(f \circ d_{B/A}) = b\psi(f).$$

So ψ is a morphism of left B-modules.
$*$ Let $d \in Der_A(B, \mathscr{B})$. By theorem 2.13 $\exists!$ $f \in Hom_B(\Omega_{B/A}, \mathscr{B})$ such that $f \circ d_{B/A} = d \Rightarrow \psi(f) = d \Rightarrow \psi$ is bijective.
Therefore

$$Der_A(B, \mathscr{B}) \cong Hom_B(\Omega_{B/A}, \mathscr{B}).$$

□

3 Localization Functor and Derivation Functor

Theorem 3.1. *Let B be a left A-algebra and $\mathscr{B}$ a symmetric $(B$-$B)$-bialgebra and S a central multiplicatively closed subset of B. If $d \in Der_A(B, \mathscr{B})$, then the map $\tilde{d}$ defined by*

$$\tilde{d}(\frac{x}{s}) = \frac{sd(x) - xd(s)}{s^2}$$

is an A-derivation of $S^{-1}B$ in $S^{-1}\mathscr{B}$.

Proof. $*$ It is easy to check that $\tilde{d}$ is defined.
Let $x, y \in B$ and $s, t \in S$ such that $\frac{x}{s} = \frac{y}{t}$.
We have $\frac{x}{s} = \frac{y}{t} \Rightarrow \exists\ s_1, s_2$ such that $\begin{cases} s_1 x = s_2 y & (a) \\ s_1 s = s_2 t & (b) \end{cases}$
From (b) we have $s_1 s = s_2 t \Rightarrow s_1 sxt = s_2 xt^2 \Rightarrow (s_1 x)st = s_2 xt^2 \Rightarrow (s_2 y)st = s_2 xt^2 \Rightarrow yst = xt^2$ because s_2 is invertible.
So we have

$$yst = xt^2. \tag{3.1}$$

From (b) we also have $s_1s = s_2t \Rightarrow s_1xs = s_2xt$.
So we have

$$s_2ys = s_2xt. \tag{3.2}$$

Likewise from (b) we have

$$(s_1t)s^2 = (s_2s)t^2. \tag{3.3}$$

From (a) we have $d(s_1x) = d(s_2y) \Rightarrow d(s_1)x + s_1d(x) = d(s_2)y + s_2d(y) \Rightarrow s_1d(x) - s_2d(y) = d(s_2)y - d(s_1)x$
So
$ts[s_1d(x)-s_2d(y)] = ts(d(s_2)y)-ts(d(s_1)x) \Rightarrow ts[s_1d(x)-s_2d(y)] = d(s_2)(tsy)-d(s_1)(tsx)$, because the $(B\text{-}B)$-bialgebra $\mathscr{B}$ is symmetric.

So, from equation (3.1), we have $ts[s_1d(x)-s_2d(y)] = d(s_2)(xt^2)-d(s_1)(xts)$.
So we have

$$ts[s_1d(x) - s_2d(y)] = xt[td(s_2) - sd(s_1)]. \tag{3.4}$$

From (b) we have $d(s_1s) = d(s_2t) \Rightarrow d(s_1)s + s_1d(s) = d(s_2)t + s_2d(t) \Rightarrow s_1d(s) - s_2d(t) = d(s_2)t - d(s_1)s \Rightarrow xt[s_1d(s) - s_2d(t)] = xt[td(s_2) - sd(s_1)] = tss_1d(x) - tss_2d(y)$.
So we have $tss_1d(x) - xts_1d(s) = tss_2d(y) - s_2xtd(t)$.
So we have

$$s_1t[sd(x) - xd(s)] = s_2s[td(y) - yd(t)]. \tag{3.5}$$

Put

$$X = s_1t \text{ and } Y = s_2s.$$

By (3.3) and (3.5) we have

$$\begin{cases} X(sd(x) - xd(s)) = Y(td(y) - yd(t)) \\ Xs^2 = Yt^2 \end{cases}$$

Therefore

$$\tfrac{sd(x)-xd(s)}{s^2} = \tfrac{td(y)-yd(t)}{t^2} \Longrightarrow \tilde{d}(\tfrac{x}{s}) = \tilde{d}(\tfrac{y}{t}).$$

So

$$\tilde{d} \text{ is well defined.}$$

$*$ Let $a \in A$, we have $\tilde{d}(a.1_{S^{-1}B}) = \tilde{d}(\frac{a}{1}) = \frac{1d(a)-ad(1)}{1^2} = 0 \Rightarrow A.1_{S^{-1}B} \subseteq Ker\tilde{d}$.

So $\tilde{d}$ is A-linear.

$*$ Let $x, y \in \mathscr{B}$ and $s, t \in S$. We have

$$\begin{aligned} d(\tfrac{x}{s})\tfrac{y}{t} + \tfrac{x}{s}d(\tfrac{y}{t}) &= \tfrac{sd(x)-xd(s)}{s^2}\tfrac{y}{t} + \tfrac{x}{s}\tfrac{td(y)-yd(t)}{t^2} = \tfrac{(sd(x)-xd(s))y}{s^2t} + \tfrac{x(td(y)-yd(t))}{st^2} \\ &= \tfrac{t(sd(x)-xd(s))y}{(st)^2} + \tfrac{sx(td(y)-yd(t))}{(st)^2}, \text{ (because } S \subseteq Z(B)) \\ &= \tfrac{[st(d(x)y+xd(y))]-[xy(d(s)t-sd(t))]}{(st)^2} = \tfrac{std(xy)-(xy)d(st)}{(st)^2} \\ &= \tilde{d}(\tfrac{xy}{st}) = \tilde{d}(\tfrac{x}{s}\tfrac{y}{t}). \end{aligned}$$

So

$$\tilde{d} \text{ is a derivation of } S^{-1}B \text{ in } S^{-1}\mathscr{B}.$$

□

Proposition 3.2. *Let B be a left A-algebra, $\mathscr{B}$ a left B-algebra and S a central multiplicatively closed subset of B. Then there are isomorphisms of $Z(B)$-modules*

$$Der_A(Z(B), \mathscr{B}) \cong Der_A(S^{-1}Z(B), S^{-1}\mathscr{B}).$$

Proof. Consider the map $\delta : Der_A(Z(B), \mathscr{B}) \longrightarrow Der_A(S^{-1}Z(B), S^{-1}\mathscr{B})$

$$d \longmapsto \tilde{d}.$$

$*$ Let $d_1, d_2 \in Der_A(Z(B), \mathscr{B})$, we have

$$\begin{aligned} \delta(d_1+d_2)(\tfrac{b}{s}) &= (\widetilde{d_1 + d_2})(\tfrac{b}{s}) = \tfrac{s(d_1+d_2)(b)-b(d_1+d_2)(s)}{s^2} = \tfrac{[sd_1(b)-bd_1(s)]+[sd_2(b)-bd_2(s)]}{s^2} \\ &= \tfrac{sd_1(b)-bd_1(s)}{s^2} + \tfrac{sd_2(b)-bd_2(s)}{s^2} = \tilde{d}_1(\tfrac{b}{s}) + \tilde{d}_2(\tfrac{b}{s}) = (\delta(d_1) + \delta(d_2)(\tfrac{b}{s}). \end{aligned}$$

So we have

$$\delta(d_1 + d_2) = \delta(d_1) + \delta(d_2).$$

$*$ Let $d \in Der_A(Z(B), \mathscr{B}), b \in Z(B)$, we have

$$\delta(b.d)(\tfrac{b'}{s}) = \widetilde{(b.d)}(\tfrac{b'}{s}) = \tfrac{s((b.d)(b'))-b'((b.d)(s))}{s^2} = \tfrac{b(sd(b')-b'd(s))}{s^2} = b\tilde{d}(\tfrac{b'}{s}) = (b.\tilde{d}).$$

So we have

$$\delta(b.d) = b.\delta(d).$$

Therefore δ is a morphism of left $Z(B)$-modules.

$*$ Consider now the map: $\psi : Der_A(S^{-1}Z(B), S^{-1}\mathscr{B}) \longrightarrow Der_A(Z(B), \mathscr{B})$

$$d \longmapsto d \circ i_S^{Z(B)}.$$

Similarly, ψ is a morphism of left $Z(B)$-modules.

$*$ We have

$$\delta \circ \psi(d)(b) = \delta(d \circ i_S^{Z(B)})(b) = \delta(d(\tfrac{b}{1})) = \tilde{d}(\tfrac{b}{1}) = \tfrac{1d(b)-bd(1)}{1^2} = d(b), \forall\, b \in Z(B).$$

So we have

$$\delta \circ \psi = 1_{Der_A(S^{-1}Z(B), S^{-1}\mathscr{B})}.$$

So the morphisms of left $Z(B)$-modules δ and ψ are inverses of each other and so

$$Der_A(Z(B), \mathscr{B}) \cong Der_A(S^{-1}Z(B), S^{-1}\mathscr{B})$$

. □

Theorem 3.3. *Let B be a A-algebra and S a central multiplicatively closed subset of B. Then*

$$Der_A(S^{-1}Z(B), \text{-}) : S^{-1}B\text{-}Alg \longrightarrow S^{-1}Z(B)\text{-}Mod$$

(which associates the left $S^{-1}Z(B)$-module $Der_A(S^{-1}Z(B), S^{-1}\mathscr{B})$ to the left B-algebra $\mathscr{B}$, and the morphism

$$Der_A(S^{-1}Z(B), S^{-1}f) : Der_A(S^{-1}Z(B), S^{-1}\mathscr{B}) \longrightarrow Der_A(S^{-1}Z(B), S^{-1}\mathscr{B}')$$
$$\tilde{d} \longmapsto S^{-1}f \circ \tilde{d}$$

to the morphism of B-algebras $f : \mathscr{B} \longrightarrow \mathscr{B}'$)
is a covariant monofunctor.

Proof. $*$ Let $f : \mathscr{B} \to \mathscr{B}'$, $g : \mathscr{B}' \to \mathscr{B}''$ and $d \in Der_A(Z(B), \mathscr{B})$.
We have,
$Der_A(S^{-1}Z(B), S^{-1}(g \circ f))(\tilde{d}) = S^{-1}(g \circ f) \circ \tilde{d}$, where $\tilde{d}$ is the extension of d to $S^{-1}A$,
$= (S^{-1}(g) \circ S^{-1}(f)) \circ \tilde{d}$, because the functor $S^{-1}()$ is covariant,
$= S^{-1}(g)\circ(S^{-1}(f)\circ\tilde{d}) = S^{-1}(g)\circ[Der_A(S^{-1}Z(B), S^{-1}(f))(\tilde{d})]$
$= Der_A(S^{-1}Z(B), S^{-1}(g))[Der_A(S^{-1}Z(B), S^{-1}(f))(\tilde{d})]$
$= [Der_A(S^{-1}Z(B), S^{-1}(g))\circ Der_A(S^{-1}Z(B), S^{-1}(f))](\tilde{d})$.

$*$ We have $Der_A(S^{-1}Z(B), 1_{S^{-1}\mathscr{B}})(\tilde{d}) = 1_{S^{-1}\mathscr{B}}\circ\tilde{d} = \tilde{d} = 1_{Der_A(S^{-1}Z(B), S^{-1}\mathscr{B})}(\tilde{d})$.
So $Der_A(S^{-1}Z(B), 1_{S^{-1}\mathscr{B}}) = 1_{Der_A(S^{-1}Z(B), S^{-1}\mathscr{B})}$.
So $Der_A(S^{-1}Z(B), \text{-})$ is a covariant functor.

$*$ Let $f : \mathscr{B} \longrightarrow \mathscr{B}'$ be a monomorphism of B-algebras.
Let $d, d' \in Der_A(Z(B), \mathscr{B})$ such that $Der_A(S^{-1}Z(B), S^{-1}f)(\tilde{d}) = Der_A(S^{-1}Z(B), S^{-1}f)(\tilde{d}')$.
Then

$$Der_A(S^{-1}Z(B), S^{-1}f)(\tilde{d})(\tfrac{b}{s}) = Der_A(S^{-1}Z(B), S^{-1}f)(\tilde{d}')(\tfrac{b}{s}), \forall\ b \in Z(B), s \in S.$$

So we have $S^{-1}f \circ \tilde{d}(\frac{b}{s}) = S^{-1}f \circ \tilde{d}'(\frac{b}{s}) \Rightarrow S^{-1}f(\tilde{d}(\frac{b}{s})) = S^{-1}f(\tilde{d}'(\frac{b}{s}))$, and since the functor $S^{-1}()$ preserves monomorphisms, $S^{-1}(f)$ is a monomorphism. So $\tilde{d}(\frac{b}{s}) = \tilde{d}'(\frac{b}{s})$, $\forall\ b \in Z(B), s \in S$. As a result $\tilde{d} = \tilde{d}'$, so $Der_A(S^{-1}Z(B), S^{-1}f)$ is a monomorphism of $S^{-1}Z(B)$-modules. Consequently, $Der_A(S^{-1}Z(B), \text{-})$ is a monofunctor. □

Proposition 3.4. *Let B be an A-algebra and S a central multiplicatively closed subset of B. Then $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.*

Proof. We have

$$Hom_{S^{-1}B}(S^{-1}\Omega_{B/A}, S^{-1}\mathscr{B}) \cong Hom_B(\Omega_{B/A}, \mathscr{B}) \cong Der_A(B, \mathscr{B}) \cong Der_A(S^{-1}B, S^{-1}\mathscr{B}) \cong Hom_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}).$$

Since the contravariant functor $Hom_{S^{-1}B}(\text{-}, S^{-1}\mathscr{B})$ is fully faithful, we have $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$. □

Theorem 3.5. *Let B be an A-algebra and S a central multiplicatively closed subset of B. Then the functor $Der_A(S^{-1}B, \text{-}) : S^{-1}B\text{-}Alg \to Ens$ is representable.*

Proof. By corollary 2.15 and proposition 3.4 we have

$$Der_A(S^{-1}B, S^{-1}\mathscr{B}) \cong Hom_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}),$$

for all $S^{-1}\mathscr{B} \in Ob(S^{-1}B\text{-}Alg)$.
So the covariant functor $Der_A(S^{-1}B, \text{-})$ is isomorphic to the covariant functor $Hom_{S^{-1}B}(\Omega_{S^{-1}B/A}, \text{-})$.
Therefore the functor $Der_A(S^{-1}B, \text{-})$ is representable. □

4 Localization Functor and Homological Functor on the Diferential Algebra

Theorem 4.1. *Let B be a finitely generated Noetherian A-algebra, S a central multiplicatively closed subset of B and $\mathscr{B}$ an $(B\text{-}B)$-bialgebra. Then there exists an isomorphism of left $S^{-1}B$-algebra,*

$$S^{-1}B \otimes_B \widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}).$$

Proof. Since B is finitely generated, by proposition 2.14, $\Omega_{B/A}$ is finitely generated. Also since $\Omega_{B/A}$ is a right finitely generated module over a Noetherian ring B, $\Omega_{B/A}$ is finitely presented.
We have the isomorphism of left $S^{-1}B$-algebras,

$$S^{-1}Hom_B(\Omega_{B/A}, \mathscr{B}) \cong Hom_{S^{-1}B}(S^{-1}\Omega_{B/A}, S^{-1}\mathscr{B}).$$

So we can deduce the following complex isomorphism:

$$S^{-1}Hom_B(P_{\Omega_{B/A}}, \mathscr{B}) \cong Hom_{S^{-1}B}(S^{-1}P_{\Omega_{B/A}}, S^{-1}\mathscr{B}).$$

So we have,

$$H^n(S^{-1}Hom_B(P_{\Omega_{B/A}}, \mathscr{B})) \cong H^n(Hom_{S^{-1}B}(S^{-1}P_{\Omega_{B/A}}, S^{-1}\mathscr{B})).$$

Since the homological functor H^n commutes with the functor $S^{-1}()$, we have on the one hand,

$$H^n(S^{-1}Hom_B(P_{\Omega_{B/A}}, \mathscr{B})) \cong S^{-1}H^n(Hom_B(P_{\Omega_{B/A}}, \mathscr{B})) = S^{-1}\widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}),$$

and on the other,

$$H^n(Hom_{S^{-1}B}(S^{-1}P_{\Omega_{B/A}}, S^{-1}\mathscr{B}))) = \widehat{Ext}^n_{S^{-1}B}(S^{-1}\Omega_{B/A}, S^{-1}\mathscr{B}).$$

Therefore $S^{-1}\widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(S^{-1}\Omega_{B/A}, S^{-1}\mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B})$. So we have

$$S^{-1}B\otimes_B\widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong S^{-1}\widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}).$$

□

Corollary 4.2. *Let B be a finitely generated Noetherian duo-algebra over A, S_R the set of regular elements of B, $S = S_R \cap Z(B)$ and $\mathscr{B}$ a B-bialgebra. Then there exists an isomorphism of left $S^{-1}B$-algebras,*

$$S^{-1}B \otimes_B \widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}).$$

Proof. Since B is a duo-ring, the set of regular elements of S_R of B is a multiplicatively closed subset, so S is a central multiplicatively closed subset of B.
So, by theorem 4.1, $S^{-1}B \otimes_B \widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B})$ as an isomorphism of left $S^{-1}B$-algebras. □

Corollary 4.3. *Let B be a finitely generated Noetherian domain duo-algebra over A, P a prime ideal of B, $S = (B\text{-}P) \cap Z(B)$ and $\mathscr{B}$ a B-bialgebra. Then there exists an isomorphism of left $S^{-1}B$-algebras,*

$$S^{-1}B \otimes_B \widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B}).$$

Proof. Since B is a duo-domain, $S = B\text{-}P$ is a multiplicatively closed subset of B, so S is a central multiplicatively closed subset of B, So, by theorem 4.1, $S^{-1}B \otimes_B \widehat{Ext}^n_B(\Omega_{B/A}, \mathscr{B}) \cong \widehat{Ext}^n_{S^{-1}B}(\Omega_{S^{-1}B/A}, S^{-1}\mathscr{B})$ as an isomorphism of left $S^{-1}B$-algebras.

□

References

[1] F. W. Anderson, Kent R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1974, 1992 Inc.

[2] M. F. Atiyah, I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Company.

[3] E. A. Behrens, Ring Theory, Accademic Press, 1972.

[4] N. Bourbaki, Elements de Mathématique, Herman, Paris, 1970.

[5] H. H. Brungs, Three Questions on Duo Rings, Pacific Journal of Mathematics, **58,** (1975), no. 2, 345–349.

[6] F. Dennis, Noncommutative algebra, Graduate Texts in Mathematics, **144,** Springer-Verlag, 1993.

[7] M. P. Eelbert, Localisation in Duo-ring, Kansas City, Missouri, USA.

[8] D. Faye et al., Functor $S^{-1}()$ and Adjoint Isomorphism, International Mathematical Forum, **11,** no. 5, (2016), 227–237.

[9] T. Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, **131,** Springer-Verlag, 2001.

[10] J. P. Lafon, Algèbre Commutative Langages géométrique et algébriques, Hermann, 1977.

[11] M. F. Maaouia: Thèse d'ètat, Faculté des Sciences et Techniques, UCAD, Dakar, Juillet, 2011.

[12] M. F. Maaouia et al., Localization in a Duo-Ring and Polynomials Algebra, Springer International Publishing, Switzerland, 2016, C. T. Gueye, M. S. Molina (eds.), Non-Associative and Non-Commutative Algebra and Operator Theory, Springer Proceedings in Mathematics and Statistics, **160,** DOI 10.1007 978-3-319-32902-4-13.

[13] M. F. Maaouia, M. Sanghare, Enveloppe plates sur un duo-anneau, Int. J. Contemp. Math. Sciences, **6,** no. 18, (2011), 869–886.

[14] : M. F. Maaouia, M. Sanghare, Anneau de valuation non Nècessairement commutatif et duo-anneau de Didekind, Global Journal of Pure and Applied Mathématiques, **8,** no. 1, (2012), 49–63.

[15] M. F. Maaouia, M. Sanghare, Localisation Dans Les Duo-Anneaux, Afrika Mathematika, 2009.

[16] M. F. Maaouia, M. Sanghare, Module de fraction-Sous-modules S-saturés et foncteurs $S^{-1}()$, International Journal of Algebra, **6,** no. 16, (2012), 775–798.

[17] Joseph J. Rotman, Advanced Modern Algebra, Graduate Studies in Mathematics, **114,** AMS, 2010.

[18] M. Sanghare, On S-duo rings, Com. in Algebra, **20,** no. 8, 1992.

[19] M. Thiaw, M. Maaouia, M. Sanghare, Functors $S^{-1}()$, $Hom_A(\text{-}, \mathscr{B})$, $\text{-} \otimes_A \mathscr{B}$ and $Ext^n_A(\text{-}, \mathscr{B})$ in the Category of A-Alg, International Journal of Theoretical and Applied Mathematics, **5,** no. 1, (2019), 1–9, DOI: 10.11648/j.ijtam.20190501.11