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On the almost periodic solutions of fuzzy cellular neural networks of high order with multiple time lags

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Abstract

In this paper, the existence and uniqueness and global exponential stability of almost periodic solutions are discussed for a class of high-order fuzzy cellular neural networks with time-varying delays. By means of some properties of almost periodic functions, exponential dichotomy hypotheses and differential inequalities, certain suitable sufficient conditions are established to guarantee the existence and uniqueness of solutions and exponential stability of solutions of the considered mathematical model. We prove two new results on the subject and give an example as an application of these results.

1 Introduction

Cellular neural networks (CNN) proposed by Chua and Yang [1] have been extensively studied in both theory and practice. However, since uncertainty is inevitable in mathematical modeling of real world problems, Yang and Yang [2] introduced fuzzy cellular neural networks (FCNN) to the literature, which integrates fuzzy logic into the traditional CNN structure and maintains the

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logic connection among cells. More recently, many neural network models have been extensively studied and successfully applied to relational memory, pattern recognition, signal processing and optimization problems (see [2-8] and references therein). Recently, studies on stability and boundedness analysis, periodic and almost periodic solutions of fuzzy cellular neural networks (FCNNs) and some other differential equations have been conducted [10,11, 24, 25, 26]. High-order neural networks have become noticeable because of the stronger approach of higher-order neural networks, the higher the convergence rate, the greater the storage capacity, and the higher the fault tolerance than the lower-grade neural networks [16-23]. Thus, inspired by the above mentioned scientific works, the following higher order fuzzy cellular neural networks with time-varying delays (HFCNNs) are considered:

$$\begin{cases} \frac{dx_i(t)}{dt} = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ + \bigwedge_{j=1}^{n} \alpha_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) + \bigvee_{j=1}^{n} \beta_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}n(t)f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t))) \\ + \bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n} \gamma_{ijl}(t)f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t))) \\ + \bigvee_{j=1}^{n}(\bigvee_{l=1}^{n} \sigma_{ijl}(t)f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t))) \\ + \bigwedge_{j=1}^{n} T_{ij}(t)u_j(t) + \bigvee_{j=1}^{n} H_{ij}u(t)\mu_j(t) + I_i(t), t > 0 \\ x_i(s) = \varphi_i(s), -\tau \le s \le 0, \end{cases} \tag{1}$$

where $i, j, l = 1, 2, ..., n$. $d_i(t) > 0$, $0 \le \tau_{ij} \le \tau$ denotes the potential (or voltage) of the cell i at time t; $d_i(t)$ is a positive constants, denotes the rate with which the cell i reset its potential to the resting state when isolated from the other cells and inputs; Time delays $\tau_{ij}(t)$ are non-negative, continuously differentiable functions, they correspond to finite speed of signal transmissions; $a_{ij}(t)$ and $b_{ijl}(t)$ are the first-order and second-order connection weights of neural network, respectively; $\alpha_{ij}(t), \beta_{ij}(t), T_{ij}(t)$ and $H_{ij}(t)$ are elements of the first-order fuzzy feedback MIN template, first-order fuzzy feedback MAX template, first order fuzzy feed-forward MIN template and first order fuzzy feed forward MAX template, respectively; $\gamma_{ijl}(t)$ and $\sigma_{ijl}(t)$ are elements of the second-order fuzzy feedback MIN template, second-order fuzzy feedback MAX template, respectively; $\bigwedge$ and $\bigvee$ denote the fuzzy AND and fuzzy OR operations, respectively; $\mu_i(t)$ and $I_i(t)$ denote input and bias of the i-th neurons, respectively; f_i are the activation functions.

2 Preliminaries

In this section $Y = (y_1, y_2, ..., y_n) \in \Re^n$ shows column vector. Define $|Y| = (|y_1|, |y_2|, ..., |y_n|)^T$ and $\|Y\| = \max_{1\leq i\leq n} |y_i|$. For matrices or vectors A and B, $A \leq B$ (resp. $A > B$) means that $A - B \geq 0$ (resp. $A - B > 0$). $BC(\Re, \Re^n)$ denotes the set of bounded and continuous functions from $\Re$ to $\Re^n$.Note that $(BC(\Re, \Re^n), \|.\|_\infty)$ is a Banach space and $\|.\|_\infty$ denotes the supremum norm $\|h\|_\infty := \sup_{t\in\Re} \|h\|$. In this work, for a given a bounded continuous h defined on $\Re$, let h^+ and h_- be defined as

$$h^+ = \sup_{t\in\Re} |h(t)|, \quad h_- = \inf_{t\in\Re} |h(t)|.$$

We suppose the following assumptions hold:

(A1) Let $d_i, a_{ij}, \tau_{ij}, \beta_{ij}, \sigma_{ij}, \gamma_{ij}, H_{ij}, T_{ij}, I_{ij}, u_{ij} \in AP(\Re, \Re^+)$.
(A2) For each $i = 1, 2, ..., n$,

$$M[d_i] = \lim_{T\to+\infty} \frac{1}{T} \int_t^{T+t} d_i(s)ds > 0$$

and there exists a continuous and bounded function $\widetilde{d}_i \in (0, +\infty)$ and a positive constant K_i such that

$$\exp^{-\int_s^t d_i(u)du} \leq K_i \exp^{-\int_s^t \widetilde{d}_i(u)du}.$$

(A3) For each $j = 1, 2, ..., n$, there exist nonnegative constants L_j^h such that

$$| h_j(x) - h_j(y) | \leq L_j^h | x - y |$$

for all $x, y \in R$.
(A4) There exist positive constants ξ_j and μ_i such that

$$\sup_{t\in}\{-d_i(t) + K_i\xi_j^{-1}[(\sum_{j=1}^{n}(|a_{ij}| + |\alpha_{ij}| + |\beta_{ij}|))\xi_j L_j^f$$

$$+\xi_j^{-1}(\sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|))\xi_l L_l^f M^f$$

$$+\xi_j^{-1}(\sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ijl}| + |\gamma_{ijl}| + |\sigma_{ijl}|))\xi_j L_j^f M^f < \mu_i.$$

Definition 2.1.[5] For any $\epsilon > 0$, there exists a number $l(\epsilon) > 0$ with the property that any interval of length $l(\epsilon)$ of the real line contains at least one points with abscissa τ such that

$$\mid f(x+\tau) - f(t) \mid < \epsilon, \qquad -\infty < x < \infty.$$

The concept of almost periodic functions is given by Harald Bohr [4], where the number τ is called an ϵ -translation number of $f(x)$ corresponding to ϵ or an ϵ translation number. $AP(X, \Re)$ means the set all almost periodic functions.

Definition 2.2. An almost periodic solution $x^*(t)$ of system (1) is said to be global exponential stable, if there exist constants $\epsilon > 0$ and $K > 0$ such that

$$\|\phi - x^*\| = \max_{1\le i\le n}\{\sup_{-\tau\le s\le 0}|\phi_i(s) - x_i^*(s)|\} \le Ke^{-\epsilon(t-t_0)} \qquad \texttt{for all} \quad t > t_0.$$

Lemma 2.1. [10] Let $x_j, \widetilde{x}_j, \theta_{ij}, \kappa_{ij} \in R, h_j \in C(\Re)$ and $i, j \in J$. Then we have

$$|\bigvee_{j=1}^{n}\kappa_{ij}h_j(x_j) - \bigvee_{j=1}^{n}\kappa_{ij}h_j(\widetilde{x}_j)| \le \sum_{j=1}^{n}|\kappa_{ij}||h_j(x_j) - h_j(\widetilde{x}_j)|$$

and

$$|\bigwedge_{j=1}^{n}\theta_{ij}h_j(x_j) - \bigwedge_{j=1}^{n}\theta_{ij}h_j(\widetilde{x}_j)| \le \sum_{j=1}^{n}|\theta_{ij}||h_j(x_j) - h_j(\widetilde{x}_j)|.$$

Lemma 2.2.[10] Let $x_j, \alpha_{ij}, \beta_{ij}, \tau_{ij} \in AP(\Re)$. Then

$$\bigwedge_{j=1}^{n} \alpha_{ij}(t) f_j(x_j(t-\tau_{ij}(t))), \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau_{ij}(t))) \in AP(\Re, \Re).$$

Lemma 2.4. [10] If $\tau(t)$ and $x(t)$ are almost periodic, then $x(t-\tau(t))$ is almost periodic.

3 Existence of almost periodic solutions

Lemma 3.1. Let $x_j, x_j, \gamma_{ijl}, \sigma_{ijl}, \tau_{ij} \in AP(\Re)$. Then

$$\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n} \alpha_{ij}(t) f_j(x_j(t-\tau_{ij}(t))) f_l(x_l(t-\tau_{ij}(t)))) \in AP(\Re, \Re)$$

and

$$\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n} \beta_{ij} f_j(x_j(t-\tau_{ij}(t))) f_l(x_l(t-\tau_{ij}(t)))) \in AP(\Re, \Re).$$

Proof. The proof of this lemma can be completed by a similar way as in [11].

Theorem 3.1. Let assumptions (A1)-(A4) hold. Then there exists a unique almost periodic solution of system (1)

Proof.

Let $\overline{x}_i(t) = \xi_i^{-1} x_i(t)$ for $i \in J$. Then system (1) can be transformed to the following form:

$$\begin{aligned}\frac{d\overline{x}_i(t)}{dt} &= -d_i(t)\overline{x}_i(t) + \xi_i^{-1}\sum_{j=1}^{n} a_{ij}(t)f_j(\overline{x}_j(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigwedge_{j=1}^{n}\alpha_{ij}(t)f_j(\overline{x}_j(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigvee_{j=1}^{n}\beta_{ij}(t)f_j(\overline{x}_j(t-\tau_{ij}(t)))\\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}n(t)f_j(\overline{x}_j(t-\tau_{ij}(t)))f_l(\overline{x}_l(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}(t)f_j(\overline{x}_j(t-\tau_{ij}(t)))f_l(\overline{x}_l(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}(t)f_j(\overline{x}_j(t-\tau_{ij}(t)))f_l(\overline{x}_l(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigwedge_{j=1}^{n}T_{ij}(t)u_j(t) + \xi_i^{-1}\bigvee_{j=1}^{n}H_{ij}u(t)\mu_j(t) + \xi_i^{-1}I_i(t).\end{aligned} \tag{2}$$

Let $\phi \in AP(\Re,\Re)$. From Lemma 2.2 and Lemma 3.1, we have

$$\begin{aligned}&+ \xi_i^{-1}\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}(t)f_j(\phi_j(t-\tau_{ij}(t)))f_l(\phi_l(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}(t)f_j(\phi_j(t-\tau_{ij}(t)))f_l(\phi_l(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigwedge_{j=1}^{n}T_{ij}(t)u_j(t) + \xi_i^{-1}\bigvee_{j=1}^{n}H_{ij}u(t)\mu_j(t) + \xi_i^{-1}I_i(t) \in AP(\Re,\Re)\end{aligned} \tag{3}$$

and

$$\begin{aligned}&\xi_i^{-1}\sum_{j=1}^{n} a_{ij}(t)f_j(\phi_j(t-\tau_{ij}(t))) + \xi_i^{-1}\bigwedge_{j=1}^{n}\alpha_{ij}(t)f_j(\phi_j(t-\tau_{ij}(t)))\\ &+ \xi_i^{-1}\bigvee_{j=1}^{n}\beta_{ij}(t)f_j(\phi_j(t-\tau_{ij}(t))) + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}n(t)f_j(\phi_j(t-\tau_{ij}(t)))f_l(\phi_l(t-\tau_{ij}(t))) \in AP(\Re,\Re\end{aligned}$$

So, from (3), $M[d_i(t)] > 0$ and Lemma 2.1 in [21] that system (2) has exactly one almost periodic solution:

$$x^{\varphi}(t) = \{x^{\varphi}(t)\} \int_{-\infty}^{t} \exp^{-\int_s^t d_i(u)du} [\xi_i^{-1} \sum_{j=1}^{n} a_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s)))$$

$$+ \xi_i^{-1} \bigwedge_{j=1}^{n} \alpha_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s))) + \xi_i^{-1} \bigvee_{j=1}^{n} \beta_{ij}(s) f_j(\phi_j(s-\tau_{ij}(s)))$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} n(s) f_j(\phi_j(s-\tau_{ij}(s))) f_l(\phi_l(s-\tau_{ij}(s))) \qquad (4)$$

$$+ \xi_i^{-1} \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl}(s) f_j(\phi_j(s-\tau_{ij}(s))) f_l(\phi_l(s-\tau_{ij}(s)))$$

$$+ \xi_i^{-1} \bigvee_{j=1}^{n} (\bigvee_{l=1}^{n} \sigma_{ijl}(s) f_j(\phi_j(s-\tau_{ij}(s))) f_l(\phi_l(s-\tau_{ij}(s)))$$

$$+ \xi_i^{-1} \bigwedge_{j=1}^{n} T_{ij}(s) u_j(s) + \xi_i^{-1} \bigvee_{j=1}^{n} H_{ij} u(s) \mu_j(s) + \xi_i^{-1} I_i(s)] ds.$$

We can now define $\Gamma : AP(\Re, \Re^n) \to AP(\Re, \Re^n)$ in the form of

$$(\Gamma\varphi)(t) = x^{\varphi}(t), \qquad \forall \varphi \in AP(\Re, \Re^n).$$

We prove that the mapping T is a contraction mapping of $AP(\Re, \Re^n)$.
By assumptions (A1)-(A4) and (4), we derive that

$$|((\Gamma\varphi)(t) - (\Gamma\psi)(t))_i| = | \int_{-\infty}^{t} \exp^{-\int_s^t d_i(u)du} [\xi_i^{-1} \sum_{j=1}^{n} a_{ij}(s) \times (f_j(\xi_j \varphi_j(s-\tau_{ij}(s)))$$

$$- f_j(\xi_j \psi_j(s-\tau_{ij}(s))) + \xi_i^{-1} \bigwedge_{j=1}^{n} \alpha_{ij}(s) [f_j(\varphi_j(s-\tau_{ij}(s))) - f_j(\psi_j(s-\tau_{ij}(s)))] +$$

$$\xi_i^{-1} \bigvee_{j=1}^{n} \beta_{ij}(s) [f_j(\varphi_j(s-\tau_{ij}(s))) - f_j(\psi_j(s-\tau_{ij}(s)))]$$

$$+ \sum_{j=1}^{n} \sum_{l=1}^{n} b_{ijl} n(s) [f_j(\varphi_j(s-\tau_{ij}(s))) f_l(\varphi_l(s-\tau_{ij}(s))) - f_j(\psi_j(s-\tau_{ij}(s))) f_l(\psi_l(s-\tau_{ij}(s)))]$$

$$+ \xi_i^{-1} \bigwedge_{j=1}^{n} (\bigwedge_{l=1}^{n} \gamma_{ijl}(s) [f_j(\varphi_j(s-\tau_{ij}(s))) f_l(\varphi_l(s-\tau_{ij}(s))) - f_j(\psi_j(s-\tau_{ij}(s))) f_l(\psi_l(s-\tau_{ij}(s)))]$$

$$+\xi_i^{-1}\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}(s)[f_j(\varphi_j(s-\tau_{ij}(s)))f_l(\varphi_l(s-\tau_{ij}(s)))-f_j(\psi_j(s-\tau_{ij}(s)))f_l(\psi_l(s-\tau_{ij}(s)))]$$

$$\leq K_i\int_{-\infty}^{t}\exp^{-\int_s^t\widetilde{d}_i(u)du}[\xi_i^{-1}\sum_{j=1}^{n}|a_{ij}(s)|L_j^f\xi_j+\xi_i^{-1}\sum_{j=1}^{n}|\alpha_{ij}(s)|L_j^f\xi_j+\xi_i^{-1}\sum_{j=1}^{n}|\beta_{ij}(s)|L_j^f\xi_i$$

$$+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|b_{ij}(s)|M^fL_l^f\xi_l+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|b_{ijl}(s)|M^fL_j^f\xi_j$$

$$+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|\gamma_{ijl}(s)|M^fL_l^f\xi_l+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|\gamma_{ijl}(s)|M^fL_j^f\xi_j$$

$$+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|\sigma_{ijl}(s)|M^fL_l^f\xi_l+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}|\sigma_{ijl}(s)|M^fL_j^f\xi_j]\|\varphi-\psi\|_\infty ds$$

$$=K_i\int_{-\infty}^{t}\exp^{-\int_s^t\widetilde{d}_i(u)du}\times[\xi_i^{-1}\sum_{j=1}^{n}(|a_{ij}(s)|+|\alpha_{ij}(s)|+|\beta_{ij}(s)|)L_j^f\xi_j$$

$$+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)|+|\gamma_{ij}(s)|+|\sigma_{ij}(s)|)M^fL_l^f\xi_l$$

$$+\xi_i^{-1}\sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)|+|\gamma_{ij}(s)|+|\sigma_{ij}(s)|)\times M^fL_j^f\xi_j]\|\varphi-\psi\|_\infty ds$$

$$\leq\int_{-\infty}^{t}\exp^{-\int_s^t\widetilde{d}_i(u)du}[\widetilde{d}_i(s)-K_i]ds\|\varphi-\psi\|_\infty$$

$$\leq\int_{-\infty}^{t}\exp^{-\int_s^t\widetilde{d}_i(u)du}[1-\frac{K_i}{\sup_{t\in R}\widetilde{d}_i(t)}]\widetilde{d}_i(s)ds\|\varphi-\psi\|_\infty$$

$$\leq[1-\frac{K_i}{\sup_{t\in R}\widetilde{d}_i(t)}]\|\varphi-\psi\|_\infty.$$

From the above statements, we see that the mapping Γ is a contraction. Thus, the mapping Γ has a unique fixed point $x^*\in AP(\Re,\Re)$, $\Gamma x^*=x^*$.

4 Global exponential stable solutions

Theorem 4.1. Under the assumptions of Theorem 3.1 and

$$\sup_{t\in R}\{-\widetilde{d}_i(t)+K_i[\sum_{j=1}^{n}(|a_{ij}(s)|+|\alpha_{ij}(s)|+|\beta_{ij}(s)|)L_j^f\xi_j \tag{5}$$

$$+\quad \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)|+|\gamma_{ij}(s)|+|\sigma_{ij}(s)|)M^f L_l^f]e^{\lambda_0\tau_{ij}(t)}\}<-\kappa_i<0,$$

where λ_0 and κ_i are positive constants, almost periodic solutions of (2) are global exponential stable.

Proof. Obviously, by Theorem 3.1, system (1) has a unique almost periodic solution $x_*(t)=x_i^*(t)$. Suppose that $x(t)=x_i(t)$ is an arbitrary solution of system (1) with the initial value $\phi(t)=\phi_i$. Let $y(t)=y_i(t)=\xi(x_i(t)-x_i^*(t))$. Then

$$\begin{aligned}
\frac{dy_i}{dt} &= -d_i(t)y_i(t)+\sum_{j=1}^{n}a_{ij}(t)\times[f_j(x_j(t-\tau_{ij}(t)))-f_j(x_j^*(t-\tau_{ij}(t)))]\\
&+\bigwedge_{j=1}^{n}\alpha_{ij}(t)[f_j(x_j(t-\tau_{ij}(t)))-f_j(x_j^*(t-\tau_{ij}(t)))]\\
&+\bigvee_{j=1}^{n}\beta_{ij}(t)[f_j(x_j(t-\tau_{ij}(t)))-f_j(x_j^*(t-\tau_{ij}(t)))] \qquad (6)\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}[f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t)))\\
&-f_j(x_j^*(t-\tau_{ij}(t)))f_l(x_l^*(t-\tau_{ij}(t)))]\\
&+\bigwedge_{j=1}^{n}(\bigwedge_{l=1}^{n}\gamma_{ijl}(t)[f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t)))\\
&-f_j(x_j^*(t-\tau_{ij}(t)))f_l(x_l^*(t-\tau_{ij}(t)))]\\
&+\bigvee_{j=1}^{n}(\bigvee_{l=1}^{n}\sigma_{ijl}(t)[f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t)))\\
&-f_j(x_j(t-\tau_{ij}(t)))f_l(x_l(t-\tau_{ij}(t)))].
\end{aligned}$$

Let $P = P_i(w)$. Then

$$P_i(w) = \sup_{t\in R}\{w - \widetilde{d}_i(t) + K_i[\sum_{j=1}^{n}(|a_{ij}(s)| + |\alpha_{ij}(s)| + |\beta_{ij}(s)|) \times L_j^f \xi_j + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)| + |\gamma_{ij}(s)| + |\sigma_{ij}(s)|)M^f L_l^f]e^{w\tau_{ij}(t)}\} \quad (7)$$

From (7), we have

$$P_i(0) = \sup_{t\in R}\{-\widetilde{d}_i(t) + K_i[\sum_{j=1}^{n}(|a_{ij}(s)| + |\alpha_{ij}(s)| + |\beta_{ij}(s)|) \times L_j^f \xi_j + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)| + |\gamma_{ij}(s)| + |\sigma_{ij}(s)|)M^f L_l^f]\}$$

$$\leq \sup_{t\in R}\{-\widetilde{d}_i(t) + K_i[\sum_{j=1}^{n}(|a_{ij}(s)| + |\alpha_{ij}(s)| + |\beta_{ij}(s)|) \times L_j^f \xi_j + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)| + |\gamma_{ij}(s)| + |\sigma_{ij}(s)|)M^f L_l^f]e^{\lambda_0\tau_{ij}(t)}\} < -\kappa_i < 0.$$

Since $P_i(w)$ is continuous, there exists a $\lambda \in (0, \min_{i\in j}(\lambda_0, \inf_{t\geq t_0} \widetilde{d}_i(t)))$ such that

$$P_i(\lambda) = \sup_{t\in R}\{w - \widetilde{d}_i(t) + K_i[\sum_{j=1}^{n}(|a_{ij}(s)| + |\alpha_{ij}(s)| + |\beta_{ij}(s)|) \times \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)| + |\gamma_{ij}(s)| + |\sigma_{ij}(s)|)M^f L_l^f]e^{\lambda\tau_{ij}(t)}\} < 0.$$

Let $\|\phi - x^*\| = \sup_{t\leq 0}\max_{i\in j}|\phi_i - x_i^*|$ and for a constant M , $M = \max_{i\in} K_i + 1$. For any $\epsilon > 0$ and for each $t \in [t_0 - r, t_\xi]$, it is clear that

$$\|y(t)\| < M(\|\phi - x^*\| + \epsilon)e^{-\lambda(t-t_0)}.$$

For every $t > 0$, we will show

$$\|y(t)\| < M(\|\phi - x^*\| + \epsilon)e^{-\lambda(t-t_0)}. \tag{8}$$

Suppose (8) is not true. So there is an $\theta > t_\xi$ such that

$$\begin{cases} \|y(\theta)\| = M(\|\phi - x^*\| + \epsilon)e^{-\lambda(\theta-t_0)} \\ \|y(t)\| < M(\|\phi - x^*\| + \epsilon)e^{-\lambda(t-t_0)} \qquad \text{for every } t \in [t_0 - r, \theta). \end{cases} \tag{9}$$

Multiple both sides of (8) by $e^{\int_{t_0}^{t} d_i(u)du}$ and integrate on the interval of $[t_0, t]$. Then, for $t = \theta$, we get

$$\begin{aligned} |y(\theta)| \;\leq\; & |y(t_0)|e^{\int_{t_0}^{\theta} d_i(u)du} + \int_{t_0}^{\theta} e^{\int_s^{\theta} d_i(u)du}\Big(\sum_{j=1}^{n} a_{ij}(s) \\ \times\; & |f_j(x_j(s-\tau_{ij}(s))) - f_j(x_j^*(s-\tau_{ij}(s)))| \\ +\; & \bigwedge_{j=1}^{n} \alpha_{ij}(s)|f_j(x_j(s-\tau_{ij}(s))) - f_j(x_j^*(s-\tau_{ij}(s)))| \\ +\; & \bigvee_{j=1}^{n} \beta_{ij}(s)|f_j(x_j(s-\tau_{ij}(s))) - f_j(x_j^*(s-\tau_{ij}(s)))| \\ +\; & \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ij}(s)|f_j(x_j(s-\tau_{ij}(s)))f_l(x_l(s-\tau_{ij}(s))) \\ -\; & f_j(x_j^*(s-\tau_{ij}(s)))f_l(x_l^*(s-\tau_{ij}(s)))| \\ +\; & \bigwedge_{j=1}^{n}\Big(\bigwedge_{l=1}^{n} \gamma_{ijl}(s)|f_j(x_j(s-\tau_{ij}(s)))f_l(x_l(s-\tau_{ij}(s))) \\ -\; & f_j(x_j^*(s-\tau_{ij}(s)))f_l(x_l^*(s-\tau_{ij}(s)))| \\ +\; & \bigvee_{j=1}^{n}\Big(\bigvee_{l=1}^{n} \sigma_{ijl}(s)|f_j(x_j(s-\tau_{ij}(s)))f_l(x_l(s-\tau_{ij}(s))) \\ -\; & f_j(x_j^*(s-\tau_{ij}(t)))f_l(x_l^*(s-\tau_{ij}(s)))|\Big)\Big)\Big)ds. \end{aligned} \tag{10}$$

By assumptions (A1) and (A4), we can conclude that

$$
\begin{aligned}
|y(\theta)| &\leq K_i(\|\phi - x^*\| + \epsilon)e^{-\int_{t_0}^{\theta}\widetilde{d}_i(u)du} + K_i\int_{t_0}^{\theta} e^{\int_s^{\theta}\widetilde{d}_i(u)du} \\
&\times (\sum_{j=1}^{n}|a_{ij}(s)||y_j(s-\tau_{ij}(s)|L^f + \sum_{j=1}^{n}|\alpha_{ij}(s)|L^f|y_j(s-\tau_{ij}(s)| \\
&+ \sum_{j=1}^{n}|\beta_{ij}(s)|L^f|y_j(s-\tau_{ij}(s)| + \sum_{j=1}^{n}\sum_{l=1}^{n}|b_{ij}(s)|L^f|y_j(s-\tau_{ij}(s)| \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}|\gamma_{ijl}(s)|L^f|y_j(s-\tau_{ij}(s)| + \sum_{j=1}^{n}\sum_{l=1}^{n}|\sigma_{ijl}(s)|L^f|y_j(s-\tau_{ij}(s)|))ds \\
&\leq K_i(\|\phi - x^*\| + \epsilon)e^{-\int_{t_0}^{\theta}\widetilde{d}_i(u)du} + K_i\int_{t_0}^{\theta} e^{\int_s^{\theta}\widetilde{d}_i(u)du} \\
&\times [\sum_{j=1}^{n}(|a_{ij}(s)| + |\alpha_{ij}(s)| + |\beta_{ij}(s)|)L^f + \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)| + |\gamma_{ijl}(s)| + |\sigma_{ijl}(s)|) \\
&\times M^f L^f]M(\|\phi - x^*\| + \epsilon)e^{-\lambda(s-\tau_{ij}(s)-t_0)}ds \\
&\leq K_i(\|\phi - x^*\| + \epsilon)e^{-\int_{t_0}^{\theta}(\widetilde{d}_i(u)-\lambda)du}e^{-\lambda(\theta-t_0)} \\
&+ K_i\int_{-\infty}^{\theta} e^{\int_s^{\theta}(\widetilde{d}_i(u)-\lambda)du}[\sum_{j=1}^{n}(|a_{ij}(s)| + |\alpha_{ij}(s)| + |\beta_{ij}(s)|)L^f \\
&+ \sum_{j=1}^{n}\sum_{l=1}^{n}(|b_{ij}(s)| + |\gamma_{ijl}(s)| + |\sigma_{ijl}(s)|) \times M^f L^f]M(\|\phi - x^*\| + \epsilon)e^{-\lambda(\theta-t_0)}e^{\tau_{ij}^{+}}ds \\
&\leq K_i(\|\phi - x^*\| + \epsilon)e^{-\lambda(\theta-t_0)}e^{-\int_{t_0}^{\theta}(\widetilde{d}_i(u)-\lambda)du}
\end{aligned}
$$

$$
\begin{aligned}
&+ K_i\int_{-\infty}^{\theta} e^{\int_s^{\theta}(\widetilde{d}_i(u)-\lambda)du}(\widetilde{d}_i(u)-\lambda)dsM(\|\phi - x^*\| + \epsilon)e^{-\lambda(\theta-t_0)} \\
&= M(\|\phi - x^*\| + \epsilon)e^{-\lambda(\theta-t_0)}[(\frac{K_i}{M} - 1)\int_{-\infty}^{\theta} e^{\int_s^{\theta}(\widetilde{d}_i(u)-\lambda)du} + 1] \\
&< M(\|\phi - x^*\| + \epsilon)e^{-\lambda(\theta-t_0)},
\end{aligned}
$$

which contradicts (9). Hence, letting $\epsilon \to 0^+$, we obtain

$$|y_i(t)| \leq M\|\varphi - x^*\|e^{-\lambda(t-t_0)}.$$

Example. In this section, we present an illustrative example. Consider a 2-dimension HFCNNs with time delays. For system 1, we take activation functions $f_i(x) = \frac{1}{2}(|x+1| - |x-1|)$ $(i = 1, 2)$. Obviously, the functions $f_i(x)$ satisfy assumption (A3) with $L_i = 0.8 (i = 1, 2)$. Let

$$D = diag\{d_1(t), d_2(t)\}_{2\times 2} = \begin{pmatrix} (1 + \frac{3}{2}\sin 30t) & 0 \\ 0 & (1 + \frac{3}{2}\sin 30t) \end{pmatrix},$$

$$A = \{a_{ij}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{18}\sin t & \frac{1}{36}\sin t \\ \frac{1}{72}\sin t & \frac{1}{108}\sin t \end{pmatrix},$$

$$\alpha = \{\alpha_{ij}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{36}\sin t & \frac{1}{18}\sin t \\ \frac{1}{72}\sin t & \frac{1}{108}\sin t \end{pmatrix}, \quad \beta = \{\beta_{ij}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{18}\sin t & \frac{1}{72}\sin t \\ \frac{1}{36}\sin t & \frac{1}{108}\sin t \end{pmatrix},$$

$$B_1 = \{b_{1jl}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{24}\cos t & \frac{1}{48}\cos t \\ \frac{1}{72}\cos t & \frac{1}{96}\cos t \end{pmatrix}, \quad B_2 = \{b_{2jl}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{24}\cos t & \frac{1}{72}\cos t \\ \frac{1}{48}\cos t & \frac{1}{108}\cos t \end{pmatrix},$$

$$R_1 = \{\gamma_{1jl}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{24}\cos t & \frac{1}{96}\cos t \\ \frac{1}{72}\cos t & \frac{1}{108}\cos t \end{pmatrix}, \quad R_2 = \{\gamma_{2jl}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{24}\cos t & \frac{1}{72}\cos t \\ \frac{1}{108}\cos t & \frac{1}{120}\cos t \end{pmatrix},$$

$$Q_1 = \{\sigma_{1jl}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{24}\cos t & \frac{1}{48}\cos t \\ \frac{1}{72}\cos t & \frac{1}{96}\cos t \end{pmatrix}, \quad Q_2 = \{\sigma_{2jl}(t)\}_{2\times 2} = \begin{pmatrix} \frac{1}{24}\cos t & \frac{1}{48}\cos t \\ \frac{1}{96}\cos t & \frac{1}{108}\cos t \end{pmatrix},$$

$$T = \{T_{ij}\}_{2\times 2} = \{H_{ij}\}_{2\times 2} = H = 0, \quad I_1(t) = \sin\sqrt{2}t, \quad I_2(t) = \sin\sqrt{3}t.$$

Then, it follows that, for $L^f = 0.8$ we have $M^f = 1$; assumptions (A1)-(A4) hold. In addition, we can take $\xi_i^{-1} = 1$, $\overline{d}_i(t) = 1$, $e^{-\int_s^t d_i(u)du} \leq e^{\frac{1}{10}} e^{-(t-s)}$, $K_i = e^{\frac{1}{10}}$, $i = 1, 2$. Then, it is clear that

$$\begin{aligned}
&\sup_{t\in R}\{-1 + e^{\frac{1}{10}}[(|a_{11}(s)| + |\alpha_{11}(s)| + |\beta_{11}(s)|)0.8 \\
&+ (|b_{11l}(s)| + |\gamma_{11l}(s)| + |\sigma_{11l}(s)|)1.6\} \\
&= \sup_{t\in R}\{-1 + e^{\frac{1}{10}}[(\left|\frac{1}{18}\sin t\right| + \left|\frac{1}{36}\sin t\right| + \left|\frac{1}{18}\sin t\right|)0.8 \\
&+ (\left|\frac{1}{24}\cos t\right| + \left|\frac{1}{24}\cos t\right| + \left|\frac{1}{24}\cos t\right|)1.6\} = -0.76979686857 < -\kappa_1 = -0.5.
\end{aligned}$$

This means that, for $i, j = 1, 2$, assumption (A4) is satisfied. In the light of the this discussion,in particular case, we can conclude that (1) has a unique almost periodic solution which is global exponential stable.

References

[1] L. O. Chua, L. Yang, Cellular neural networks theory, IEEE Transactions on Circuits Systems, **5**, (1988), 1257–1272.

[2] T. Yang, L. B. Yang, Global stability of fuzzy cellular neural network, IEEE Trans. Circuits Syst. I Fundam. Theory Appl. **43**, (1996), 880–883.

[3] T. Yang, L. B. Yang, C. W. Wu, L. O. Chua, Fuzzy cellular neural networks theory, Proceedings of IEEE International Workshop on Cellular Neural Networks and Applications, **1**, (1996), 181–186.

[4] H. Bohr, On the Explicit Determination of the Upper Limit of an Almost-Periodic Function, J. London Math. Soc., **1**, (1926), 109112.

[5] C. Corduneanu, Almost Periodic Functions, J. Math. Biol., **39**, (1989), Chelsea, New York.

[6] A. M. Fink, Almost Periodic Differential Equations, Lecture Notes in Mathematics, **377**, (1974), Springer, Berlin.

[7] K. Yuan, K. D. Cao, J. M. Deng, Exponential stability and periodic solutions of fuzzy cellular neural networks with timevarying delays, Neurocomputing **69**, (2006), 1619–1627.

[8] T. Yoshizawa, Stability theory and the existence of periodic solutions and almost periodic solutions, Applied Mathematical Sciences, **14**, (1975), Springer, Berlin.

[9] M. C. Tan, Y. N. Zhang, New sufficient conditions for global asymptotic stability of Cohen-Grossberg neural networks with time-varying delays, Nonlinear Analysis Real World Applications, **10**, (2009), 2139–2145.

[10] Z. Huang, Almost periodic solutions for fuzzy cellular neural networks with time-varying delays, Neural Comput. and Appl., **28**, (2017), 2313-2320.

[11] Z. Huang, Almost periodic solutions for fuzzy cellular neural networks with multi-proportional delays, Int. J. Mach. Learn. and Cyber., **8**, (2017), 1323-1331.

[12] M. C. Tan, Dynamic analysis of delayed fuzzy cellular neural networks with time-varying coefficients, Advances in Neural Networks, **5551**, (2009), 433–439.

[13] M. C. Tan, Global asymptotic stability of fuzzy cellular neural networks with unbounded distributed delays, Neural Process Lett., **31**, (2010), 147–157.

[14] T. W. Huang, Exponential stability of delayed fuzzy cellular neural networks with distributed delay, Physics Letters A., **351**, (2006), 48–52.

[15] Q. H. Zhang, R. G. Xiang, Global asymptotic stability of fuzzy cellular neural networks with time-varying Delays, Physics Letters A., **372**, (2008), 3971-3977.

[16] C. J. Li, C.D., Li, X. F. Liao, T. W. Huang, Impulsive effects on stability of high-order BAM neural networks with time delays, Neurocomputing, **74**, (2011), 1541-1550.

[17] W. C. Deniel, J. L. Liang, L. James, Global exponential stability of impulsive high-order BAM neural networks with time-varying delays, Neural Networks, **19**, (2006), 1581-1590.

[18] F. L. Ren, J. D. Cao, LMI-based criteria for stability of highorder neural networks with time-varying delay, Nonlinear Anal., **7**, (2006), 194-210.

[19] B. J. Xu, X. Z. Liu, X. X. Liao, Global exponential stability of high order Hopfield type neural networks, Applied Mathematics and Computation, **174**, (2006), 98–116.

[20] S. Xu, M., Tan, H. Liao, Convergence behavior of high-order fuzzy cellular neural networks with time-varying delays, 25th Chinese Control and Decision Conference, (2013).

[21] Y. Xu, New results on almost periodic solutions for CNNs with time-varying leakage delays, Neural Comput. Appl., **25**, (2014), 12931302.

[22] J. L. Qiu, Dynamics of high-order Hopfield neural networks with time delays, Neurocomputing, **73**, (2010), 820-826.

[23] C. D. Zhang, H. G. Zhang, Z. S. Wang, Novel exponential stability criteria of high-Order neural networks with time varying delays, IEEE Trans. Syst. Man. Cybern. B Cybern., **41**, (2011), 486-496.

[24] H. J. Jiang, B. J. Guo, Z. D. Teng, Exponential stability of high-Order fuzzy cellular neural networks with time-varying delays, Advances on Neural Networks,**551**, (2009), 413–422.

[25] C. Tunç, On the properties of solutions for a system of non-linear differential equations of second order. Int. J. Math. Comput. Sci. **14,** no. 2, (2019), 519-534.

[26] C. Tunç, O. Tunç, On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order. Journal of Advanced Research, **7,** no. 1, (2016), 165–168.