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The zero sum Ramsey numbers of the join of two graphs

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Abstract

We present an upper bound of the zero sum Ramsey number of the join of a pair of graphs. Also, we use this result to find the exact value of the join of a matching with regular graph with small Ramsey number.

1 Introduction

Let G be a simple graph with finite vertex set $V(G)$ and edge set $E(G)$. The order of G, written as n, denotes the number of vertices. The graph K_n, the *complete graph* of n vertices, is a graph where vertices are pairwise adjacent. Let $\mathbb{Z}_q$ be the cyclic additive group of order q. A $\mathbb{Z}_q$-coloring of the edges of a graph G is a function $f : E(G) \to \mathbb{Z}_q$. If $\sum_{e\in E(G)} f(e) = 0$, we say that G is a *zero sum* graph with reference to f and q. The *zero sum Ramsey number of G with respect to $\mathbb{Z}_q$*, denoted by $R(G, \mathbb{Z}_q)$, is the smallest integer t such

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that for each $\mathbb{Z}_q$-coloring of the edges of the complete graph K_t, there is a subgraph G' in K_t isomorphic to G which is zero sum with reference to q and the restriction of the given coloring to $E(G')$. The graph G' is called a *zero sum copy* of G. $R(G, \mathbb{Z}_q)$ exists if and only if $q \mid |E(G)|$ because if $q \not| \; |E(G)|$ then for any t the constant coloring $f : E(K_t) \to \mathbb{Z}_q$ given by $f(e) \equiv 1$ avoids a zero sum copy (mod q) of G. On the other hand, suppose that q divides $|E(G)|$. By Ramsey's theorem, for all t sufficiently large, for any coloring of $E(K_t)$ with q or fewer colors there will be a monochromatic copy of G in k_t: if that constant color on the edges of that copy is an element of $\mathbb{Z}_q$ then clearly the sum of weights on the edges of the copy is zero mod q. Thus $R(G, \mathbb{Z}_q)$ is well defined when q divides $|E(G)|$.

Zero sum Ramsey graph theory, which was started by Bialostocki and Dierker in 1990 [1], concerns coloring the edges of complete graphs with the elements of a given group, and the appearance of zero sum substructures (instead of monochromatic substructures). For a reference about most of the zero sum Ramsey results in the context of graph theory and comprehensive literature see the survey by Caro [3] and the ongoing survey by Radziszowski [5]. A connection with a new invariant in graph theory can be found in Fujita at al [4].

In this paper, we study the zero sum Ramsey number for a join of graphs. Let G_i be any graph with vertex set V_i and edge set E_i, $i = 1, 2$. The *join* $G = G_1 + G_2$ has the vertex set $V_1 \cup V_2$ and the edge set

$$E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}.$$

Furthermore we calculate the zero sum Ramsey number $R(G+M, \mathbb{Z}_q)$ where G is a regular graph and M is a matching, both with number of edges multiple of q.

2 Zero sum Ramsey number of join of graphs

In this section we show our main result which consists in showing an upper bound of the zero sum Ramsey number for the join of graphs.

Theorem 2.1. *Let G be a k-regular graph of n vertices. If $q \in \mathbb{Z}^+$, such that q divides $|E(G)|$ with k and q coprime, then for any graph H with $|E(H)|$ a multiple of q and $R(H, \mathbb{Z}_q) - |V(H)| \leq R(G, \mathbb{Z}_q)$ we have*

$$R(G + H, \mathbb{Z}_q) \leq R(G, \mathbb{Z}_q) + |V(H)|.$$

Proof: Let G be a k-regular graph of order n and let q be a positive integer coprime with k such that $q|\frac{nk}{2}$. Let H be a graph of m vertices, with $|E(H)|$ multiple of q. Let us denote by N and M the zero sum Ramsey numbers of G and H, respectively. Let $f : E(K_{N+m}) \to \mathbb{Z}_q$ be a $\mathbb{Z}_q$-coloring of K_{N+m}. We have to show that a zero sum copy of $G + H$ exists. By hypothesis, $N + m \geq R(H, \mathbb{Z}_q)$. Hence by definition of $R(H, \mathbb{Z}_q)$ there exists a zero sum (mod q) copy of H and we denote its vertex set by H'. For $K = K_{N+m} \setminus H'$, we define a new $\mathbb{Z}_q$-coloring, $g : E(K) \longrightarrow \mathbb{Z}_q$ as follows: If $e = (u, v)$ is an edge of K, then

$$g(e) = f(e) + k^{-1}(a(u) + a(v)),$$

where $a(u)$ and $a(v)$ are the sums mod q of the f-weights of the edges connecting u, v to H' and k^{-1} is the multiplicative inverse of k in $\mathbb{Z}_q$. Since $|V(K)| = N$, by definition of $R(G, \mathbb{Z}_q)$, there exists a zero sum copy of G with respect to g and we denote its vertex set by G'. We have an equation with coloration f.

$$0 = \sum_{e \in E(G')} g(e) = \sum_{e \in E(G')} [f(e) + k^{-1}(a(u) + a(v))].$$

Therefore

$$\sum_{e \in E(G+H)} f(e) = \sum_{e \in E(H')} f(e) + \sum_{e \in E(G')} f(e) + \sum_{u \in V(G')} a(u).$$

Recollect that G is regular of degree k. We have

$$\sum_{uv = e \in E(G')} [a(u) + a(v)] = k \sum_{u \in V(G')} a(u).$$

Operating we have

$$\begin{aligned} \sum_{e \in E(G+H)} f(e) &= \sum_{e \in E(H')} f(e) + \sum_{e \in E(G')} f(e) + k^{-1} \sum_{e \in E(G')} [a(u) + a(v)] \\ &= \sum_{e \in E(H')} f(e) + \sum_{e \in E(G')} g(e) \equiv 0 \pmod{q}. \end{aligned}$$

■

Corollary 2.2. *Let n, k, t, q be positives integers such that n, t are multiples of q and k is a coprime with q. If G is a k-regular graph on n vertices and tK_2 is a matching with t edges, then*

$$R(G + tK_2, \mathbb{Z}_q) \leq R(G, \mathbb{Z}_q) + 2t. \tag{2.1}$$

Proof: For t a multiple of q, we have $R(tK_2, \mathbb{Z}_q) = 2t + q - 1$ by [2]. Thus, $R(tK_2, \mathbb{Z}_q) - |tK_2| = q - 1$. Let us note that $R(G, \mathbb{Z}_q) \geq |G| = n$ for all graph G. Since n is multiple of q, then

$$R(tK_2, \mathbb{Z}_q) - |tK_2| \leq R(G, \mathbb{Z}_q).$$

Thus, by Theorem 2.1, for $H = tK_2$, we have (2.1). ■

Theorem 2.3. *Let n, k, t, q be positive integers satisfying $q > 2$, q is a co-prime with k such that does not divide $2k - 1$ and n, t are multiples of q. Let tK_2 be a matching with t edges. If G is a k-regular graph on n vertices with $R(G, \mathbb{Z}_q) = n + 2$, then*

$$R(G + tK_2, \mathbb{Z}_q) = n + 2t + 2.$$

Proof: First, let us observe that $R(G + tK_2, \mathbb{Z}_q) \leq R(G, \mathbb{Z}_q) + 2t$ by Corollary 2.2. It remains to show $R(G + tK_2, \mathbb{Z}_q) \geq R(G, \mathbb{Z}_q) + 2t$. Let $m = n + 2t + 1$. Let us construct a $\mathbb{Z}_q$-coloring of the edge of complete graph K_m without zero sum copy of $G + tK_2$.

For $u, v \in V(K_m)$, let $f : E(K_m) \to \mathbb{Z}_q$ be the coloration defined for each e as:

$$f(e) = \begin{cases} 1 & \text{if } u \text{ or } v \text{ is incident to } e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Suppose that there exists a zero sum copy of $G + tK_2$ in K_m. Let us denote the copies of G and tK_2 by M_1, M_2, respectively. Let A be the set of edges with one endpoint in M_1 and the other one in M_2. Let us note that

$$E(G + tK_2) = E(G) \cup A \cup E(tK_2).$$

The value of $\sum_{e \in E(G+tK_2)} f(e)$ depends on u, v. In what follows we will use the convention that the name of a graph can also stand for its vertex set. For instance, we may state that $|M_2| = 2t$.

Case 1. For $u, v \in M_2$: We see that $\sum_{e \in E(G)} f(e) = 0$, because neither u nor v is incident with any edge in G. On the other hand, there are $2n$ edges in A with f-weight 1 and the others have f-weight 0. Since n is a multiple of q, it follows that

$$\sum_{e \in A} f(e) = 2n \equiv 0 \pmod{q}.$$

Hence

$$\sum_{e \in E(G+tK_2)} f(e) \equiv \sum_{e \in E(tK_2)} f(e) \pmod{q}.$$

Since M_2 is a matching, then $\sum_{e\in E(tK_2)} f(e) = 1$ if the nodes u, v are incident with e, otherwise this sum equals 2.

Case 2. For $u, v \in M_1$: This situation is similar to the one in Case 1, the sum over $E(G + tK_2)$ is congruent mod q to sum over $E(G)$. Since G is k-regular, then

$$\sum_{e\in E(G)} f(e) = \begin{cases} 2k-1 & \text{if } u \text{ and } v \text{ are adjacent in } M_1,\\ 2k & \text{otherwise.} \end{cases}$$

Since q is a coprime with k such that does not divide $2k - 1$ and $q > 2$, then neither $2k - 1$ nor $2k$ is equal to zero mod q.

Case 3. For $u \in M_1$ and $v \in M_2$: We can assert that $\sum_{e\in E(tK_2)} f(e) = 1$ and $\sum_{e\in E(G)} f(e) = k$. It is straightforward that

$$\sum_{e\in A} f(e) = n + 2t - 1.$$

Then

$$\sum_{e\in E(G+tK_2)} f(e) = k + n + 2t - 1 + 1 \equiv k \not\equiv 0 \pmod q.$$

Case 4. If $v \in K_m \backslash (M_1 \cup M_2)$ then $u \in M_1 \cup M_2$, because $|K_m \backslash (M_1 \cup M_2)| = 1$. In A, the edges adjacent to u have f-weight 1 and the remain edge in A has f-weight 0. If $u \in M_2$, then

$$\sum_{e\in E(G+tK_2)} f(e) = 1 + n \equiv 1 \pmod q,$$

on the other hand, if $u \in M_1$, we have

$$\sum_{e\in E(G+tK_2)} f(e) = k + 2t \equiv k \pmod q.$$

In conclusion, there exists a $\mathbb{Z}_q$-coloring of K_{n+2t+1} that does not admit a zero sum copy of $G + tK_2$. ■

Now we now use Theorem 2.3 to calculate the exact value of the zero sum Ramsey number with respect to $\mathbb{Z}_3$ of the join of two matchings.

Corollary 2.4. *If t, s are positive integer multiples of 3, then*

$$R(tK_2 + sK_2, \mathbb{Z}_3) = 2t + 2s + 2$$

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