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A note on Pythagorean Triples

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Abstract

Some relations among Pythagorean triples are established. The main tool is a fundamental characterization of the Pythagorean triples through a chatetus which allows to determine relationships with Pythagorean triples having the same chatetus raised to an integer power.

1 Introduction

Let x, y and z be positive integers satisfying

$$x^2 + y^2 = z^2.$$

Such a triple (x, y, z) is called Pythagorean triple and if, in addition, x, y and z are co-prime, it is called primitive Pythagorean triple. First, let us recall a recent novel formula that allows to obtain all Pythagorean triples as follows.

Theorem 1.1. *(x, y, z) is a Pythagorean triple if and only if there exists $d \in C(x)$ such that*

$$x = x, \quad y = \frac{x^2}{2d} - \frac{d}{2}, \quad z = \frac{x^2}{2d} + \frac{d}{2}, \tag{1.1}$$

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with x positive integer, $x \geq 1$, and where

$$C(x) = \begin{cases} D(x), & \text{if } x \text{ is odd}, \\ \\ D(x) \cap P(x), & \text{if } x \text{ is even}, \end{cases}$$

with

$$D(x) = \left\{ d \in \mathbb{N} \quad \text{such that } d \leq x \text{ and } d \text{ divisor of } x^2 \right\} ,$$

and if x is even with $x = 2^n k$, $n \in \mathbb{N}$ and $k \geq 1$ odd fixed, with

$$P(x) = \left\{ d \in \mathbb{N} \quad \text{such that } d = 2^s l, \text{ with } l \text{ divisor of } x^2 \text{ and } s \in \{1, 2, \ldots, n-1\} \right\} .$$

We want to find relations between the primitive Pythagorean triple (x, y, z) generated by any predetermined x positive odd integer using (1.1) and the primitive Pythagorean triple generated by x^m with $m \in \mathbb{N}$ and $m \geq 2$. In this paper we take care of relations only for the case in which the primitive triple (x, y, z) is generated with $d \in C(x)$ only with $d = 1$ and the primitive triple (x^m, y', z') is generated with $d_m \in C(x^m)$ only with $d_m = 1$ obtaining formulas that give us y' and z' directly from x, y, z. This is the first step to investigate on other relations between Pythagorean triples.

2 Results

The following theorem holds.

Theorem 2.1. *Let (x, y, z) be the primitive Pythagorean triple generated by any predetermined positive odd integer $x \geq 1$ using (1.1) with $z - y = d = 1$ and let (x^m, y', z') be the primitive Pythagorean triple generated by x^m, $m \in \mathbb{N}$, $m \geq 2$, using (1.1) with $z' - y' = d_m = 1$, we have the following formulas*

$$\begin{aligned} y' &= y \left[1 + \sum_{p=1}^{m-1} x^{2p} \right] , \\ z' &= y \left[1 + \sum_{p=1}^{m-1} x^{2p} \right] + 1 , \end{aligned} \tag{2.1}$$

for every $m \in \mathbb{N}$ and $m \geq 2$.

Moreover we have

$$z\left[(-1)^{m-1}+\sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right]=\begin{cases} y' & \text{if } m \text{ is even,}\\ z' & \text{if } m \text{ is odd,}\end{cases} \tag{2.2}$$

and

$$z\left[(-1)^{m-1}+\sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right]+(-1)^{m-2}=\begin{cases} z' & \text{if } m \text{ is even,}\\ y' & \text{if } m \text{ is odd.}\end{cases} \tag{2.3}$$

Proof. Let x be a positive odd integer that we consider as $x = 2n+1$, $n \in \mathbb{N}$, so that using (1.1) with $d = z - y = 1$ it gives the primitive Pythagorean triple

$$x = 2n+1, \quad y = 2n^2+2n, \quad z = 2n^2+2n+1, \tag{2.4}$$

while considering x^m, $m \in \mathbb{N}$, $m \geq 2$, using (1.1) with $d_m = z' - y' = 1$ it gives the primitive Pythagorean triple

$$x^m, \quad y' = \frac{x^{2m}-1}{2}, \quad z' = \frac{x^{2m}+1}{2}. \tag{2.5}$$

Comparing (2.4) and (2.5) we obtain

$$\begin{aligned} y' &= \frac{(2n+1)^{2m}-1}{2} = \frac{[(2n+1)^2-1]}{2}\left[(2n+1)^{2(m-1)}+(2n+1)^{2(m-2)}+\ldots+1\right] \\ &= \frac{(4n^2+4n)}{2}\left[1+\sum_{p=1}^{m-1}(2n+1)^{2p}\right] = (2n^2+2n)\left[1+\sum_{p=1}^{m-1}(2n+1)^{2p}\right] = y\left[1+\sum_{p=1}^{m-1}x^{2p}\right], \end{aligned}$$

which is the first part of (2.1), and because $d_m = z' - y' = 1$ we also obtain

$$z' = y\left[1+\sum_{p=1}^{m-1}x^{2p}\right]+1,$$

which is the second of (2.1).
Moreover, if m is odd, using (2.4) and (2.5) we obtain

$$\begin{aligned} z' &= \frac{(2n+1)^{2m}+1}{2} = \frac{[(2n+1)^2+1]}{2}\left[(2n+1)^{2(m-1)}-(2n+1)^{2(m-2)}+\ldots-(2n+1)^2+1\right] \\ &= (2n^2+2n+1)\left[1+\sum_{p=1}^{m-1}(-1)^{m-1-p}(2n+1)^{2p}\right] = z\left[1+\sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right], \end{aligned}$$

which is the second case of (2.2), and because $d_m = z' - y' = 1$ we obtain also

$$y' = z\left[1 + \sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right] - 1\,,$$

which is the second case of (2.3).
Finally, if m is even, we prove that

$$y' = z\left[-1 + \sum_{p=1}^{m-1}(-1)^{m-1-p}x^{2p}\right]\,, \tag{2.6}$$

which is the first case of (2.2) and because $d_m = z' - y' = 1$ we also obtain

$$z' = z\left[-1 + \sum_{p=1}^{m-1-p}(-1)^{m-1-p}x^{2p}\right] + 1\,,$$

which is the first case of (2.3).
To do that we use (2.4) and (2.5) to write

$$\frac{(2n+1)^{2m} - 1}{2} = (2n^2 + 2n + 1)\left[-1 + \sum_{p=1}^{m-1}(-1)^{m-1-p}(2n+1)^{2p}\right]\,, \tag{2.7}$$

and we prove that (2.7) is an identity. In fact

$$(2n+1)^{2m} - 1 = (4n^2 + 4n + 2)\left[-1 + \sum_{p=1}^{m-1}(-1)^{m-1-p}(2n+1)^{2p}\right]\,,$$

$$(2n+1)^{2m} - 1 = \left[(2n+1)^2 + 1)\right]\left[-1 + \sum_{p=1}^{m-1}(-1)^{m-1-p}(2n+1)^{2p}\right]\,,$$

$$(2n+1)^{2m} = -(2n+1)^2 + \sum_{p=1}^{m-1}(-1)^{m-1-p}(2n+1)^{2(p+1)} + \sum_{p=1}^{m-1}(-1)^{m-1-p}(2n+1)^{2p}\,,$$

$$\begin{aligned}(2n+1)^{2m} = -(2n+1)^2 &+ \Big[(-1)^{m-2}(2n+1)^4 + (-1)^{m-3}(2n+1)^6 \\ &+(-1)^{m-4}(2n+1)^8 + \ldots - (2n+1)^{2(m-1)} + (2n+1)^{2m}\Big] \\ &+\Big[(-1)^{m-2}(2n+1)^2 + (-1)^{m-3}(2n+1)^4 + (-1)^{m-4}(2n+1)^6 \\ &+(-1)^{m-5}(2n+1)^8 + \ldots - (2n+1)^{2(m-2)} + (2n+1)^{2(m-1)}\Big]\,, \end{aligned} \tag{2.8}$$

and, because m is even, after simplifying (2.8) we get

$$(2n-1)^{2m} = (2n-1)^{2m}\,,$$

so we proved that (2.7) is an identity. Therefore, (2.6) holds. Consequently, formulas (2.1), (2.2) and (2.3) have thus been proved.
Obviously, because $z-y=d=1$, we can also obtain other relations between (x,y,z) and (x^m,y',z'); for example, (2.1) is equivalent to

$$y' = z + y\sum_{p=1}^{m-1} x^{2p} - 1\,,$$

$$z' = z + y\sum_{p=1}^{m-1} x^{2p}\,.$$

Similarly, we can obtain other relations from (2.2) and (2.3). □

We illustrate formulas (2.1), (2.2) and (2.3) by the following example.

Example 2.1. We give the following table that can be extended for each primitive triples x, y, z, and x^s, y', z' with $x-y=1$ and $x'-y'=1$.
Using (2.1) we obtain

$x=3$	$y=4$	$z=5$
$x = 3^2$	$y' = 4(1+3^2) = 40$	$z' = 41$
$x = 3^3$	$y' = 4(1+3^2+3^4) = 364$	$z' = 365$
$x = 3^4$	$y' = 4(1+3^2+3^4+3^6) = 3280$	$z' = 3281$
$x = 3^5$	$y' = 4(1+3^2+3^4+3^6+3^8) = 29524$	$z' = 29525$
$x = 3^6$	$y' = 4(1+3^2+3^4+3^6+3^8+3^{10}) = 265720$	$z' = 265721$
$\vdots$	$\vdots$	$\vdots$

While using (2.2) and (2.3) we obtain

$x = 3$	$y = 4$	$z = 5$
$x = 3^2$	$y' = 5(-1 + 3^2) = 40$	$z' = 41$
$x = 3^3$	$z' = 5(1 - 3^2 + 3^4) = 365$	$y' = 364$
$x = 3^4$	$y' = 5(-1 + 3^2 - 3^4 + 3^6) = 3280$	$z' = 3281$
$x = 3^5$	$z' = 5(1 - 3^2 + 3^4 - 3^6 + 3^8) = 29525$	$y' = 29524$
$x = 3^6$	$y' = 5(-1 + 3^2 - 3^4 + 3^6 - 3^8 + 3^{10}) = 265720$	$z' = 265721$
$\vdots$	$\vdots$	$\vdots$

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