International Journal of Mathematics and
Computer Science, **15**(2020), no. 2, 711–716

On rings and symmetric generalized biderivations

Ahlam Fallatah

Department of Mathematics
College of Science
Taibah University
Madinah, Saudi Arabia

email: afallatah@taibahu.edu.sa

(Received January 20, 2020, Accepted February 25, 2020)

Abstract

In the present paper, our aim is to prove the following result: let R be a prime ring of a characteristic different from two. If Δ_1, Δ_2 are two symmetric generalized biderivations on R with associated biderivation D such that $[\Delta_1(x,x), \Delta_2(x,x)] = 0$ for all $x \in R$, then the following results hold:

1. R is commutative.
2. Δ_1 and Δ_2 act as left bi-multipliers on R.

1 Introduction

The idea of symmetric bi-derivations was introduced by Maksa [3] who showed showed [4] that symmetric bi-derivations are related to general solutions of some functional equations. The notion of additive commuting mappings is closely connected with the notion of bi-derivations. Every commuting additive mapping $f : R \longrightarrow R$ gives rise to a bi-derivation on R. Namely linearizing $[x, f(x)] = 0$ for all $x, y \in R$ $(x, y) \mapsto [f(x), y]$ is a bi-derivation. Now we introduce the concept of symmetric bi-derivations as follows:

Key words and phrases: Prime ring, Generalized biderivations, multipliers.
AMS (MOS) Subject Classifications: 16W20, 16W25, 16N80
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

Definition 1.1. (Symmetric mapping) A mapping $D : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$.

Definition 1.2. (Bi-additive mapping) Let R be a ring. A mapping $D : R \times R \to R$ is called bi-additive if it is additive in both arguments.

Definition 1.3. (Trace) A mapping $f : R \to R$ defined by $f(x) = D(x, x)$, where $D : R \times R \to R$ is a symmetric mapping, is called the trace of D.

Remark 1.1. 1. The trace f of D satisfies the relation $f(x + y) = f(x) + f(y) + D(x, y) + D(y, x)$ for all $x, y \in R$.

2. If D is symmetric, then the trace f of D satisfies the relation $f(x+y) = f(x) + f(y) + 2D(x, y)$ for all $x, y \in R$.

Definition 1.4. (Biderivation) A bi-additive mapping $D : R \times R \longrightarrow R$ is called a bi-derivation if for every $x \in R$, the map $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a derivation of R; that is, $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$ and $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$.

Following [6], we initiate the idea of generalized biderivation on rings given as:

Definition 1.5. (Generalized biderivation) A biadditive mapping $\Delta : R \times R \longrightarrow R$ is said to be a generalized biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of R associated with the function $y \mapsto D(x, y)$ as well as if for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of R associated with the function $x \mapsto D(x, y)$ for all $x, y \in R$. It also satisfies $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$ for all $x, y, z \in R$.

Example 1.1. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in S \right\}$, where S is any commutative ring. Consider $\Delta : R \times R \longrightarrow R$ be generalized biderivation with associated map $D : R \times R \longrightarrow R$ defined as $\Delta\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \right) = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & 0 \end{pmatrix}$, and

$$D\left(\begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ b_2 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & b_1 b_2 \end{pmatrix}.$$

In this paper, we prove some theorems on symmetric generalized biderivations of prime ring generalizing the results proved in [1, 2, 5].

2 Main Theorems

To prove our main theorems, we need the following lemma:

Lemma 2.1. *[1] Let R be a prime ring of characteristic different from two and let I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation on R with associated biderivation D such that $D(\Delta(x,y),z) = 0$ for all $x, y, z \in I$, then either R is commutative or $D = 0$. Moreover, Δ acts as a left bimultiplier on R.*

Theorem 2.1. *Let R be prime ring of a characteristic different from 2. If Δ_1, Δ_2 are generalized biderivations with associated biderivation D such that $[\Delta_1(y,y),r] + [r,\Delta_2(y,y)] = 0$ for all $y, r \in R$, then the following results follow:*

1. *R is commutative.*

2. *Δ_1 and Δ_2 act as left bi-multipliers.*

Proof. By hypothesis, we have

$$[\Delta_1(y,y),r] + [r,\Delta_2(y,y)] = 0 \text{ for all } y, r \in R. \tag{2.1}$$

Linearization in y yields

$$\begin{aligned}[\Delta_1(y,y),r] &+ 2[\Delta_1(y,z),r] + [\Delta_1(z,z),r] + [r,\Delta_2(y,y)] \\ &+ [r,\Delta_2(z,z)] + 2[r,\Delta_2(y,z)] = 0 \text{ for all } y, z, r \in R.\end{aligned} \tag{2.2}$$

Using the characteristic condition and (2.1), we get

$$[\Delta_1(y,z),r] + [\Delta_2(y,z),r] = 0 \text{ for all } y, z, r \in R. \tag{2.3}$$

Substitute zu for z in (2.3) to get

$$[\Delta_1(y,z)u,r]+[zD(y,u),r]+[\Delta_2(y,z)u,r]+[zD(y,u),r] = 0 \text{ for all } y, z, u, r \in R. \tag{2.4}$$

This implies that

$$\begin{aligned}&\Delta_1(y,z)[u,r] + [\Delta_1(y,z),r]u + z[D(y,u),r] + [z,r]D(y,u) + \Delta_2(y,z)[u,r] \\ &+[\Delta_2(y,z),r]u + [z,r]D(y,u) + z[D(y,u),r] = 0 \text{ for all } u, y, z, r \in R.\end{aligned} \tag{2.5}$$

Replace u by ur in (2.5) to obtain

$$\begin{aligned}&\Delta_1(y,z)[u,r]r2[D(y,u),r]r+2u[D(y,r),r]+z[u,r]D(y,r)+[z,r]D(y,u)r+[z,r]uD(y,r)\\&+\Delta_2(y,z)[u,r][z,r]D(y,u)r+[\Delta_2(y,z),r]u+[z,r]D(y,u)r+[z,r]uD(y,r)\\&+z[D(y,u),r]r+z[u,r]D(y,r)+zu[D(y,r),r]=0 \text{ for all } u,y,z,r\in R.\end{aligned} \tag{2.6}$$

After simplification and using the characteristic of R is not two, we get

$$zu[D(y,r),r]+z[u,r]D(y,r)+[z,r]uD(y,r)=0 \text{ for all } u,y,z,r\in R. \tag{2.7}$$

Replacing z by tz in (2.7), we have

$$[t,r]zuD(y,r)=0 \text{ for all } u,y,z,r,t\in R. \tag{2.8}$$

Primeness of R implies that either $[t,r]=0$ or $D(y,r)=0$ for all $t,y,r\in R$. The first case shows that R is commutative. If we take $D(y,r)=0$ for all $y,r\in R$, then the generalized biderivations Δ_1,Δ_2 reduces to the left bi-multiplier. This complete the proof. □

Theorem 2.2. *Let R be prime ring of a characteristic not equal to two. If Δ_1,Δ_2 are two symmetric generalized biderivations on R with associated biderivation D such that $[\Delta_1(x,x),\Delta_2(x,x)]=0$ for all $x\in R$, then the following condition holds:*

1. *R is commutative.*
2. *Δ_1 and Δ_2 acts as a left bi-multiplier on R.*

Proof. By hypothesis, we have

$$[\Delta_1(x,x),\Delta_2(x,x)]=0 \text{ for all } x\in R. \tag{2.9}$$

Linearize (2.9) in x to get

$$\begin{aligned}[\Delta_1(x,x),\Delta_2(x,x)] &+ [\Delta_1(x,x),\Delta_2(y,y)]+2[\Delta_1(x,x),\Delta_2(x,y)]\\&+[\Delta_1(y,y),\Delta_2(x,x)]+[\Delta_1(y,y),\Delta_2(y,y)]\\&+2[\Delta_1(y,y),\Delta_2(x,y)]+2[\Delta_1(x,y),\Delta_2(x,x)]\\&+2[\Delta_1(x,y),\Delta_2(y,y)]+2[\Delta_1(x,y),\Delta_2(x,y)]=0 \text{ for all } x,y\in R.\end{aligned} \tag{2.10}$$

By given condition in hypothesis, we arrive at

$$\begin{aligned}[\Delta_1(x,x),\Delta_2(y,y)] &+ 2[\Delta_1(x,x),\Delta_2(x,y)]\\&+[\Delta_1(y,y),\Delta_2(x,x)]+2[\Delta_1(y,y),\Delta_2(x,y)]\\&+2[\Delta_1(x,y),\Delta_2(x,x)]+2[\Delta_1(x,y),\Delta_2(y,y)]\\&+2[\Delta_1(x,y),\Delta_2(x,y)]=0 \text{ for all } x,y\in R.\end{aligned} \tag{2.11}$$

Substitute $-y$ for y in (2.11) to find

$$\begin{aligned}[\Delta_1(x,x),\Delta_2(y,y)] &- 2[\Delta_1(x,x),\Delta_2(x,y)] \\ &+ [\Delta_1(y,y),\Delta_2(x,x)] - 2[\Delta_1(y,y),\Delta_2(x,y)] \\ &- 2[\Delta_1(x,y),\Delta_2(x,x)] - 2[\Delta_1(x,y),\Delta_2(y,y)] \\ &+ 2[\Delta_1(x,y),\Delta_2(x,y)] = 0 \text{ for all } x,y \in R. \end{aligned} \tag{2.12}$$

Addition of equations (2.11) and (2.12) and the use of characteristic restriction yield

$$[\Delta_1(x,x),\Delta_2(y,y)]+[\Delta_1(y,y),\Delta_2(x,x)]+2[\Delta_1(x,y),\Delta_2(x,y)] = 0 \text{ for all } x,y \in R. \tag{2.13}$$

Again linearize the above equation in x to get

$$\begin{aligned}[\Delta_1(x,x),\Delta_2(y,y)] &+ [\Delta_1(y,y),\Delta_2(y,y)] + 2[\Delta_1(x,y),\Delta_2(y,y)] \\ &+ [\Delta_1(y,y),\Delta_2(x,x)] + [\Delta_1(y,y),\Delta_2(y,y)] \\ &+ 2[\Delta_1(y,y),\Delta_2(x,y)] + 2[\Delta_1(x,y),\Delta_2(x,y)] \\ &+ 2[\Delta_1(y,y),\Delta_2(x,y)] + 2[\Delta_1(y,y),\Delta_2(y,y)] \\ &+ 2[\Delta_1(x,y),\Delta_2(y,y)] = 0 \text{ for all } x,y \in R. \end{aligned} \tag{2.14}$$

From equation (2.13) and (2.14), we obtain

$$4[\Delta_1(y,y),\Delta_2(x,y)] + 4[\Delta_1(x,y),\Delta_2(y,y)] = 0 \text{ for all } x,y \in R. \tag{2.15}$$

Since characteristic of R is not two, we have

$$[\Delta_1(y,y),\Delta_2(x,y)] + [\Delta_1(x,y),\Delta_2(y,y)] = 0 \text{ for all } x,y \in R. \tag{2.16}$$

Putting xz in place of z in (2.16), we get

$$\begin{aligned}[\Delta_1(y,y),\Delta_2(x,y)z] &+ [\Delta_1(y,y),xD(z,y)] + [\Delta_1(x,y)z,\Delta_2(y,y)] \\ &+ [xD(z,y),\Delta_2(y,y)] = 0 \text{ for all } x,y,z \in R. \end{aligned} \quad lll \tag{2.17}$$

Simplification of (2.17) and the use of (2.16) imply that

$$\begin{aligned}\Delta_2(x,y)[\Delta_1(y,y),z] &+ [\Delta_1(y,y),x]D(z,y) + x[\Delta_1(y,y),D(z,y)] + \Delta_1(x,y)[z,\Delta_2(y,y)] \\ &+ x[D(z,y),\Delta_2(y,y)] + [x,\Delta_2(y,y)]D(z,y) = 0 \text{ for all } x,y,z \in R. \end{aligned} \tag{2.18}$$

Applying (2.18) to the resulting equation after replacing z by $z\Delta_2(y,y)$ in (2.18) and we see that

$$\begin{aligned}[\Delta_1(y,y),x]zD(\Delta_2(y,y),y) &+ xz[\Delta_1(y,y),D(\Delta_2(y,y),y)] + x[\Delta_1(y,y),z]D(\Delta_2(y,y),y) \\ &+ xz[D(\Delta_2(y,y),y),\Delta_2(y,y)] + x[z,\Delta_2(y,y)]D(\Delta_2(y,y),y) \\ &+ [x,\Delta_2(y,y)]zD(\Delta_2(y,y),y) = 0 \text{ for all } x,y,z \in R. \end{aligned} \tag{2.19}$$

If we substitute rx for x in (2.19) and use (2.19), then the last equation takes the form

$$[\Delta_1(y,y),r]xzD(\Delta_2(y,y),y)+[r,\Delta_2(y,y)]xzD(\Delta_2(y,y),y)=0 \text{ for all } r,x,y,z\in R. \tag{2.20}$$

The equation above implies that $\{[\Delta_1(y,y),r]+[r,\Delta_2(y,y)]\}xzD(\Delta_2(y,y),y)=0$ for all $r,x,y,z\in R$. By primeness of R we bring that for all $r,x,y,z\in R$ either $\{[\Delta_1(y,y),r]+[r,\Delta_2(y,y)]\}=0$ or $xzD(\Delta_2(y,y),y)=0$. In the first case, the conclusion follows from Theorem 2.1. Next consider the case $xzD(\Delta_2(y,y),y)=0$ for all $x,y,z\in R$. Consequently, we get the result by applying Lemma 3.1. □

References

[1] A. Ali, V. D. Filippis, F. Shujat, Results concerning symmetric generalized biderivations of prime and semiprime rings, Matematiqki Vesnik, **66,** no. 4, (2014), 410–417.

[2] F. Shujat, Symmetric generalized biderivation on prime rings, Bol. soc. Paran. Math., (2018).

[3] G. Maksa, A remark on symmetric biadditive functions having nonnegative diagonalization, Glasnik. Mat., 15 (35), (1980), 279–282.

[4] G. Maksa, On the trace of symmetric biderivations, C. R. Math. Rep. Acad. Sci., **9,** (1987), 303–307.

[5] J. Vukman, Symmetric biderivations on prime and semiprime rings, Aequationes Math., **38,** (1989), 245–254.

[6] N. Argac, On prime and semiprime rings with derivations, Algebra Colloq., **13,** no. 3, (2006), 371–380.