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Algebraic Independent Numbers of Certain types of Simple Graphs

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Abstract

The set of all vertices in the graph G in which every two vertices are not adjacent is called independent set of vertices, and the maximum number of independent vertices by $B_0(G)$, similarly the set of all edges in the graph G in which every two edges are not incident is called independent set of edges, and the maximum number of independent edges denoted by $B_1(G)$.

In this article we will investigate and determine the independent numbers of vertices and edges of certain types of graphs by classic graph definitions and by using characteristic polynomial of graphs.

1 Introduction

The relationship between the graph theory and algebra is very strong. Using linear algebra as a tool in the graph theory makes the study of graph theory developed exponentially.

A graph G consists of a nonempty set of vertices $V(G)$ with a prescribed set of edges $E(G)$, and the order of the graph G is the number of vertices in it, while the number of edges is called the size of the graph G.

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A graph is called **simple** if it has no loops, no parallel edges, and the valancy of vertex is the number of edges incident to it, the graph G is called k**-regular** if the valancy of every vertex in G equal K. Also, the graph G is called **complete graph** if every vertex in it is adjacent with all rest vertices and is denoted by K_n, a **tree** is a **connected graph** with no cycles. A graph G is called **connected** if there exists at least one path between every two vertices is G. **A complete bigraph** $K_{m,n}$ is a graph with two disjoint sets of vertices such that every vertex in first set adjacent to all vertices in the second set. **A star graph** $S(1, n)$ is a tree with no root and n end vertices. **A cycle graph** C_n is a closed path with n vertices. **A path graph** P_n is a path with n vertices. **A wheel graph** $W(1, n)$ is a graph which consists of star graph and cycle graph. An edge is called a **loop** if its incident with itself, if more than one edges incident with certain two vertices they called **parallel** edges, a graph G is called **connected** graph if there exist at least one path between any two vertices in G, a **tree** is a connected graph without cycles.

The set of vertices in G in which every two vertices are not adjacent is called independent set of vertices and the maximum number of vertices is denoted by $B_0(G)$ and called vertex independent number and the maximum number of independent edges is called edge independent number, denoted by $B_1(G)$. The adjacency matrix $A(G) = [a_{ij}]_{n\times n}$ of a labeled graph G with n vertices is an $n \times n$ matrix in which,

$$a_{ij} = \begin{cases} 1, & \text{if } u \text{ index } i \text{ adjcent to } u \text{ index } j \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

Let $A = A(G)$ be the adjacency matrix of the graph G. Then the determinate of the matrix $[\lambda I A]$ is called the characteristic polynomial of the graph G, denoted by $P(\lambda)$.

2 Independent Numbers of Graphs

In the following sequel we will investigate and find the independent numbers of vertices and edges of well-known graphs using graph definitions and using characteristic polynomials of Graphs.

Theorem 2.1 (Sitthiwirattham [3]). *If the graph G has no vertex independent, then $B_0(G) = 1$, and if it has no edge independent, then $B_1(G) = 1$.*

Theorem 2.2. *Let G be a path graph P_n. Then , $B_0(G) = [\frac{(n+1)}{2}]$, and $B_1 = [\frac{n}{2}]$.*

Proof. Let G be a path graph P_n, since every non-end vertex in G adjacent with predecesson and subsequent vertex, so if n is even then the number of independent vertices is $\frac{n}{2}$, and if n is odd, we have $\frac{(n+1)}{2}$ independent vertices, thus $B_0(G) = [\frac{(n+1)}{2}]$. Similarly for edges, G have $n-1$ edges, we have $B_1 = \frac{n}{2}$, if n is even and if n is odd, $B_1(G) = [\frac{n}{2}]$, hence $B_1(G) = [\frac{n}{2}]$. □

Theorem 2.3. *Let G be a cycle graph C_n. Then, $B_0(G) = B_1(G) = [\frac{n}{2}]$.*

Proof. Let G be a cycle graph C_n, so G is a closed path with n vertices and n edges. Now every vertex in G has valancy two, and adjacent with predecessor and successor vertices, so every vertex independent (nonadjacent) with the follow vertices of predessor and successor vertices, consequently, if n is even, the number of those vertices will be $\frac{n}{2}$ and if n is odd, the number of those vertices is $\frac{n-1}{2}$. Thus $B_o(G) = [\frac{n}{2}]$. is even and $[\frac{n}{2}]$. Following the same argument for independent edges, we have $B_1(G) = [\frac{n}{2}]$. □

Theorem 2.4. *Let K_n be a complete graph. Then, the vertex independent number $B_0(K_n) = 1$, and the edge independent number $B_1(K_n) = [\frac{n}{2}]$.*

Proof. Let G be a complete graph K_n, since every vertex of G adjacent with all other vertices, so there is no vertex independent, consequently $B_0(G) = 1$, by (Theorem 2.1). On the other hand, if we neglect the inner edges in G and consider the outer edges which is isomorphic to the cycle graph C_n, thus $B_1(G) = [\frac{n}{2}]$. (Theorem 2.3). □

Theorem 2.5. *Let G be a star graph $S(1, n)$ with $n + 1$ vertices. Then $B_0(G) = n$, and $B_1(G) = 1$.*

Proof. Let G be a star graph $S(1, n)$, with $n + 1$ vertices, G consists of one root vertex adjacent with n end vertices, thus number of independent vertices equal n, and since all edges are incident, then the number of independent edges is one. □

Theorem 2.6. *Let G be a bigraph $K_{m,n}$. Then, $B_0(G) = max(m, n)$ and $B_1(G) = min(m, n)$.*

Proof. Let G be a bigraph $K_{m,n}$, so G consists of two disjoint sets of vertices and every set of vertices are not adjacent with itself, so the number of independent vertices is the set with largest cardinality, i.e $max(m, n)$. Now by bigraph construction the valancy of every vertex in the first set is n and the second set is m, so number of independent edges in G equal to the least valancy which equal $min(m, n)$. □

Theorem 2.7. *Let G be a wheel graph $W(1,n)$. Then $B_0(G) = B_1 = [\frac{n}{2}]$.*

Proof. Let G be a wheel graph, so G consists of $C_n \cup S(1,n)$, removing the root vertex of G_m the graph will be C_n so by theorem (2.3). $B_0(G) = B_1(G) = [\frac{n}{2}]$. □

3 Characteristic polynomials of graphs and independent numbers

In the following sequel we will discuss and construct a relation between characteristic polynomials of graphs and the independent numbers of vertices and edges of graphs.

Theorem 3.1 (Biggs [1]). *Let G be a simple graph with characteristic polynomial,*

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda_{n-1} + ... + a_0$$

Then

1. $a_{n-1} = 0$,
2. $|a_{n-1}| = |E(G)|$,
3. $|a_{n-3}| =$ *twice number of* C_3 *in* G.

Theorem 3.2 (Wrikat [4]). *The characteristic polynomial of a complete graph K_n is*

$$P(\lambda) = (\lambda + 1)^{n-1}(\lambda - n + 1)$$

Theorem 3.3. *Let G be a complete graph K_n. Then $B_0(G) = |a_n|$ and $B_1(G) = (|a_0| - [\frac{n-2}{2}])$.*

Proof. Let G be a complete graph K_n. Let A be the adjacency matrix of G, so A is an $n \times n$ matrix with zeros in the main diagonal and ones elsewhere, let its characteristic polynomial $P(\lambda) = a_n\lambda^n + a_{n-1}\lambda_{n-1} + ... + a_0$, since $|a_n| = 1$ So independent vertices equal one. $B_1[G] = [\frac{n}{2}]$ (Theorem 2.4), Now $(|a_0| - [\frac{n-2}{2}]) = ((n-1)[\frac{n-2}{2}]) = [\frac{n}{2}]$. □

Example 3.1. *Let $G = K_8$.*

$$P(\lambda) = \lambda^8 - 28\lambda^6 - 112\lambda^5 - 210\lambda^4 - 224\lambda^3 - 140\lambda^2$$

So $B_0(G) = 1$, $B_1(G) = (|a_0| - [\frac{n-2}{2}]) = (7 - 3) = 4 = [\frac{8}{2}]$.

Theorem 3.4. *Let G be a star $S(1,n)$. Then $B_0(G) = |a_n - 2| = n$ and $B_1(G) = |a_n| = 1$.*

Proof. Let G be a star graph with $n+1$ vertices Since number of end vertices equal number of edges and they are independent, so $B_0(G) = |a_n - 1|$,(Theorem 3.1). On the other hand, all edges are incident, thus $B_1(G) = |a_n| = 1$. □

Example 3.2. *Let G be a star graph $S(1,4)$, the characteristic polynomial of G is $P(\lambda) = -\lambda^5 + 4\lambda^3$, thus $B_0(G) = 4$, $B_1 = 1$.*

Theorem 3.5 (Wrikat [4]). *Let G be a star graph $S(1,n)$. Then the characteristic polynomial of G is*

$$P(\lambda) = \lambda^{n-2}(\lambda^2(n-1))$$

Theorem 3.6. *Let G be a cycle graph C_n. Then, $B_0(G) = B_1(G) = [\frac{|a_{n-2}|}{2}]$.*

Proof. Let G be a cycle graph C_n with n vertices and $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0$, G consists of n vertices and n edges which is a closed path, $|a_{n-2}|$ represents number of edges (Theorem 3.1), so $B_0(G) = B_1(G) = [\frac{|a_{n-2}|}{2}]$. □

Example 3.3. *Let $G = C_6$ be a cycle graph with 6 vertices, the characteristic polynomial of C_6, $P(\lambda) = \lambda^6 - 6\lambda^4 + 9\lambda^2 - 4$, $|a_{n-2}| = 6 = |E(G)| = |V(G)|$. $B_0(G) = 6/2 = 3$, $B_1(G) = 6/2 = 3$.*

Theorem 3.7. *Let G be a wheel graph. Then $B_0(G) = B_1(G) = [\frac{|a_{n-2}|}{4}]$, where the characteristic polynomial of G is $P(\lambda) = \lambda^n + a_{n-1}\lambda_{n-1} + ... + a_0$.*

Proof. Let G be a wheel graph $W(1,n)$ with n vertices and $2n$ edges, removing the root vertex, since it is adjacent with all vertices in G, we have a cycle graph C_n in which $B_0(G) = B_1(G) = [\frac{n}{2}]$, but number of edges in $G = 2n = |a_{n-2}|$,(Theorem 3.1) , thus $B_0(G) = B_1(G) = [|a_{n-2}|/4]$. □

Example 3.4 (Schwenk[2]). *Let G be a wheel graph with 8 vertices $W(1,7)$, it is characteristic polynomial is*

$$P(\lambda) = \lambda^8 14\lambda^6 14\lambda^5 + 35\lambda^4 + 42\lambda^3 12\lambda^2 30\lambda 7$$

$B_0(G) = B_1(G) = 12/4 = 3$.

Theorem 3.8. *Let G be a complete bigraph $K_{m,n}$, with characteristic polynomial $P(\lambda)$. Then $B_0(G) = [\frac{|a_{n-2}|}{B_1}]$ and $B_1(G) = [\frac{|a_{n-2}|}{B_0}]$.*

Proof. Let G be a complete bigraph $K_{m,n}$, since the characteristic polynomial of G takes the form $\lambda^{m+n}+a_{n-2}\lambda^{m+n-2}+...+a_0$, where a_{n-2} represents number of edges in G which is mn, so by (Theorem 2.6), $B_0(G) = max(m,n) = \frac{mn}{min(m,n)} = [\frac{|a_{n-2}|}{min(m,n)}]$ and, $B_1(G) = min(m,n) = \frac{mn}{max(m,n)} = \frac{|a_{n-2}|}{max(m,n)}$. □

Example 3.5. *Let G be a complete bigraph $K_{3,4}$. The characteristic polynomial of G is $P(\lambda) = -\lambda^7 + 12\lambda^5$. Therefore, the number of edges is $|a_{n-2}| = 12$. Thus $B_0(G) = 12/3 = 4$, and $B_1(G) = \frac{12}{4} = 3$.*

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