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Computing the number of subgroups and normal, cyclic, maximal, minimal and Sylow subgroups of the direct product Group $D_{2n} \times C_2$

Hayder Shelash, Muayad G. Mohsin, Mohsin Shalan Abdulhussein

Department of Mathematics
Faculty of Computer Sciences and Mathematics
University of Kufa
Najaf, Iraq

email: hayder.ameen@uokufa.edu.iq, muaidg.alabdi@uokufa.edu.iq,
mohsins.alakayshee@uokufa.edu.iq

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Abstract

In this paper, we compute the number of subgroups, normal, cyclic subgroups, maximal, minimal and Sylow subgroups of the group $G = D_{2n} \times C_2$.

1 Introduction

Recently, the applications of computational group theory (in short, CGT) have been widely studied and one of the most important aspects which has been considered is computing the number of subgroups. Cavior [2] and Calhoun [1] computed the number of subgroups of the Dihedral group D_{2n} to be $\tau(n)+\sigma(n)$, where $\tau(n)$ is the number of divisors of n and $\sigma(n)$ is the sum of divisors of n. Marius Tărnăuceanu [11] computed the number of subgroups for some of finite groups. For more details we refer the reader to [5], [6, 7] [3, 4, 8, 9, 10, 11].

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In this paper, we study and compute the number of subgroups and normal, cyclic, maximal, minimal and Sylow subgroups of the group $C_2 \times D_{2n}$ denoted by $Sub(G)$, $N(G)$, $Cy(G)$, $Max(G)$, $Min(G)$ and $Syl(G)$ We recall that $\pi(n)$ is the number of odd prime divisors of n.

2 Basic Properties and Terminology

Shelash and Ashrefi [8, 9] introduced the method of computing the order subgroup table of the finite group G. Suppose $|G| = 2^r \prod_{1\leq i\leq s} p_i^{\alpha_i}$ is a prime factorization of $|G|$, $2 < p_1 < p_2 < \cdots < p_s$ and δ is all odd divisors of $m = \prod_{1\leq i\leq s} p_i^{\alpha_i}$. Then

$$\delta_{ij} = \begin{cases} 2^{j-1} & i = 1 \\ 2^{j-1}\delta_i & i \neq 1 \end{cases} .$$

Table 1. Order subgroups table when $|G| = n = 2^r \prod_{1\leq i\leq s} p_i^{\alpha_i}$.

j	1	2	3	$\ldots$	$r+1$
	1	2	4	$\ldots$	2^r
1	1	2	4	$\ldots$	2^{r+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
δ_i	δ_i	$2\delta_i$	$4\delta_i$	$\ldots$	$2^r\delta_i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

It is easy to see that the rows and columns are equal exactly to $\tau(2^r)$ and $\tau(m) = \prod_{1\leq i\leq s}(\alpha_i + 1)$ respectively. Recall that $\tau(n)$ is the number of all divisors of n and $\sigma(n)$ is the sum of all divisors of n. The Frattine subgroup is $\phi(G) = \bigcap_{M \lessdot G} M$, where M, the maximal subgroup of G, has a significant contribution in CGT. It is well-known that $Max(G) = Max(\frac{G}{\phi(G)})$, $R(G)$ is the intersection of all normal maximal subgroups and $L(G)$ is the intersection of all Self-Normalizing subgroups. We can write $\phi(G) = R(G) \cap L(G)$.

3 Main Results

In this section, we will present an algorithm to compute the number of subgroups of the group $C_2 \times D_{2n}$.

3.1 Algorithm to Compute the number of subgroups

The Dihedral group is defined as $\langle a, b \mid a^n = b^2 = e \mid bab = a^{-1}\rangle$ with subgroups $\langle a^d\rangle, \forall d|n$ and $\langle a^d, a^j b\rangle, \forall d|n, 1 \leq j \leq d$. Likewise, the cyclic group C_2 is defined as $\langle c^2 = e\rangle$.

Now, we define the following product group $C_2 \times D_{2n} = \langle a^n = b^2 = c^2 = e, bab = a^{-1}, [a, c] = [b, c] = e\rangle$ and the center $Z(C_2 \times D_{2n}) = \langle a^n, c\rangle$ if n is an even number and $Z(C_2 \times D_{2n}) = \langle c\rangle$ otherwise. Then $\dfrac{C_2 \times D_{2n}}{Z(C_2 \times D_{2n})} \simeq D_n$ if n is even and D_{2n} otherwise.

The group $C_2 \times D_{2n}$ has eight different types of the subgroups which will be computed in the following theorem.

Theorem 3.1. *Suppose $n = 2^r m$ where m is odd number and $r \geq 0$. Then the number of all subgroups of the group $C_2 \times D_{2n}$ is equal to*

$$Sub(C_2 \times D_{2n}) = 2\tau(n) + \tau(\frac{n}{2}) + 3\sigma(n) + 2\sigma(\frac{n}{2})$$

Proof. It is well-known that the lattice subgroups of the Dihedral group are $\langle a^i\rangle$ and $\langle a^i, a^j b\rangle$ for all $i \mid n$; $1 \leq j \leq i$. From the following diagram we can see all the possible types of subgroup in group $C_2 \times D_{2n}$:

1. A subgroups $G_1(i) = \langle a^i\rangle$, $\forall i|n$;
2. A subgroups $G_2(i) = \langle a^i, c\rangle$, $\forall i|n$;
3. A subgroups $G_3(i, j) = \langle a^i, a^j b\rangle$, $\forall i|n$, $1 \leq j \leq i$;
4. A subgroups $G_4(i, j) = \langle a^i, a^j bc\rangle$, $\forall i|n$, $1 \leq j \leq i$;
5. A subgroups $G_5(i, j) = \langle a^i, a^j b, c\rangle$, $\forall i|n$, $1 \leq j \leq i$;
6. A subgroups $G_6(i, j) = \langle a^i c, a^j b\rangle$, $\forall i|\frac{n}{2}$, $1 \leq j \leq i$;
7. A subgroups $G_7(i, j) = \langle a^i c, a^j bc\rangle$, $\forall i|\frac{n}{2}$, $1 \leq j \leq i$;
8. A subgroups $G_8(i) = \langle a^i c\rangle$, $\forall i|\frac{n}{2}$.

The first type of subgroups, $Sub(G_1) = \tau(n)$, since for each divisor i of n , there exists exactly one subgroup of this type of cyclic subgroups $\tau(n)$ which is contained in G_1.

The second type of subgroups $Sub(G_2) = \tau(n)$ is a similar argument as Part(1).

The subgroups of the third type $Sub(G_3) = \sigma(n)$. It is easy to see that $\langle a^i, a^j b\rangle = \langle a^u, a^v b\rangle$ if and only if $i = v$ and $j = u$. Since $i|n$, all divisors of n are $n, \frac{n}{2}, \cdots, 1$ and $\sum_{i|n} i = \sigma(n)$.

The fourth type and fifth of gubgroups are $Sub(G_4) = Sub(G_5) = \sigma(n)$.

The subgroups of the sixth type is $Sub(G_6) = \sigma(\frac{n}{2}$, $Sub(G_7) = \sigma(\frac{n}{2})$ and $Sub(G_8) = \tau(\frac{n}{2})$. Finally, the subgroups of the group $C_2 \times D_{2n}$ is

$$Sub(C_2 \times D_{2n}) = 2\tau(n) + \tau(\frac{n}{2}) + 3\sigma(n) + 2\sigma(\frac{n}{2}).$$

□

Corollary 3.2. *Let n be an odd number. Then*

1. $Sub(C_2 \times D_{2n}) = 2\tau(n) + 3\sigma(n)$;
2. $Sub(C_2 \times D_{2n}) = Sub(D_{2(2n)})$

Proof. (1) We have $\tau(\frac{n}{m}) = \sigma(\frac{n}{m}) = 0$ when $m \nmid n$.

Suppose that n is odd number. Then

$$\begin{aligned} Sub(C_2 \times D_{2n}) &= 2\tau(n) + \tau(\frac{n}{2}) + 3\sigma(n) + 2\sigma(\frac{n}{2}) \\ &= 2\tau(n) + 3\sigma(n) \end{aligned}$$

(2) Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}$ is odd number, then

$$\begin{aligned} Sub(C_2 \times D_{2n}) &= 2\tau(p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) + 3\sigma(p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) \\ &= \tau(2)\tau(p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) + \sigma(2)\sigma(p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) \\ &= \tau(2p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) + \sigma(2p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}) \\ &= \tau(2n) + \sigma(2n) \\ &= Sub(D_{2(2n)}) \end{aligned}$$

□

Next, we will compute the numbers of subgroups for each order in group $C_2 \times D_{2n}$. Suppose that n is an odd number. The number of subgroups for all order in group $C_2 \times D_{2n}$ is given by the following table:

Table 2. Number of Subgroup when n is odd number

j	1	2	4
	1	2	2^2
c_0	1	$2n+1$	n
c_1	1	$2(\frac{n}{c_1})+1$	$\frac{n}{c_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_t	1	$2(\frac{n}{c_t})+1$	$\frac{n}{c_t}$

Hence, $c_i \mid n$ for each $i \in \{1, 2, \cdots, t\}$.

Suppose that n is a even number. The number of subgroups for each order in group is given by the following table:

Table 3. Number Subgroup of Each Orders Subgroups

j	1	2	$3 \leq j \leq r+1$	$r+2$	$r+3$
	1	2	2^{j-1}	2^{r+1}	2^{r+2}
c_0	1	$2n+3$	$3(\frac{n}{2^{j-3}})+3$	$3(\frac{n}{2^{r-1}})+1$	$\frac{n}{2^r}$
c_1	1	$2(\frac{n}{c_1})+3$	$3(\frac{n}{2^{j-3}c_1})+3$	$3(\frac{n}{2^{r-1}c_1})+1$	$\frac{n}{2^r c_1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_i	1	$2(\frac{n}{c_i})+3$	$3(\frac{n}{2^{j-3}c_i})+3$	$3(\frac{n}{2^{r-1}c_i})+1$	$\frac{n}{2^r c_i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_t	1	$2(\frac{n}{c_t})+3$	$3(\frac{n}{2^{j-3}c_t})+3$	$3(\frac{n}{2^{r-1}c_t})+1$	$\frac{n}{2^r c_t}$

3.2 The Number Normal Subgroups of Group $C_2 \times D_{2n}$

In this section, we will compute the number normal subgroups of the group $C_2 \times D_{2n}$.

In the next table, we explain the location of the normal subgroups.

Theorem 3.3. *Let $n = 2^r m$ be integer number, where m is an odd number and $r \geq 0$. Then the number of normal subgroups is:*

$$NSub(C_2 \times D_{2n}) = \begin{cases} 2\tau(n) + \tau(\frac{n}{2}) + 11 & r \geq 1 \\ \\ 2\tau(n) + 3 & r = 0 \end{cases}.$$

Proof. Suppose that n is odd number. Then it is clear that the first type and second type of the subgroups are normal and the fifth type of subgroups is normal when $i = 1, 2$ and $1 \leq j \leq i$. Thus $NSub(C_2 \times D_{2n}) = 2\tau(n) + 3$ and the number of normal subgroups table we can explain in the following table:

Table 4. Number of Normal Subgroup when n is odd number

	1	2	2^2
c_0	1	1	0
c_1	1	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_t	1	3	1

Let n be an even number. The number of normal subgroups in group $C_2 \times D_{2n}$ are:

1. A subgroups $G_1(i) = \langle a^i \rangle$, where $i|n$, $NSub(G_1) = \tau(n)$;
2. A subgroups $G_2(i) = \langle a^i, c \rangle$, where $i|n$, $NSub(G_2) = \tau(n)$;
3. A subgroups $G_3(i,j) = \langle a^i, a^j b \rangle$, where $i = 1, 2$, $NSub(G_3) = \sigma(2)$;
4. A subgroups $G_4(i,j) = \langle a^i, a^j bc \rangle$, where $i = 1, 2$ $NSub(G_4) = \sigma(2)$;
5. A subgroups $G_5(i,j) = \langle a^i, a^j b, c \rangle$, where $i = 1, 2$ and $1 \leq j \leq i$,$NSub(G_5) = \sigma(2)$;
6. A subgroups $G_6(i,j) = \langle a^i c, a^j b \rangle$, where $i = 1$ $NSub(G_6) = 1$;
7. A subgroups $G_7(i,j) = \langle a^i c, a^j bc \rangle$, where $i = 1$, $NSub(G_7) = 1$;
8. A subgroups $G_8(i) = \langle a^i c \rangle$ where $i|\frac{n}{2}$, $NSub(G_8) = \tau(\frac{n}{2})$.

Table 5. Number of Normal Subgroup when n is even number

	1	2	2^2	2^3	$\cdots$	2^{r-1}	2^r	2^{r+1}	2^{r+2}
c_0	1	3	3	3	$\cdots$	3	3	1	0
c_1	1	3	3	3	$\cdots$	3	3	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_{t-1}	1	3	3	3	$\cdots$	3	3	1	0
c_t	1	3	3	3	$\cdots$	3	7	7	1

□

Corollary 3.4. *If $NSub(D_{2(2n)})$ with n an odd number. Then*

$$NSub(D_{2(2n)}) = NSub(C_2 \times D_{2n}).$$

Proof. Suppose that n is odd number. Then by theorem 3.3 the number

$$\begin{aligned} NSub(C_2 \times D_{2n}) &= 2\tau(n) + 3 \\ &= \tau(2)\tau(n) + 3 \\ &= \tau(2n) + 3 \\ &= NSub(D_{2(2n)}) \end{aligned}$$

□

Theorem 3.5. *The number of cyclic subgroups of the group $C_2 \times D_{2n}$ is given by :*

$$CSub(C_2 \times D_{2n}) = \begin{cases} \tau(n) + \tau(\frac{n}{2^r}) + \tau(\frac{n}{2}) + 2n & 2 \mid n \\ \tau(n) + \tau(\frac{n}{2^r}) + 2n & 2 \nmid n \end{cases}$$

Proof. Suppose that n is even number. It is clear that the subgroup $G_1 = \langle a^i \rangle$ where $i|n$ have exactly $\tau(n)$ of cyclic subgroups, and the second subgroups have $\tau(\frac{n}{2^r})$ of cyclic subgroups. When $i = n$ in subgroups G_3 and G_4 we have cyclic subgroups such that $CSub(G_3) = n$ and $CSub(G_4) = n$.

The subgroup G_8 is cyclic such that $CSub(G_8) = \tau(\frac{n}{2})$.

The fifth, sixth and seventh subgroups are not cyclic.

In case n is odd number, the proof is obvious.

3.3 The Maximal Subgroups, Minimal Subgroups and Sylow Subgroups :

Here we will compute all maximal subgroups of the group $C_2 \times D_{2n}$. It is well known that the maximal subgroup M have prime index $[|G| : |M|] = p$; if there does not exist a proper subgroup in K, then K is minimal subgroup and the number of Sylow subgroups of order p^n is $n_p(G) = |Syl_p(G)| = [G : N_G(H)]$.

Theorem 3.6. *Let $n = 2^r m$ be an integer and $r > 0$. Then the following assertions hold:*

1− *The number of maximal subgroups is :*

$$Max(C_2 \times D_{2n}) = \begin{cases} \pi(2n) + 5 & 2 \mid n \\ \pi(2n) + 3 & 2 \nmid n \end{cases} .$$

2− *The number of minimal subgroups is:*

$$Min(C_2 \times D_{2n}) = \begin{cases} \pi(n) + 2n + 2 & 2 \mid n \\ \pi(n) + 2n + 1 & 2 \nmid n \end{cases}.$$

3− *The number of Sylow subgroups be*

$$Sy(C_2 \times D_{2n}) = \pi(m) + m$$

Proof.

- If there exists a subgroup $H \subset G$ and the index $[|G| : |H|] = p$ is a prime number, then H is a maximal subgroup of G.

  In case n is an even number in the following subgroups, we have: $Max(G_2) = \langle a, c\rangle$, $Max(G_3) = \langle a, ab\rangle$, $Max(G_4) = \langle a, abc\rangle$, $Max(G_6) = \langle ac, ab\rangle$, $Max(G_7) = \langle ac, abc\rangle$ and $Max(G_5) = \langle a^p, a^j b, c\rangle, 1 \leq j \leq p$, where $p \in \pi(2n)$.

  In case n is an odd number there is no maximal subgroup for G_6 and G_7.

- It is clear that for any subgroup of $\langle a^j b\rangle$ and $\langle a^j bc\rangle$ when $1 \leq j \leq n$ is minimal subgroup and for any subgroup of $\langle a^{\frac{n}{p_i}}\rangle$ with p_i divide n is a minimal subgroup. The subgroup $\langle c\rangle$ is a minimal subgroup.

- Since the subgroup $\langle a^m, a^j b, c\rangle$ is of order 2^n, it is Sylow 2−subgroup and $1 \leq j \leq m$ and the number of all Sylow 2−subgroups is equal to m. On the other hand, the subgroup of type $\langle a^{\frac{n}{p_i^{\alpha_i}}}\rangle$ of order $p_i^{\alpha_i}$ is Sylow p_i- subgroup for any p_i divide m. Finally, the number of Sylow subgroups of the group $C_2 \times D_{2n}$ is given by $Sy(C_2 \times D_{2n}) = \pi(m) + m$

□

Corollary 3.7. *Let p_i be an odd prime number and $p_i \in \pi(n)$. Then $\langle a^i, a^j b, c\rangle$, where $1 \leq j \leq i$ are Self-Normalizing .*

Proof. Easy to check.

□

Corollary 3.8. *The following hold:* (1) $R(C_2 \times D_{2n}) = \begin{cases} \langle a^2\rangle & 2 \mid n \\ \langle a\rangle & 2 \nmid n \end{cases}$

(2) $L(C_2 \times D_{2n}) = \langle a^{p_1 p_2 \cdots p_s}, c\rangle$ *, where p_i is odd prime for all i*

(3) $\phi(C_2 \times D_{2n}) = \begin{cases} \langle a^{2p_1 p_2 \cdots p_s}\rangle & 2 \mid n \\ \langle a^{p_1 p_2 \cdots p_s}\rangle & 2 \nmid n \end{cases}$

Proof. (1) If n is odd number, then $R(C_2 \times D_{2n}) = \langle a, c\rangle \cap \langle a, ab\rangle \cap \langle a, abc\rangle = \langle a\rangle$; If n is even number, then$R(C_2 \times D_{2n}) = \langle a, c\rangle \cap \langle a, ab\rangle \cap \langle a, abc\rangle \cap \langle a^2, ab, c\rangle \cap \langle a^2, a^2b, c\rangle = \langle a^2\rangle$;
(2) $L(C_2 \times D_{2n}) = \bigcap_{1\le j\le p}\langle a^p, a^jb, c\rangle = \langle a^{\prod_{1\le i\le s} p_i}, c\rangle$, where $m = \prod_{1\le i\le s} p_i^{\alpha_i}$, p_i is odd prime for each i.
(3) follows from (1) and (2). □

We summarize using the following table:

Table 5. Location of Certain subgroup

j	1	2	·	r	$r+1$	$r+2$	$r+3$
c_i	1	2	·	2^{r-1}	2^r	2^{r+1}	2^{r+2}
1		Min					Syl_2
p_1	Min						
$\vdots$	$\vdots$						
p_s	$\boxed{Min}$						
$\vdots$							
$\prod_{1\le i\le s} p_i^{\alpha_i-1}$				$\boxed{\phi(G)}$		$\boxed{L(G)}$	
$\vdots$							
$p_1^{\alpha_1}$	$\boxed{Syl_{p_1}}$						
$\vdots$	$\vdots$						
$p_s^{\alpha_s}$	$\boxed{Syl_{p_s}}$						
$\vdots$							
$\frac{m}{p_s}$							$\boxed{Max}$
$\vdots$							$\vdots$
$\frac{m}{p_1}$							$\boxed{Max}$
m				$\boxed{R(G)}$		$\boxed{Max}$	

$t = \prod_{1\le i\le s} p_i^{\alpha_i-1}$

References

[1] W. C. Calhoun, Counting the subgroups of some finite groups, *Amer. Math. Monthly,* **94,** no. 1, (1987), 54–59.

[2] S. R. Cavior, The subgroups of the dihedral groups, *Math. Mag.*, **48,** (1975).

[3] M. Hormozi, K. Rodtes, Symmetry classes of tensors associated with the semi−dihedral groups SD_{8n}, *Colloq. Math.,* **131,** no. 1, (2013) 59–67.

[4] G. James, M. Liebeck, *Representations and Characters of Groups*, 2nd Edition, Cambridge University Press, New York, 2001.

[5] David W. Jensen, M. K. Keane , A Number-Theoretic Approach to Subgroups of Dihedral Groups, *SUT J. Math.,* **37,** no. 1, (1990).

[6] J. M. Oh, Y. Kim, K. W. Hwang, The number of chains of subgroups in the lattice of subgroups of the dicyclic group, *Discrete Dyn. Nat. Soc.,* 2012 Art. ID 760246, page 17.

[7] J. M. Oh , The number of chains of subgroups of a finite cycle group, *European J. Combin.,* **33,** no. 2, (2012), 259–266.

[8] Haider Baker Shelash, Ali Reza Ashrafi, Computing maximal and minimal subgroups with respect to a given property in certain finite groups, *Quasigroups Related Systems,* **27,** (2019), no. 1, 133–146.

[9] H. B. Shelash, A. R. Ashrafi, *Computing the number of subgroups, normal subgroups and characteristic subgroups in certain finite groups*, Ph.D Thesis, 2018.

[10] R. Schmidt, *Subgroup Lattices of Groups*, De Gruyter Expositions in Mathematics, 3rd Edition, De Gruyter, New York, 1994.

[11] M. Tarnauceanu, The number of chains of subgroups of a finite elementary Abelian p−group, *Politehn. Univ. Bucharest Sci. Bull. Ser. A, Appl. Math. Phys.,* **77,** no. 4, (2015), 65–68.

[12] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.7.5, 2014.