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Impulsive Jump-Diffusion Models For Pricing Securities

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Abstract

In this paper, we consider impulsive stochastic differential equations with jump, diffusion and impulse variables for pricing European call and put options using various types of strike prices. The risk involved in mortgages regarding natural disasters like cyclone, hurricane, earthquake and tsunamis etc. and the yo-yo movement of energy prices need to be hedged. Pay-off to hedge disasters are designed to predict the approximate strike price for pricing disaster pruned options. It is suggested that for effective pricing of energy and weather derivatives options impulses, jumps and diffusion are essential. The existence of strong solutions to the models established and numerical solution to the models are sought using Impulsive Euler-Maruyama and Milton numerical schemes. Monte Carlo simulation is utilized to price options at various scenarios. The criteria for mean square stability and asymptotic stability of the numerical solutions for the models used are also obtained.

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1 Introduction

Impulsive systems are now gradually finding applications in finance in form of applications of stochastic processes with impulsive states (see [11]) or stochastic differential equations with rapid jumps that act for short period of time ([13]).

In recent times, a lot of numerical algorithms are being developed for impulsive systems (see for examples [13, 15, 20, 21]); hence there are a lot of potential applications of numeric simulation for solving problems in the field of impulsive systems. Simulation of the behavior of market using impulsive stochastic differential equations is a relatively new area of research that will attract the attention of Scientists and Financial Engineers in the near future.

A decade ago, the global economy was affected by a liquidity crunch and, as a result, prediction of the state of the economy of many nations were inaccurate. This may be due to failure of some existing models that do not adequately capture the real-time behavior of the markets or national economies of many nations [11, 14, 16]. There were many arguments for the global economic recessions. Perhaps one of the most accepted reasons for such a crunch was due to the new strange phenomena that have affected markets or economies which are not adequately captured in the existing models ([11]). In view of this predicament there is dire need to explore and deploy new models to predict the market behavior effectively. New models may be deployed for pricing options that are completely different from the existing ones or modifications of the generic models.

Furthermore, some challenges faced with the fundamental assumptions often made to develop most existing financial models is the absence of arbitrage opportunity in the market especially in the organized markets. This simply means that, “transaction costs” are not allowed, “no short selling” and the underlying processes governing the models are assumed to be martingale in nature ([9,10,12,16]). In practice, sometimes, this term may creep in and affect the behavior of the underlying assets in the market. Other assumptions often made in some models which make the models redundant are “no jump” or “no impulse” takes place, whereas, in reality, jump and impulses may happen during the life circle of investments ([1, 5, 6, 11]).

It is noteworthy that the market may also be affected by jumps or shocks that happen so rapidly for short moments (impulses) ([11]) and dividends paid on bonds may be lumps over a series of time. The scenario where an insurance premium is charged for hedging, the behavior of the asset can best be explored from the impulsive point of view. Therefore, there is a need to evolve a new asset pricing mechanism to study the dynamics of assets such as forward rates and bond processes because of an impulsive nature of market.

The risk involved in mortgages and natural disasters like cyclone, hurricane, earthquake and tsunamis etc. may be hedged. Moreover, in order to develop models for effective pricing of options for energy and weather derivatives, impulses, jumps and diffusion are essential.

In the literature, there are numerous generic models in use for studying the behavior of a market using rate modeling. Let us briefly review some of them: The Black-Scholes-Merton model is a diffusion model with jump that is driven by the Poisson process. Heath-Jarrow-Morton (HJM) for modeling forward interest rate is another example of a diffusion model which is driven by a Brownian process with a risk neutral measure assumption and it is useful in the rate modeling of time structure like exchange rate of currencies ([6, 7, 17-19]).

Cox-Ingersoll-Rose (CTR) describes the evolution of interest rates and the Hull-White model is often useful in future interest rate modeling for continuous process. There are other forms of generalized rate models that incorporate the jump parameters (see [18]) and some of the models are in affine functional forms ([5]).

The motivation for this paper is to come up with a simulation framework for pricing European options using models from an impulsive family. To be precise, we will consider a class of impulsive models with underlying process driven by diffusion and jump processes. The models have potential for modeling the rate structure of an economy which depends on external or local exchange rates with sharp jump or volatility.

2 Preliminaries and Statement of the Problem

Let $(\Omega, \Im, p)$ be a filtered probability space with information generated by $\{\Im_t\}_{t\geq 0} \subset \Im$ the Brownian (Wiener) filtration; i.e., the flow of σ- algebra,$\Im_t = \sigma\left(\Im_t^0 \bigcup \aleph\right)$, where $\Im_t^0 = \sigma\left(B_s, s \leq t\right)$ and $\aleph = \{A \in \Im : p(A) = 0\}$. We will assume that the filtered probability space $(\Omega, \{\Im_t\}_{t\geq 0}, p)$ satisfies the usual condition often referred to as a stochastic basis describing the probability of uncertainty and the structure of the flow of incoming information.

Let $\Re = (-\infty, +\infty)$, $L^2(\Re) = \left\{f : \int_{\Re} |f|^2 dx < \infty\right\}$and $W^{k,2}(\Re)$ be a Sobolev space of order k defined on the subsets of the functional space

$L^2(\Re^n)$ such that the function and its weak derivative, up to some order k, have an L^2-norm that is finite. Then $W^{k,2}(\Re^n)$ admits a norm
$|f|_{k,2} = \left(\sum_{i=0}^{2} \int_R |f^{(i)}|_2^2\right)^{1/2} < \infty$, equipped with the norm $|\bullet|_{k,2}$ and $W^{k,2}(\Re)$ is a separable Banach space. $W_0^{k,2}(\Re)$ is a set in $W^{k,2}$with zero trace. By standard results in Functional Analysis $W^{k,2} = H^2(\Re)$ and $W_0^{k,2} = H_0^2(\Re)$ where $H^2(\Re)$ is the set of Fourier series in $L^2(\Re)$ whose coefficients decay sufficiently rapidly. Finally, $C([0,T] \times \Re)$ is a space of continuous functions in$[0,T]$ and have values in$\Re$. $C_0^\infty(\Re)$ will be the space of smooth functions with compact support in $\Re$ and essential supremum norm $|\cdot|_{C_0^\infty(\Re)}$

Definition 2.1. *A process $\{L(t), t \geq 0\}$ with values in $\{\Re, B(\Re)\}$ is said to be a Levy process if it is stochastically continuous, adaptive process starting almost surely from* 0, *with stationary independent increments, and cadlag trajectories defined by the classical Levy-Ito decomposition (LID) theorem as follows:*

$$L_t = Mt + Q\omega_t + \int x[N(t,dx) - tv(dx)] + \int_{|x|>1} xN(t,dx) \tag{2.1}$$

where $M\varepsilon\Re^+, Q\varepsilon\Re^+$and $\{\omega_t, t \geq 0\}$is a Poisson measure. By the same (LID) theorem, the Levy jump process can also be written as $L_t = \gamma t + \sigma\omega_t + \lambda J_{t-}$. Let $J_{t-} \sim N(t)$ be the Poisson distribution with intensityλ. γ quantifies the deterministic part of the model, σquantifies the volatility of the model and J_{t-} takes the form $J_{t-} = \sum_{s\varepsilon[0,t\},\Delta x_s \neq 0} \Delta x_s$.

A compound Poisson process is a stochastic process $J(t)$ of the form $J(t) = \sum_{i=1}^{N(t)} y_i$, where $N(t)$ is a standard Poisson process and y_i are independent and identically distributed random variables.

Let $t_{k+1} > t_k, k = 0,1,2,...$ for $k \to \infty$, is an asset at time t. This will be denoted by S_t. If lumped with the rate a_j at time $t = t_j$,then $S_{t_j} = S_{t_j^-} - a_j S_{t_{j-1}} = (1-a_j)S_t$for $0 < t_0 < t_1 < t_2 < \cdots < t_k < T$. T is the maturity time.

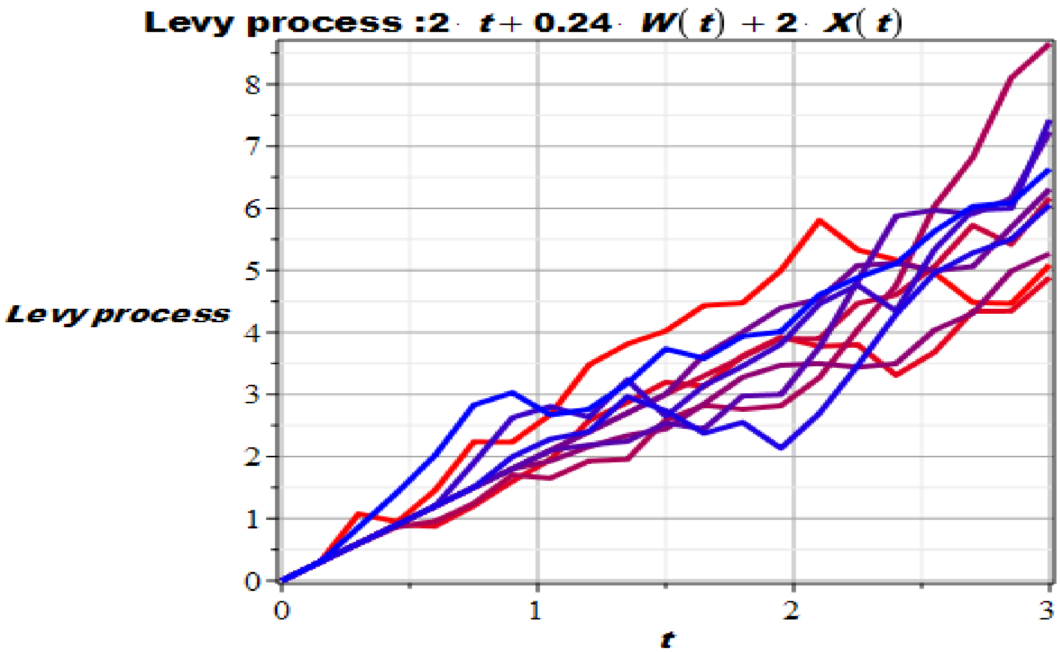

Figure 1: Levy process generated from $2t + 0.24w_t + 2X_t$

3 Impulsive Stochastic Differential Equations

Consider the impulsive stochastic differential equations (ISDES)

$$\left.\begin{array}{c} dS_t = a\left(t, S_t\right)dt + b\left(t, S_t\right)dw_t + J\left(t, S_{t^-}\right)dL_t, t \neq t_k, k = 0, 1, 2, \ldots \\ \Delta S_{t_k} = (1 - c_k)S_{t_k}, k = 0, 1, 2, \cdots, t = t_k \end{array}\right\} \tag{3.2}$$

For some $T\varepsilon\Re = (-\infty, \infty), a(.,.), b(.,.), J(.,.) \in C([0, T] \times \Re^+)$, where L_t is the Levy process and c_k are some impulsive parameters, where $a\left(t, S_t\right)$ is the drift, $b\left(t, S_t\right)$ is the diffusion, $J\left(t, S_{t^-}\right)$ is the jump and ΔS_{t_k} is the impulsive part of the impulsive stochastic differential equations and S_t is the stock price at period t. T is the maturity time for the European option.
We define the norm in $H^2(\Re)$ as $|\phi|_{H_0^2} = E[\int_0^T e^{-ct}\left|\phi_t\right|^2]^{1/2}$ for every $\phi\varepsilon H_0^2$. Clearly, by some standard results in Mathematical Analysis, we have

$$e^{-cT}\left\|\phi\right\|_{H_0^2} \leq \left\|\phi\right\|_{H_c^2} \leq \left\|\phi\right\|_{H_0^2}.$$

Lemma 3.1. ***(Doob's inequality)*** *If X is a non-negative sub-martingale,* *then*

$$p\left(\sup_{s\leq t} X_s > x\right) \leq \frac{E[X_t]}{x}$$

and for $\alpha > 1, E\left(\sup_{s\leq t} {X_s}^{\alpha}\right) \leq (\frac{\alpha}{1-\alpha})^{\alpha} E({X_t}^{\alpha})$.

Lemma 3.2. ***(Markov inequality)*** *The random variable* X *has finite expected value, then for any* $a > 0$*the probability that* $|X| \geq a$ *is bounded above by* $(E|X|)/a$*, is,* $P(|X| \geq a) \leq \frac{E|X|}{a}$.

Lemma 3.3. ***(Martingale convergence theorem)***
Let X*be a sub-martingale satisfying* $Sup_t E\left(|X_t|\right) < \infty$*then* $\sup_{t\to\infty} X_t$*exists almost surely (a. s).*

Lemma 3.4. *Let* F *be a closed subset of the Banach space* X *and* $T : F \to F$ *a contraction map. Then there exists a unique fixed point* $\overline{s} \in F$ *of* T*. In addition, suppose* s_0 *is arbitrary in F. Then* $\{Ts_t = s_{t-1}, t \in \{0, 1, 2, ..\}\} \to \overline{s}$ as $n \to \infty$*, then*
$|\overline{s} - s_t| \leq \frac{\lambda}{1-\lambda} |\overline{s} - s_0|, t \in \{0, 1, 2, \cdots\}$ *is valid, (see [4]).*

Definition 3.5. *An adaptive process* $\{X_t, t \geq 0\}$ *is said to be a martingale process if*

1. $E[\|X_t\|] < \infty$, for all t > 0.
2. $E[X_t|\Im_s] = X_t,$ for all $0 \leq s < t < \infty$

We note that a process is Martingale if it has tendency to rise and fall.

Definition 3.6. *The strong solution to the impulsive stochastic differential equation* (3.2) *is an* $\Im$*-adapted process* S_t *such that*
$\int_0^t \left(|a(t, S_t)|^2 + |b(t, S_t)|^2 + |S_t|^{2k}\right) < \infty$ *almost surely and*

$$S(t) = \left.\begin{array}{l} S_0 + \int_0^t \prod_{0<t_k<t}(1 + c_k)a(s, S_s)ds \\ + \int_0^t \prod_{0<t_k<t}(1 + c_k)b(s, S_s)d\omega_s \\ + \int_0^t \prod_{0<t_k<t}(1 + c_k)J(s, S_s)dL_s \end{array}\right\} \quad (3.3)$$

for $t\varepsilon[0, T]$, where S_{s^-} is the asset price at the jump state t^-.

4 Method

We will make use of the following methods throughout this paper:

4.1 Monte Carlos Methods

Let $S = \{s_l, l = 0.1, 2, \cdots\}$ be a sample whose elements are independent, identically distributed with statistical average $S_n = \frac{1}{n}\sum_{i=1}^{n} s_i$ such that $E\{s_i\} = s$ and the variance $\text{var}\{\text{s}_\text{i}\} = \sigma_s < \infty$. Then by the strong law of large numbers, $P(\lim_{n\to\infty} S_n) = 1$. The Monte Carlo simulation requires that $E(S_n) \to S \pm z_{\frac{\delta}{2}} \frac{1}{\sqrt{n}} S_s, z_{\frac{\delta}{2}} \sim N(0,1)$ [6, 9, 16, 17] and the rate of convergence is of order $n^{-\frac{1}{2}}$. We note that it does not depend on the dimension of the problem being solved.
The pay-off for the model is $E(\max(s_l - K, 0))$ and the European call and put option prices are:

$$\text{Call Price } C_p = e^{-rT} E(\max(s_T - K, 0)) \tag{4.4}$$

$$PutPriceP_p = e^{-rT} E(\max(0, s_T - K),), \tag{4.5}$$

where s_T is the solution of equation (3.2) evaluated at the maturity time $t = T$.
In real life option prices for pricing derivatives are pruned to natural disasters like floods, pestilence and earthquakes and yo-yo movement of energy prices. The pay-off for pricing such options should be able to predict the approximate strike prices for the disaster pruned options. A typical strike price, can be in the form of $K = k_0 + k_1 e^{-r_0 T}$, where $S_t$is the asset price, k_0, k_1 and r_0 are constants. Therefore, the call and put prices for equation (**??**) will be given by:

$$CallPriceC_p = e^{-rT} E\left(\max(s_T - k_0 + k_1 e^{-r_0 T}, 0)\right) \tag{4.6}$$

$$PutPriceP_p = e^{-rT} E\left(\max(0, s_T - k_0 + k_1 e^{-r_0 T})\right) \tag{4.7}$$

The choice of k_o, k_1 and r_0 must be made such that C_p is maximum and P_p is minimum. K can also chose to be a piecewise polynomial or even by the use of the barrier option or the use of some kind of time series representation for the strike price.
The error bound for the Euler iteration scheme ([6]) is

$$E[\sum_l |S_{nl} - S_T|] \le Ch^{-1/2}$$

where $C > 0, k$ and q are constants such that $|S_T| \leq N(1 + |T|^q)$.
Now consider the Impulsive Merton Jump Diffusion model as follows:

$$\left.\begin{array}{l} \frac{dS_t}{S_{t-}} = (r + \lambda\gamma)dt + \sigma(1 + \lambda)dw_t + \lambda dJ_{t-} \\ \Delta S_{t-} = (1 + c_i)S_{t-} \\ S_0 = S_{t=0}, S_T = S_{t=T} \\ 0 < t_0 < t_1 < \cdots < t_i < T, \end{array}\right\} \tag{4.8}$$

where r is the risk-neutral rate, σ is the volatility of assert, γ is the volatility jump size of λ and $w_t$is the Wiener process. $a(t, S_t) = (r + \lambda\gamma)$ is the drift parameter, $\sigma(t) = \sigma(1 + \lambda)$ is the volatility parameter. J_t is a compound Poisson process of the form $\sum_{t=0}^{N(t)}(J_t - 1)$ such that $\log(J_i)$ is independent and lognormal distributed with mean γ and standard deviation λ.

Black Scholes' Equation

The equation (4.8) is the standard Black Scholes' equation when $\lambda = 0$ and $\Delta S_{t-} = 0$. The solution of Black Scholes' equation at time $t = T$; i.e., at the maturity time or expiration date is
$S_T = S_0 \exp\left((r - 1/2\sigma^2) - \sqrt{T z_i}\right), z_i \sim N(0, 1)$ (See [10, 18])
The solution of impulsive Merton jump diffusion model is (See [9, 11])

$$S_T = \prod_{i=1}^{N} (1 + c_i)\, S_0 \exp(r - \frac{\gamma\sigma^2\lambda(1 + \lambda)^2 T}{2}) \exp \sigma(1 + \lambda)\sqrt{T z_i} \prod_{i=1}^{N(t)} (1 + y_i) \tag{4.9}$$

SVJJ model

Stochastic volatility with jumps (SVJJ) processes are stochastic volatility models that incorporate systematic fluctuations, or 'jumps'. We state the Impulsive SVJJ model as follows:

$$\left.\begin{array}{l} \frac{dS_t}{S_{t-}} = (r - \lambda\mu)dt + \sigma\sqrt{v_t}dw_{1t} + (J - 1)dN_t \\ dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v_t}dw_{2t} + \delta dN_t \\ \Delta S_{t-} = (1 + c_i)S_{t-} \\ S_0 = S_{t=0}, S_T = S_{t=T} \\ 0 < t_0 < t_1 < \cdots < t_i < T, \end{array}\right\} \tag{4.10}$$

where r is the risk-neutral drift, θ is the long-run mean of the variance process. κ is the speed of mean reversion of the variance process. σ is the volatility of the variance process v_t, δ is the volatility jump size, $w_t = (w_{1t}, w_{2t})$ is a two-dimensional Wiener process, N_t is a Poisson process, independent of w_t, with constant intensity λ and J is a lognormal random variable with mean α and variance β .

4.2 Numeric Approximations

To obtain numerical solution to the ISDE we will first of all discretize the interval $[0, T]$. Let $\delta t = {}^T\!/_l$ for some positive integer l and the numerical approximation to S_t be represented by S_l.

4.2.1 Stochastic integrals

Let a suitable function in the integral $\int_0^T h(s)ds = \lim\limits_{\delta t_l} \sum_{l=0}^{N-1} h(t_l)(t_{l+1} - t_l)$ be given.

We will approximate the stochastic integral as
$\int_0^t \xi_s dw_s \approx \sum \xi_l(w_{l+1} - w_l)$. The stochastic integral satisfies the isometry property $E\left(\int_0^t \xi_s dw_s\right)^2 = E\int_0^t \xi_s^2 dw_s$.

Consider the discrete points $t_l = l\delta t, l = 0, 1, 2, \cdots, N$

Let $H_k^1(t) = \int_0^t \prod_{0<t_k<t}(1+c_k)a(s, S_s)ds, H_k^1(t) = \int_0^t \prod_{0<t_k<t}(1+c_k)b(s, S_s)d\omega_s$ and

$$H_k^3(t) = \int_0^t \prod_{0<t_k<t}(1 + c_k)J(s, S_s)dL_s.$$

Therefore

$$S_t = S_0 + \int_0^t H_k^1(s)ds + \int_0^t H_k^2(s)dw_s + \int_0^t H_k^3(s)dL_s, t\varepsilon[0, T]. \qquad (4.11)$$

The numerical integral will be carried out using the Euler's integration for $\int_0^t H_k^1(s)ds$ and Stratonivich integration realization for $\int_0^t H_k^2(s)dw_s$. Therefore,

$$\left.\begin{array}{l} S_{l+1} = S_l + \sum_{l=0}^N \omega_i H_k^1(t_l)(t_{l+1} - t_l) \\ \qquad + \sum_{l=0}^N \omega_i^2 H_k^2(\frac{t_{l+1}+t_l}{2})(\frac{w_{t_{l+1}} - w_{t_l}}{2} + \Delta z)(w_{t_{l+1}} - w_{t_l}) \\ \qquad + \sum_{l=0}^N \omega_i^3 H_k^3(t_l)(L_{t_{l+1}} - L_{t_l}) + O(\tau^p) \end{array}\right\} \qquad (4.12)$$

$$\Delta z \sim N(0, \delta t/4)$$

By some standard results (for example [6, 10, 17]) it follows that
$E(\frac{w_{t_{l+1}} + w_{t_1}}{2} + \Delta z_l)(w_{t_{l+1}} - w_{t_1}) = \frac{(w_T^2 + w_0^2)}{2} + \sum_0^{N-1} \Delta z_l(w_{t_{l+1}} - w_{t_1})$ (See [7])

In addition, $E(\sum_0^{N-1} \Delta z_l(w_{t_{l+1}} - w_{t_1})) = T, \mathrm{Var}(\sum_0^{N-1} \Delta z_l(w_{t_{l+1}} - w_{t_1})^2)) = O(\delta t)$

The approximation solutions to the Black Scholes (BS) equation is

$$S_{k+1} = S_k \exp\left((r - \frac{\sigma^2}{2})h + \sigma\sqrt{h}Z_k\right), Z_k \sim N(0, 1), k = 0, 1, 2, \ldots \qquad (4.13)$$

The approximation solutions to the Impulsive Black Scholes (IBS) equation:

$$S_{k+1} = \prod_{i=k}(1+c_i)S_k \exp\left((r-\frac{\sigma^2}{2})h + \sigma\sqrt{h}Z_k\right), Z_k \sim N(0,1), k=0,1,2,\ldots \tag{4.14}$$

The approximation solutions for the Impulsive Merton Jump Diffusion (IMJD) equation:

$$S_{k+1} = \prod_{k=i}(1+c_i)S_k \exp\left((r-\lambda\gamma-\frac{\sigma^2(1+\lambda)^2}{2})h + \sigma(1+\lambda)\sqrt{h}Z_k\right)\prod_{k=1}^{N(t)}(1+y_k) \tag{4.15}$$

$Z_k \sim N(0,1), k=0,1,2,\ldots,$ where $Z_k, k=1..N$ are independent standard normal random variate with zero mean and variance one.$r$and $\sigma$are positive constants over the period$[kh, h(1+k)]$.

3.3 Impulsive Euler-Maruyama

$$S_{k+1} = S_k + a(t_k,S_k)h + b(t_k,S_k)\sqrt{hZ_k} + \left(\gamma h + \sigma\sqrt{hZ_k} + \lambda\right)N_k + \sum_k c_k\Delta S_k \tag{4.16}$$

3.4 Milstein's algorithm

$$S_{k+1} = S_k + a(t_k,S_k)h + b(t_k,S_k)\sqrt{hZ_k} + \frac{1}{2}\beta^2 S_{k-1} + \left(\gamma h + \sigma\sqrt{hZ_k} + \lambda\right)N_k + \sum_k c_k\Delta S_k \tag{4.17}$$

where $\sum_k c_k\Delta S_k$ account for the contribution of impulsive moments to the numerical approximation of the solution. The weak criterion is $\sup_x |P[\ \widehat{S}_T -S] - P[S_T - S]|$ which is the maximum discrepancy between the cumulative distribution function of the terminal values of $\widehat{S}_T$ and S.
A discretization scheme has a weak order of convergence θ if the error is of order $m^{-\theta}$ as the number of steps goes to infinity [6, 22]. The Euler scheme is of weak order 1 while the Milstein scheme is of weak order 2. That is the leading term in the estimate. $E[f(\widehat{S}\ (h))] - E(f(S_t))$is of order$h$, $f(\widehat{S})$ is the Monte Carlo estimator of $f(\widehat{S}\ (h))$. The mean square error is proportional to$C^{-2\theta/(2\theta+1)}$.

4.3 Stability of the method

Let r, γ and λ be complex numbers. If $\sigma = 0$ and $\lambda = 0$, then the model is deterministic. Therefore, $S_t = S_0 \exp rt$ and $\lim_{t\to\infty} S_t = 0$ if $Re\{r\} < 0$. If $\sigma \neq 0$

and $\lambda \neq 0$, then we consider the mean square stability and asymptotic stability of the method. Let S_0 be the initial stock price, then the solution of the equation (4.9) is asymptotic stable when $\lim\limits_{t\to\infty} S_t = 0 \Leftrightarrow Re\{r - \frac{\lambda\gamma\sigma^2(1+\lambda)^2}{2}\} < 0$. This is mean square stable if $\lim\limits_{t\to\infty} ES_t^2 = 0 \Leftrightarrow Re\{r\} + \frac{\lambda\gamma\sigma^2(1+\lambda)^2}{2}\} < 0$.

5 Major Results

We start by establishing the result on existence and uniqueness of strong solution to the equation (3.3).

Theorem 5.1. *Let $S_0 \in L^2$ be a random variable independently and identically distributed and assume that the functions $a(t,0), b(t,0) \in L^2(\Re^+)$, $J(0) \in C^\infty(\Re^+)$ and that for some $k > 0$*

$$\begin{aligned}&|a(s,S_{1t}) - a(s,S_{2t})| + |b(s,S_{1t}) - b(s,S_{2t})| + |J(s,S_{1t-}) - J(s,S_{2t-})| \\ &\leq k_1 |S_{1t} - S_{2t}|\end{aligned}$$

For all $t \in [0,T], x, y \in \Re^+, k_1 = \frac{k}{\prod_l (1+c_l)}$.
*Then, for all $T > 0$, there exists a unique strong solution to the equation (**??**) in H^2. Moreover,*

$$E[\sup_{t\leq T} |S_t|^2] \leq c(1 + E|S_0|^2)e^{cT}$$

For some constant $\xi = \xi(T,k)$ depending on T, c_k and k.

Proof
We make use of the Banach–Cacciopolis Theorem; i.e., Lemma 4 and Doob's maximum inequality. We must establish the following:

1. $T(S_t) \in H^2$ for all $S_t \in H^2$
2. T is a contraction map using the assumption that a, b and J satisfy the Lipchitz condition in the hypothesis.

We proceed as follows: Define the map T on $H^2([0,T] \times \Re^+)$ by

$$\left.\begin{aligned} T(S_t) = S_0 &+ \int_0^t \prod_{0<t_k<t}(1+c_k) a(s,S_s)ds \\ &+ \int_0^t \prod_{0<t_k<t}(1+c_k) b(s,S_s)d\omega_s \\ &+ \int_0^t \prod_{0<t_k<t}(1+c_k) J(s,S_s)dL_s \end{aligned}\right\} \tag{5.18}$$

We first of all check that $T(H^2) \subset H^2$ for $a, b, J \in H^2$ and $S_t \in H^2$. We proceed as follows:

$$\begin{aligned}|T(S_t)|^2_{H_0^2} = 4T\,|S_0|^2_{L^2} &+ 4E\left[\left|\int_0^T\int_0^t \prod_{0<t_k<t}(1+c_k)a(s,S_s)ds\right|^2 dt\right]\\ &+ 4E\left[\left|\int_0^T\int_0^t \prod_{0<t_k<t}(1+c_k)b(s,S_s)ds\right|^2 dw_s\right] \\ &+ 4E\left[\left|\int_0^T\int_0^t \prod_{0<t_k<t}(1+c_k)J(s-,S_{s-})ds\right|^2 dL_t\right]\end{aligned} \tag{5.19}$$

By the Lipchitz condition on a, b and J in S, we have

$$\begin{aligned}&|a(t,S_t)|^2 + |b(t,S_t)|^2 + |J(t-,S_{-t})|^2\\ &\leq \mathrm{k}(1 + |b(0,S_t)|^2 + |S_t|^{2k})\end{aligned}$$

It follows that,

$$E\left[\left|\int_0^T\int_0^t \prod_{0<t_k<t}(1+c_k)a(s,S_s)ds\right|^2 dt\right] \leq kE\left[1 + |b(t,0)|^2 + |S_t|^{2k}\right] < \infty,$$

$S_t \in H^2, b(t,0) \in L^2[0,T]$

Similarly,

$$E\left[\left|\int_0^T\int_0^t \prod_{0<t_k<t}(1+c_k)b(s,S_s)ds\right|^2 dt\right] \leq kE\left[1 + |b(t,0)|^2 + |S_t|^{2k}\right] < \infty,$$

$S_t \in H^2, b(t,0) \in L^2[0,T]$

$$E\left[\left|\int_0^T\int_0^t \prod_{0<t_k<t}(1+c_k)J(s-,S_{s-})ds\right|^2 dt\right] \leq kE\left[1 + |b(t,0)|^2 + |S_t|^{2k}\right] < \infty,$$

$S_t \in H^2, b(t,0) \in L^2[0,T]$. Therefore, $T(H_0^2) \subset L^2[0,T]$ and H_0^2 is closed in $L^2[0,T]$.

Furthermore,

$$\begin{aligned}E\,|T(S_{1t}) - T(S_{2t})|^2 &\leq \prod_{0<t_k<t}(1+c_k)\left[E\int_0^t |a(s,S_{1t}) - a(s,S_{1t})|^2\,ds\right]\\ &+ 3\prod_{0<t_k<t}(1+c_k)\left[E\int_0^t |b(s,S_{1t}) - b(s,S_{1t})|^2\,ds\right]\\ &+ 3\prod_{0<t_k<t}(1+c_k)\left[E\int_0^t |J(s-,S_{1t-}) - J(s-,S_{1t-})|^2\,ds\right]\\ &\leq 3(1+T)k\int_0^t E\,|S_{1t} - S_{1t}|\,ds\end{aligned}$$

Therefore, by the definition of a norm in H_0^2 we have

$$\begin{aligned}|T(S_{1t}) - T(S_{2t})|_{H_0^2} &\leq 3(1+T)k \int_0^T e^{-ct} \int_0^t E\left|S_{1s} - S_{2s}\right|^2 dtds \\ &\leq 3kT(1+T)\left|S_{1s} - S_{2s}\right|_{H_0^2}\end{aligned}$$

Then T satisfies the Lipchitz condition and it is a contraction map in H_0^2 if $3kT(1+T) < 1$. See [2, 3].
Finally,

$$E\left[\sup_{u\leq t}\left|S_u\right|^2\right]$$

$$= \prod_{t_0<t_k<t} |(1+c_k)|\, E\left[\sup_{u\leq t}\left[S_0 + \int_0^u a(s,S_s)ds + \int_0^u b(s,S_s)dw_s + \int_0^u J(s-,S_{s-})dL_s\right]\right]$$

$$\leq 3\prod_{t_0<t_k<t} |(1+c_k)| \left[E\left|S_0\right|^2 + E\left[\int_0^T \left|a(s,S_s)\right|^2 ds\right] + E\left[\int_0^T \left|b(s,S_s)\right|^2 dw_s\right] + E\left[\sup_{u\leq t}\int_0^T \left|S_s\right)\right|^2 ds\right]\right]$$

Therefore, by Doob maximum inequality,

$$E\left[\sup_{u\leq t}\left|S_u\right|^2\right] \leq \xi(k,T)\left(1 + E\left|S_0\right| + \int_0^u E\left[\sup_{u\leq s}\left|{S_u}^2\right|\right] ds\right)$$

By the impulsive analogue of the Gronwall-Bellman inequality, (See [13]) we have

$$E\left[\sup_{u\leq t}\left|S_u\right|^2\right] \leq \xi(k,T)(1 + E\left|S_0\right|^2)e^{cT} < \infty$$

Therefore, by Banach-Cacciopolis Theorem (Lemma 2.4), there exists a unique fixed point S_t to the operator in equation (5.18) such that if $\{S_t^n\}$ is a sequence of asset points and S_0 is arbitrary, then $\{TS_t = S_{t-1}, t \in \{0,1,2,\cdots\}\} \xrightarrow{P} S_t$ and $\left|\overline{S} - S_t\right| \leq \frac{\xi(k,T)}{1-\xi(k,T)}\left|\overline{S} - S_0\right|, t \in \{0,1,2,\cdots\}, \xi(k,T) < 1$.
It is not difficult to show that the fixed point is in fact the strong solution to the ISDE. Note that the convergence is convergence in probability.
This ends the proof.
Figure 2 Monte Carlo simulation for stock path for the Black Scholes model when the initial stock price is $S_0 = 100$ dollars, rate price $r = 0.15$, market volatility, $\sigma = 0.2$ and 10^5 replications.

Figure 3 and 4 are the Monte Carlo simulation for stock path at different scenarios for the Black Scholes model and impulsive Black Scholes model respective.

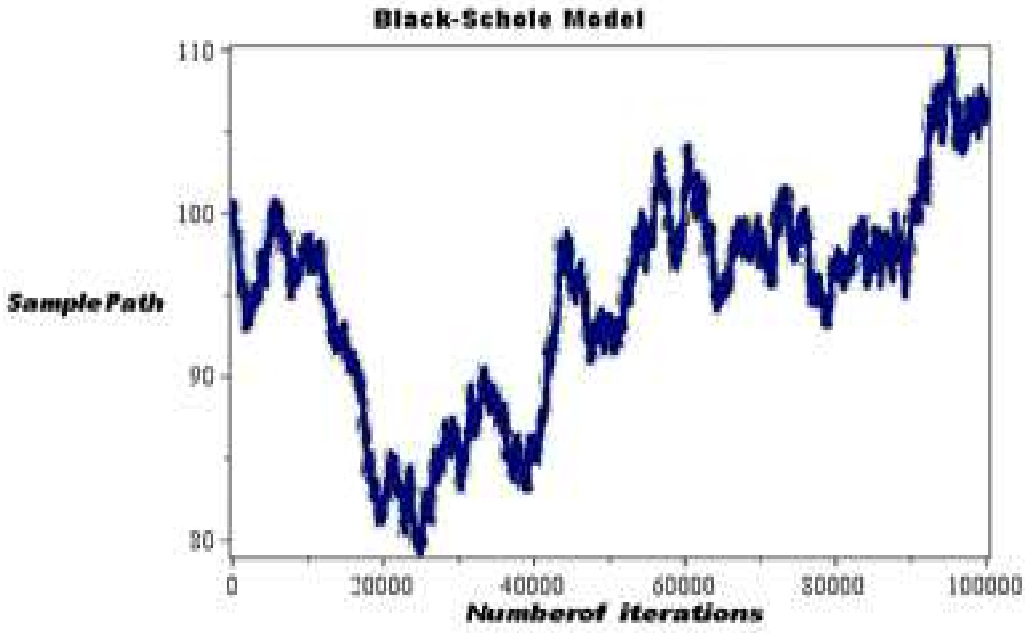

Figure 2: Monte Carlo simulation for stock path for the Black Scholes model for $S_0 = 100, r = 0.15, \sigma = 0.2$.

5.1 Examples

We will compute the call and put prices of European options using the Black Scholes model in the financial toolkit in the Maple 2018 using the following market parameters $S_0 = 100, r = 0.3, \sigma = 0.05, d = 0.02, c_i = 0.02, T = 1.00, K = 100$. Invoking the Maple command
> Black Scholes Price $(S_0, K, T, \sigma, d,' call')$ we found that the call price is 13.61 dollars. The put price for same market parameter is obtained by invoking
> Black Scholes Price $(S_0, K, T, \sigma, d,' put')$ and the put price is 9.73 dollars. To Find the call price and put price for impulsive Black Scholes model, we make use of $\prod_k (1 + c_i) \approx \exp -(c_k)^k$.
Monte Carlos simulation for the Call and put prices for Impulsive Black Scholes model can be obtained using modified Maple code:S := Black Scholes Process $(S_0.exp(-k^N), \sigma, r, d)$;

Using the code:DiscountFactor(r,T) ExpectedValue(max(S(T)-K,0),timesteps =100,replications= 10^5, output=value);

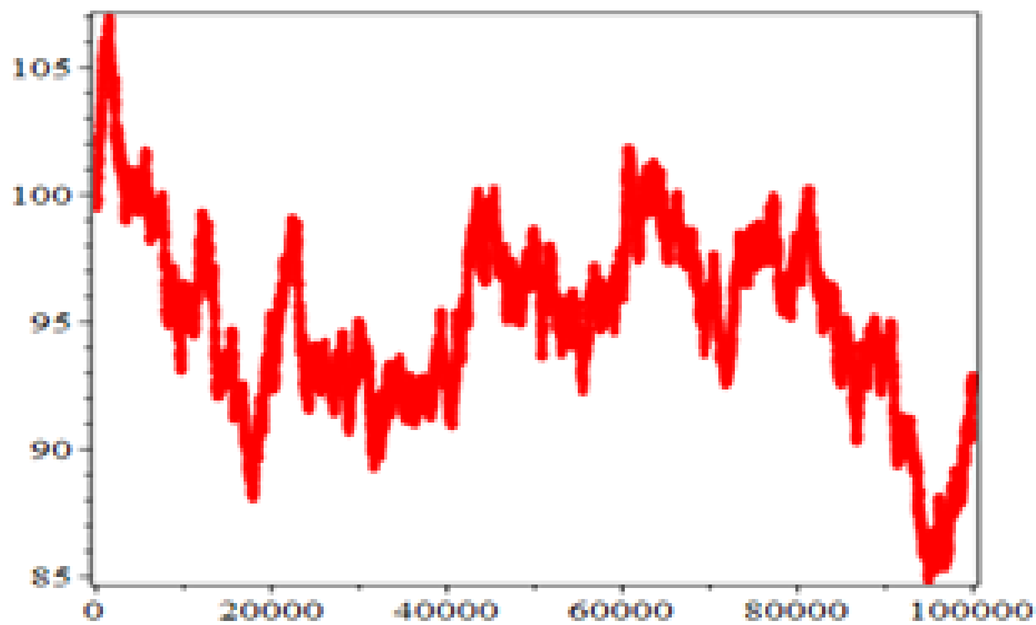

Figure 3: Monte Carlo simulation for stock path for the Black Scholes model for $S_0 = 100, r = 0.3, \sigma = 0.05$

The call price is found to be 13.7069.
Using the Maple code DiscountFactor(r,T) ExpectedValue(max(S(T)-K,0),

timesteps=100,replications= 10^5, output=value);
The Put Price is found to be 9.81 US dollars.
For $S_0 = 100, r = 0.02, \sigma_1 = 0.02, d = 0.02, a = 0.1, b = 0.25, \lambda_1 = 2.0$and $\lambda_2 = 2.0$. The numeric simulation of stock price carried out using the Euler approximation is shown in the Fig.5. The trajectory of the solution path to the Merton jump diffusion (MJD) model with jumps in the interval $[0, 1]$ is displayed in the Fig.6. The corresponding solution path to the impulsive Merton jump diffusion (IMJD) model is shown in the Fig.7.

We observed that the initial stock price was $S_0 = 100$dollars for MJD model and the price continuously far increase above S_0 (see Fig.5) for some jumps in the interval $[0, 1]$. The price for IMJD model goes below 100 dollars and even as far as 70 dollars (see Fig.6).
Figure 8 is the Monte Carlo Simulation for Impulsive Merton jump diffusion model with 20000 replications using the parameters:$S_0 = 100, r = 0.02, \sigma_1 =$

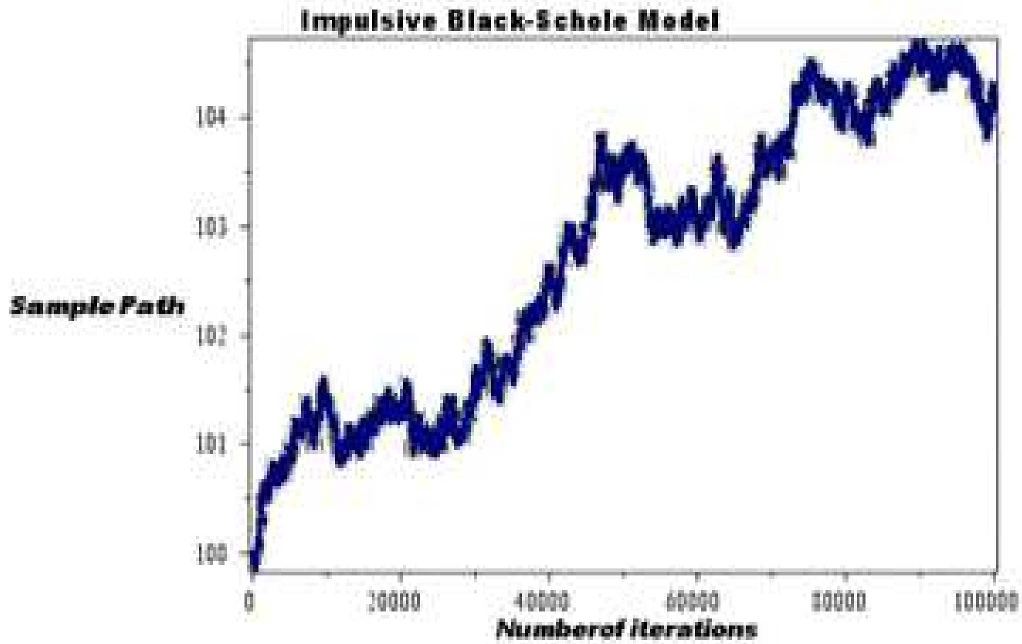

Figure 4: Monte Carlo simulation for stock path for the Impulsive Black Scholes model for $S_0 = 100, r = 0.3, \sigma = 0.05, c_i = 0.02, T = 1.00, K = 100.$

$0.02, d = 0.02, a = 0.1, b = 0.25, \lambda_1 = 2.0.$

5.2 Linear difference equations

Let $a(t_k, S_k) = rS_k, b(t_k, S_k) = \sigma S_k$ using linear difference equations, we will derive solution and carry out stability analysis for Impulsive Euler-Maruyama method (IEMM). Therefore, IEMM becomes:

$$\begin{aligned} S_{k+1} &= S_k + rS_k + \sigma\sqrt{hZ_k}S_k + (\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h S_{k-} + \textstyle\sum_k c_k S_{k-} \\ &= \left(1 + r + \sigma\sqrt{hZ_k} + (\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h\right) S_k + \textstyle\sum_k c_k S_{k-} \\ &= A^k S_0 + \textstyle\sum_k A^{k-1} c_k S_{k-}. \\ A &:= \left(1 + r + \sigma\sqrt{hZ_k} + (\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h\right) \end{aligned}$$

For Milstein method (MM)

$$\begin{aligned} S_{k+1} &= S_k + rS_k + \sigma\sqrt{hZ_k}S_k + \tfrac{1}{2}\beta^2 S_k + (\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h S_{k-} + \textstyle\sum_k c_k S_{k-} \\ &= \left(1 + r + \sigma\sqrt{hZ_k} + \tfrac{1}{2}\beta^2 + (\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h\right) S_k + \textstyle\sum_k c_k S_{k-} \\ &= B^k S_0 + \textstyle\sum_k B^{k-1} c_k S_{k-}. \\ B &:= \left(1 + r + \sigma\sqrt{hZ_k} + \tfrac{1}{2}\beta^2 + (\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h\right) \end{aligned}$$

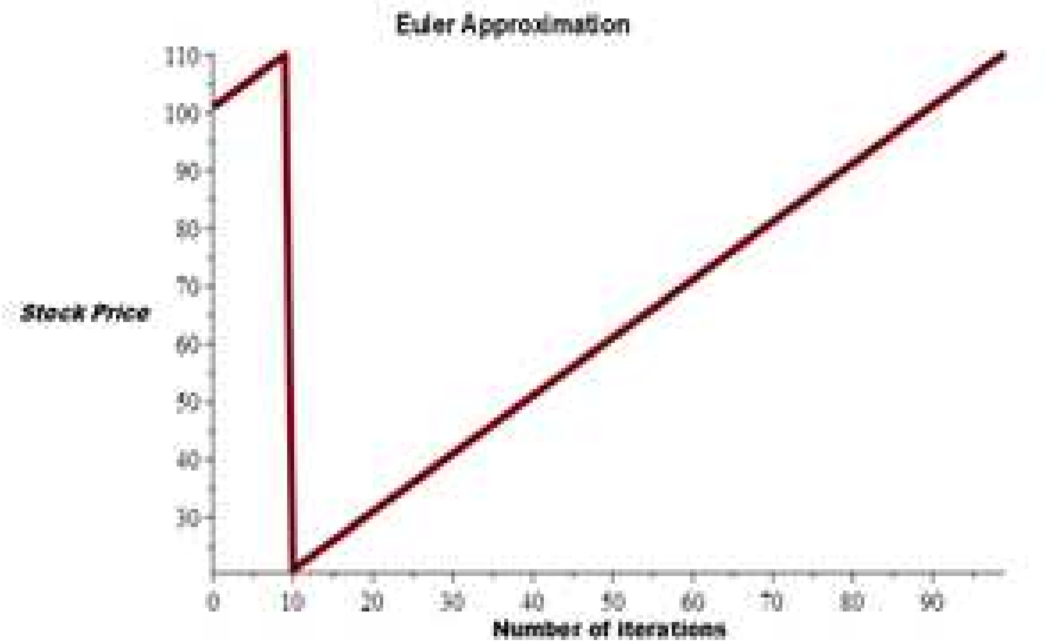

Figure 5: Euler approximation

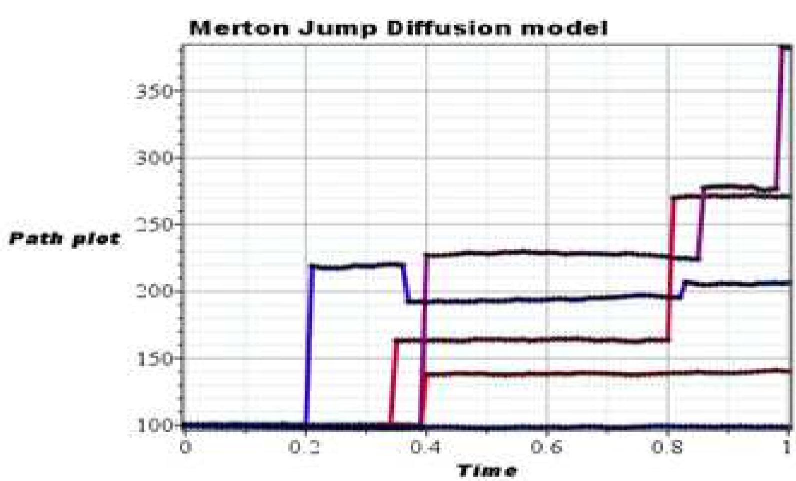

Figure 6: The trajectory of stock price of Merton jump diffusion model

5.3 Stability Analysis for IEMM and MM

We will determine criteria for stability for IEMM and MM methods. We note that IEMM will be asymptotical stable whenever $Re\{A\} < 0, \lim_{k\to\infty} \sum_k c_k S_{k-}$

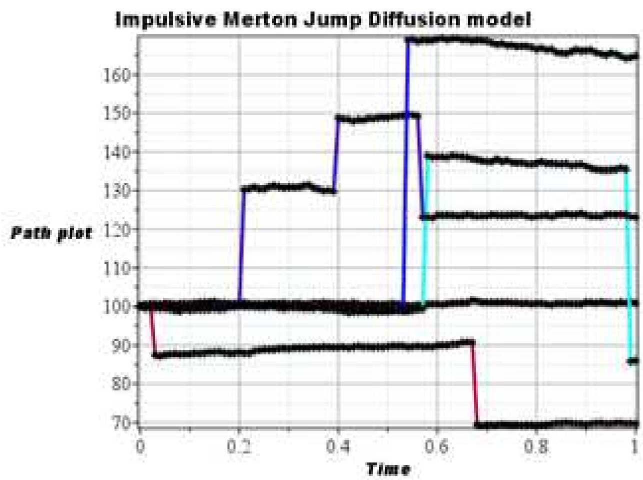

Figure 7: The trajectory of stock price of impulsive Merton jump diffusion model

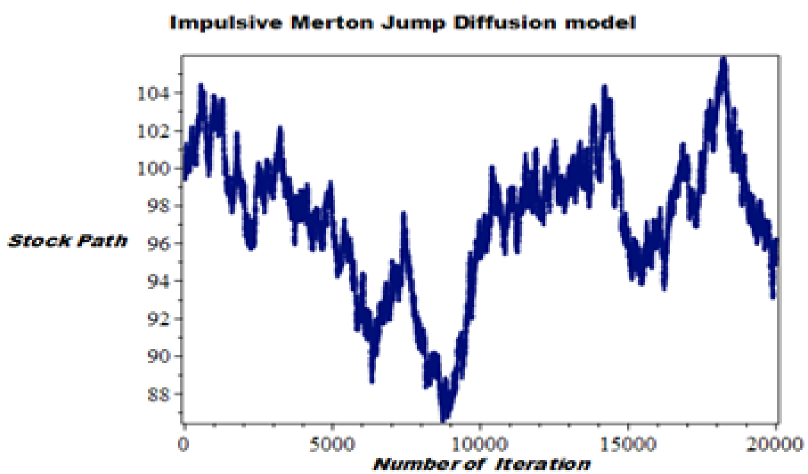

Figure 8: Monte Carlo Simulation for Impulsive Merton jump diffusion model

for $MM, Re\{B\} < 0, \lim_{k\to\infty} \sum_k c_k S_{k-}$. We find out that IEMM will be mean square stable $\left(\left|1 + r + \sigma\sqrt{hZ_k}\right|^2 + \left|(\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h\right|^2 \right) < 1, \lim_{j\to\infty} \sup |c_j| =$ 0 and MM will be mean square stable if

$$\left(\left|1 + r + \sigma\sqrt{hZ_k} + \frac{1}{2}\beta^2\right|^2 + \left|(\gamma h + \sigma\sqrt{hZ_k} + \lambda)N_h\right|^2 \right) < 1, \lim_{j\to\infty} \sup |c_j| = 0.$$

When the range of values of h for the above condition are satisfied, it gives the region of the stability of the method.

6 Conclusion

Impulsive system theory is extended to stochastic system with application to pricing European options for financial derivatives using different types of pay-off. In real life we noted that there is the need to evolve some class of option pricing natural disasters derivatives such as flooding, pestilence and earthquakes and yo-yo movement of energy prices. We make use of some pay-off that should able to predict the approximate strike price for pricing disaster derivatives. The existence of strong solutions to the models established and numerical solutions to the models was established using Impulsive Euler-Maruyama and Milton numerical schemes. Monte Carlo simulation was utilized to obtain the price options at various scenarios.

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