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The q-analogue of Sigmoid Function in the space of univalent λ-Pseudo star-like Functions

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Abstract

In the present investigation, a modified q-sigmoid function is defined and the Fekete-Szegö coefficient functional $|a_3 - \mu a_2^2|$ for certain normalized analytic functions f defined on the open unit disk. As an application of the main result, we pointed out the initial coefficients and Fekete-Szegö inequality for a subclasses of starlike functions related to sigmoid functions.

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1 Introduction and Preliminaries

The theory of hypergeometric function may be generalized along the lines of q-numbers, resulting in the formulation of q-analogues. Heine $(1821 - 1881)$ introduced the hypergeometric function of the form:-

$$1+\frac{(q^{\alpha}-1)(q^{\beta}-1)}{(q-1)(q^{\gamma}-1)}z+\frac{(q^{\alpha}-1)(q^{\alpha+1}-1)(q^{\beta}-1)(q^{\beta+1}-1)}{(q-1)(q^{2}-1)(q^{\gamma}-1)(q^{\gamma+1}-1)}z^{2}+\cdots,$$

(see [9], [10]) which was subsequently extended by Jackson (1870 - 1960), Bailey (1893 - 1961), Slater, Andrews, Agrawal and many others up to the present day.

Due to the potential usefulness of q-series and q-polynomial in a wide range of variety of fields, the q-calculus is a part of the theory of special function in which many authors had studies the q-analogue of some special functions like q-Gamma, q-Beta, q-Bernoulli, q-Zeta functions and so on.

Meanwhile, the Sigmoid function which is an example of activation function and a special function is gaining much attention in the recent years.

Authors like [11],[8],[6], [5],[4] and [12] have developed results connecting sigmoid functions and univalent functions theory. All these gave us the impetus to define a modified q-sigmoid function and calculate the initial coefficients of λ-pseudo related to q-sigmoid function and establish the Fekete-Szegö functional.

Let A be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}a_{n}z^{n}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=[z\in\mathbb{C}:|z|<1]$ satisfying the condition $f(0)=0$ and $f'(0)=1$. Furthermore, let S denote the family of all functions in A which are univalent in $\mathbb{U}$. For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write

$$f(z)\prec g(z),$$

such that $z\in\mathbb{U}$ if there exists a Schwartz function $w(z)$ analytic in $\mathbb{U}$ with $w(0)=0$ and $|w(z)|<1 \quad (z\in\mathbb{U})$ such that $f(z)=g(w(z))$.

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ [7].

Babalola [2] defined a new subclass λ-pseudo starlike function of order $\beta (0 \leq \beta < 1)$ satisfying the analytic condition

$$\Re\left(\frac{z(f'(z))^{\lambda})}{f(z)}\right) > \beta \qquad (z \in \mathbb{U}, \lambda \geq 1 \in \mathbb{R}) \tag{1.2}$$

and denoted by $L_{\lambda}(\beta)$.

Murugusundaramoorthy [8] denoted the class of λ-pseudo starlike functions satisfying the condition (1.2) and related with the sigmoid function $L_{\lambda}^{\beta}(\phi)$ where

$$\phi(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right]^m . \tag{1.3}$$

Here, we denote the class of λ-pseudo starlike functions in relation to $q-$Sigmoid function by $L_{\lambda}^{\beta}(\gamma_q)$.

Definition 1.1. *[6], [8] The sigmoid function is defined as*

$$G(z) = \frac{1}{1+e^{-z}} = \frac{1}{2} + \frac{z}{4} - \frac{z^3}{48} + \frac{z^5}{480} - \frac{17z^7}{80640} + \dots . \tag{1.4}$$

The modified sigmoid function is of the form

$$\gamma(z) = \frac{2}{1+e^{-z}} = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{17z^7}{40320} + \dots .$$

Definition 1.2. *[1], [3] For any fixed real number $q > 0$, non-negative integer r, the q-integers of the number r is defined by*

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q}, & q \neq 1 \\ r, & q = 1 \\ 0, & r = 0 \end{cases}$$

Definition 1.3. *[1],[3] The q-fractional is defined in the following:*

$$[r]_q! = \begin{cases} [r]q[r-1]q \cdots [1]q & \\ 1, & r = 0. \end{cases}$$

Definition 1.4. *[1],[3] A q-analogue of the ordinary exponential function* $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ *is defined by*

$$e_q^z := \sum_{k=0}^{\infty} \frac{z^k}{[k]q!}.$$

Definition 1.5. *[1],[3] The q-Bernoulli numbers* $\mathfrak{B}_{k,q}$ *is defined by the generating function*

$$\frac{z}{e_q^z - 1} = \sum_{k=0}^{\infty} \mathfrak{B}_{k,q} \frac{z^k}{[k]q!}, \qquad |z| < 2\pi.$$

Motivated by the above definitions, we have

Definition 1.6. *The q-Sigmoid function*

$$G_q(z) = \frac{1}{1 + e_q^{-z}}. \tag{1.5}$$

Definition 1.7. *The modified q-Sigmoid function*

$$\gamma_q(z) = \frac{2}{1 + e_q^{-z}} = 1 + \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{n=1} \frac{(-1)^n}{[n]_q!} z^n \right]^k \right). \tag{1.6}$$

2 Main results

Theorem 2.1. *Let* $f \in L_\lambda^\beta(\gamma_q)$, $(\lambda \geq 1 \in \mathbb{R})$, *then*

$$|a_2| \leq \frac{1-\beta}{(2\lambda - 1)2[1]_q!}$$

and

$$|a_3| \leq \frac{(1-\beta)([2]_q! - 2([1]_q!)^2)(2\lambda-1)^2 + (1-\beta)^2[2]_q!(4\lambda - 2\lambda^2 - 1)}{4([1]_q!)^2[2]_q!(2\lambda-1)^2(3\lambda-1)}.$$

Proof 2.1. *Suppose* $f \in L_\lambda^\beta(\gamma_q)$. *By definition, there exists* $\gamma_q(z) \in P$ *such that*

$$\frac{z(f'(z))^\lambda}{f(z)} = \beta + (1-\beta)\gamma_q(z)$$

where the function $\gamma_q(z)$ is the modified q-Sigmoid function given by

$$\gamma_q(z) = 1+\frac{1}{2[1]_q!}z+\left(\frac{1}{4([1]_q!)^2}-\frac{1}{2[2]_q!}\right)z^2+\left(\frac{1}{2[3]_q!}-\frac{1}{2[1]_q![2]_q!}+\frac{1}{(2[1]_q!)^3}\right)z^3+\dots.$$

Thus,

$$z(f'(z))^\lambda = f(z)[\beta+(1-\beta)\gamma_q(z)]$$

We have

$$z+2\lambda a_2z^2+[3\lambda a_3+2\lambda(\lambda-1)a_2^2]z^3+[4\lambda a_4+6\lambda(\lambda-1)a_2a_3+\frac{4}{3}\lambda(\lambda-1)(\lambda-2)a_2^3]z^4+\dots$$

$$= z+\left(a_2+\frac{1-\beta}{2[1]_q!}\right)z^2+\left(\frac{1-\beta}{4([1]_q!)^2}-\frac{1-\beta}{2[2]_q!}+\frac{a_2(1-\beta)}{2[1]_q!}+a_3\right)z^3$$

$$+\left(\frac{1-\beta}{2[3]_q!}-\frac{1-\beta}{2[1]_q![2]_q!}+\frac{1-\beta}{(2[1]_q!)^3}+\frac{a_2(1-\beta)}{4([1]_q!)^2}-\frac{a_2(1-\beta)}{2[2]_q!}+\frac{a_3(1-\beta)}{2[1]_q!}+a_4\right)z^4+\dots.$$

Comparing the coefficients of z^2 and z^3, we obtain

$$a_2 = \frac{1-\beta}{(2\lambda-1)2[1]_q!} \tag{2.1}$$

and

$$a_3 = \frac{(1-\beta)([2]_q!-2([1]_q!)^2)(2\lambda-1)^2)+(1-\beta)^2[2]_q!(4\lambda-2\lambda^2-1)}{4([1]_q!)^2[2]_q!(2\lambda-1)^2(3\lambda-1)}. \tag{2.2}$$

Theorem 2.2. *Let $f \in L_\lambda^\beta(\gamma_q)$ and $\mu \in \mathbb{R}$. Then*

$$|a_3-\mu a_2^2| \le \frac{(1-\beta)^2}{4([1]_q!)^2(2\lambda-1)^2}\left|\frac{([2]_q!-2([1]_q!)^2)(2\lambda-1)}{[2]_q!(3\lambda-1)(1-\beta)}+\frac{4\lambda-2\lambda^2-1}{3\lambda-1}-\mu\right|.$$

Proof 2.2. *From (2.1) and (2.2) we have*

$$a_3-\mu a_2^2 = \frac{(1-\beta)([2]_q!-2([1]_q!)^2)(2\lambda-1)^2+(1-\beta)^2[2]_q!(4\lambda-2\lambda^2-1)}{4([1]_q!)^2[2]_q!(2\lambda-1)^2(3\lambda-1)}-\mu\left(\frac{(1-\beta)}{2[1]_q!(2\lambda-1)}\right)^2,$$

$$a_3-\mu a_2^2 = \frac{(1-\beta)^2}{4([1]_q!)^2(2\lambda-1)^2}\left(\frac{([2]_q!-2([1]_q!)^2)(2\lambda-1)^2}{[2]_q!(3\lambda-1)(1-\beta)}+\frac{[2]_q!(4\lambda-2\lambda^2-1)}{[2]_q!(3\lambda-1)}-\mu\right)$$

and thus

$$|a_3-\mu a_2^2| \le \frac{(1-\beta)^2}{4([1]_q!)^2(2\lambda-1)^2}\left|\frac{([2]_q!-2([1]_q!)^2)(2\lambda-1)^2}{[2]_q!(3\lambda-1)(1-\beta)}+\frac{4\lambda-2\lambda^2-1}{3\lambda-1}-\mu\right|.$$

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