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On initial Coefficients Estimates for Certain New Subclasses of Bi-Univalent Functions Defined by a Linear Combination

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Abstract

In the present paper, we introduce and study coefficient bounds for certain new subclasses of bi-univalent functions. For this purpose, we make use of a linear combination of the functions: $\frac{zf''(z)}{f'(z)}$ and $f'(z)$. For a function belonging to the normalized univalent function class S, we obtain estimates on the initial Taylor-Maclaurin coefficients $|a_2|, |a_3|$ and $|a_4|$ for the functions in two new subclasses of the function class Σ of bi-univalent functions defined on the open unit disc $U = \{z \in C : |z| < 1\}$.

1 Introduction

Let M denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$ and satisfy the normalization condition $f(0) = f^{'}(0) - 1 = 0$.
Also let S be the class of the functions $f \in M$ of the form given by (1), which are univalent in U.
The Kobe One-Quarter Theorem [3] shows that the image of U under every function f from S contains a disc of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \ (z \in U)$$

and

$$f^{-1}(f(w)) = w, \ \ (|w| < r_0(f), r_0(f) \geq 1/4),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \quad (1.2)$$

A function $f(z) \in M$ is said to be bi-univalent in U [3], if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1).
In 1967, Lewin [5] investigated the class Σ and proved that $|a_2| < 1.51$; in 1969, Netanyahu [9] showed that $\max_{f\in\Sigma} |a_2| \leq 4/3$ and in 1980, Brannan and Clunie [1] conjectured that $|a_2| \leq \sqrt{2}$. However, the problem of coefficient estimate for $|a_n|, (n = 3, 4, \cdots .)$ is still open.
Recently, several researchers [4,6,7,8,10,12,13] obtained the coefficients $|a_2|, |a_3|$ of bi-univalent functions for the different subclass of the function class Σ . Motivated by this, the author introduces a new subclass of the function class Σ.

The objective of the present paper is to introduce two new subclasses $K_{\Sigma}^{\lambda,\alpha}$ and $K_{\Sigma}^{\lambda,\beta}$ of the function class Σ of bi-univalent functions defined on the open unit disc U and obtain improved estimates on the initial coefficient $|a_2|, |a_3|$ and $|a_4|$ for the functions in these subclasses. But the problem of coefficient estimate for $|a_n|, (n \in N - \{1, 2, 3, 4\})$ is still open. In order to prove our main results, we need the following lemma [11]:

Lemma 1.1. *If $h \in P$, then $|t_k| \leq 2$ for each k, where P is the family of all functions h analytic in U for which $Re\{h(z)\} > 0$,*

$$h(z) = 1 + t_1z + t_2z^2 + t_3z^3 + \cdots \ \ for \ \ z \in U$$

2 Main results

2.1 Coefficient estimates of the function class $K_{\Sigma}^{\lambda,\alpha}$

Definition 2.1. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $K_{\Sigma}^{\lambda,\alpha}$ if the following conditions are satisfied*

$$f \in \Sigma_m \text{ and } |arg((1-\lambda)f^{'}(z) + \lambda(1 + (\frac{zf^{''}(z)}{f^{'}(z)}))| < \frac{\alpha\pi}{2}, \tag{2.3}$$

$$(0 \leq \lambda < 2, 0 < \alpha \leq 1, z \in U)$$

.

$$f \in \Sigma_m \text{ and } |arg((1-\lambda)g^{'}(w) + \lambda(1 + (\frac{zg^{''}(w)}{g^{'}(w)}))| < \frac{\alpha\pi}{2}, \tag{2.4}$$

$$(0 \leq \lambda < 2, 0 < \alpha \leq 1, w \in U),$$

where the function g is given by (1.2).

By appropriate parameters λ, we can get several known subclasses of bi-univalent function class Σ. For example, we note that for $\lambda = 0$ the class $K_{\Sigma}^{\lambda,\alpha}$ reduces to the class $H_{\Sigma}(\alpha)$ introduced and studied by Srivastava et al. [14]. On the other hand, in the special case, when $\lambda = 1$, the class $K_{\Sigma}^{\lambda,\alpha}$ reduces to the class $K_{\Sigma}(\alpha)$ introduced and studied by Brannan and Taha [2].

Theorem 2.2. *Let the function $f \in \Sigma$ given by (1.1) be in the class $K_{\Sigma}^{\lambda,\alpha}$, $(0 \leq \lambda < 1, 0 < \alpha \leq 1, z \in U)$. Then*

$$|a_2| \leq \frac{\sqrt{2}\alpha}{\sqrt{\alpha(1-\lambda)+2}}, \tag{2.5}$$

$$|a_3| \leq \frac{2\alpha}{3(1+\lambda)} + \alpha^2, \tag{2.6}$$

and

$$|a_4| \leq \frac{\alpha}{(1+2\lambda}[1 + 2(\alpha-1)(1 + (\alpha-2)/3)]$$

$$+\frac{5\sqrt{2}\alpha^2}{\sqrt{(\alpha(1-\lambda)+2)}}(\frac{1}{3(1+\lambda)} + \frac{2\lambda}{4(1+2\lambda)}) \tag{2.7}$$

Proof. It follows from inequalities (2.3) and (2.4) that

$$(1-\lambda)f^{'}(z)+\lambda(1+\frac{zf^{''}(z)}{f^{'}(z)})=[s(z)]^{\alpha} \tag{2.8}$$

$$(1-\lambda)g^{'}(w)+\lambda(1+\frac{wg^{''}(w)}{g^{'}(w)})=[t(w)]^{\alpha}, \tag{2.9}$$

where $s(z)$ and $t(w)$ in P and have the forms

$$s(z)=1+s_1z+s_2z^2+s_3z^3+\cdots \tag{2.10}$$

and

$$t(w)=1+t_1w+t_2w^2+t_3w^3+\cdots \tag{2.11}$$

Now, equating the coefficients in (2.8) and (2.9), we obtain

$$2a_2=\alpha s_1 \tag{2.12}$$

$$3(1+\lambda)a_3-4\lambda a_2^2=\alpha s_2+\frac{\alpha(\alpha-1)}{2}s_1^2 \tag{2.13}$$

$$4(1+2\lambda)a_4-18\lambda a_2a_3-8\lambda a_2^3=\alpha s_3+\alpha(\alpha-1)s_1s_2\frac{\alpha(\alpha-1)(\alpha-2)}{6}s_1^3 \tag{2.14}$$

$$-2a_2=\alpha t_1 \tag{2.15}$$

$$-3(1+\lambda)a_3-2\lambda a_2^2=\alpha t_2+\frac{\alpha(\alpha-1)}{2}t_1^2 \tag{2.16}$$

and

$$-4(1+2\lambda)a_4+2(10+11\lambda)a_2a_3-4(5+3\lambda)a_2^3=\alpha t_3+\alpha(\alpha-1)t_1t_2\frac{\alpha(\alpha-1)(\alpha-2)}{6}t_1^3 \tag{2.17}$$

From Equations (2.12) and (2.15), we obtain

$$s_1=-t_1 \tag{2.18}$$

and

$$8a_2^2=\alpha^2(s_1^2+t_1^2) \tag{2.19}$$

Now, from Equations (2.13), (2.16) and (2.19), we obtain

$$[2(3-\lambda)-4(\alpha-1)]a_2^2=\alpha^2(s_2+t_2)$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(s_2+t_2)}{2[\alpha(1-\lambda)+2]}$$

Applying Lemma 1.1, for the coefficient s_2 and t_2, we immediately have Equations (2.5).
Now, in order to find the bound on $|a_3|$ we subtract Equation (2.16) from Equation (2.13) and then use Equations (2.18) and (2.19) to obtain

$$(1+\lambda)a_3 - 4\lambda a_2^2 + 3(1+\lambda)a_3 + 2(3+\lambda)a_2^2 = \alpha s_2 + \frac{\alpha(\alpha-1)}{2}s_1^2 - \alpha t_2 - \frac{\alpha(\alpha-1)}{2}t_1^2$$

Equivalently,

$$6(1+\lambda)(a_3 - a_2^2) = \alpha(s_2 - t_2) \tag{2.20}$$

Therefore, we have

$$a_3 = \frac{\alpha(s_2-t_2)}{6(1+\lambda)} + \frac{\alpha^2(s_1^2-t_1^2)}{8}$$

Applying Lemma 1.1 once again for the coefficient s_1, s_2, t_1 and t_2, we immediately have Equation (2.6).
Now, in order to find the bound on $|a_4|$, we subtract Equation (2.17) from Equation (2.14) to obtain

$$4(1+2\lambda)a_4 - 18\lambda a_2 a_3 + 8\lambda a_2^3 + (4+8\lambda)a_4 + 2(10+11\lambda)a_2 a_3 + 4(5+3\lambda)a_2^3$$

$$= \alpha(s_3 - t_3) + \alpha(\alpha-1)(s_1 s_2 - t_1 t_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}(s_1^3 - t_1^3)$$

Equivalently

$$8(1+2\lambda)a_4 - a_2[20(1+2\lambda)a_3 - 20(1+\lambda)a_2^2]$$

$$= \alpha(s_3 - t_3) + \alpha(\alpha-1)(s_1 s_2 - t_1 t_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}(s_1^3 - t_1^3)$$

Then, by using Equations (2.18), (2.19) and (2.20), we have

$$8(1+2\lambda)a_4 - a_2[\frac{20\alpha(1+2\lambda)(s_2-t_2)}{6(1+\lambda)} + 20\lambda a_2^2]$$

$$= \alpha(s_3 - t_3) + \alpha(\alpha-1)(s_1 s_2 - t_1 t_2) + \frac{\alpha(\alpha-1)(\alpha-2)}{6}(s_1^3 - t_1^3)$$

Now, applying Lemma 1.1 once again for the coefficient s_1, s_2, s_3, t_1, t_2 and t_3, we have Equation (2.7).
This completes the proof of Theorem 2.2. □

2.2 Coefficient estimates of the function class $K_{\Sigma}^{\lambda,\beta}$

Definition 2.3. *A function $f \in \Sigma$ given by (1.1) is said to be in the class $K_{\Sigma}^{\lambda,\beta}$ if the following conditions are satisfied*

$$Re\{(1-\lambda)f^{'}(z)+\lambda(1+(\frac{zf^{''}(z)}{f^{'}(z)}))\}>\beta, \tag{2.21}$$

$$(0 \le \lambda \le 1, 0<\beta \le 1, z \in U)$$

.

$$Re\{(1-\lambda)g^{'}(w)+\lambda(1+(\frac{zg^{''}(w)}{g^{'}(w)}))\}>\beta, \tag{2.22}$$

$$(0 \le \lambda \le 1, 0<\beta \le 1, w \in U),$$

where the function g is an extension of f^{-1} to U given by (1.2).

Theorem 2.4. *Let the function $f \in \Sigma$ given by (1.1) be in the class $K_{\Sigma}^{\lambda,\beta}$, $(0 \le \lambda \le 1, 0<\beta \le 1, z \in U)$. Then,*

$$|a_2| \le \frac{\sqrt{2}\beta}{\sqrt{\alpha(1-\lambda)+2}}, \tag{2.23}$$

$$|a_3| \le \frac{(1-\beta)[2+(\lambda-3)(1-\beta)]}{3(1+\lambda)}, \tag{2.24}$$

and

$$|a_4| \le \frac{(1-\beta)}{2(1+2\lambda}$$

$$+\frac{\sqrt{2}(1-\beta)}{\sqrt{(\alpha(1-\lambda)+2)}}[\frac{5(1-\beta)}{3(1+\lambda)}+\frac{5(1-\beta)^2(\lambda-3)}{6(1+\lambda)}-\frac{5(1+\lambda)(1-\beta)^2}{2}] \tag{2.25}$$

Proof. It follows, from inequalities (2.21) and (2.22), that there exist $s(z) \in P$ and $t(w) \in P$ such that

$$(1-\lambda)f^{'}(z)+\lambda(1+\frac{zf^{''}(z)}{f^{'}(z)})=\beta+(1-\beta)s(z), \tag{2.26}$$

$$(1-\lambda)g^{'}(w)+\lambda(1+\frac{wg^{''}(w)}{g^{'}(w)})=\beta+(1-\beta)t(w), \tag{2.27}$$

where $s(z)$ and $t(w)$ have the forms (2.10) and (2.11), respectively.

Now, equating the coefficient in (2.26) and (2.27), we obtain

$$2a_2 = (1-\beta)s_1, \tag{2.28}$$

$$3(1+\lambda)a_3 - 4\lambda a_2^2 = (1-\beta)s_2, \tag{2.29}$$

$$4(1+2\lambda)a_4 - 18\lambda a_2 a_3 - 8\lambda a_2^3 = (1-\beta)s_3, \tag{2.30}$$

$$-2a_2 = (1-\beta)t_1, \tag{2.31}$$

$$-3(1+\lambda)a_3 - 2\lambda a_2^2 = (1-\beta)t_2, \tag{2.32}$$

and

$$-4(1+2\lambda)a_4 + 2(10+11\lambda)a_2a_3 - 4(5+3\lambda)a_2^3 = (1-\beta)t_3 \tag{2.33}$$

From Equations (2.28) and (2.31), we obtain

$$s_1 = -t_1 \tag{2.34}$$

and

$$8a_2^2 = (1-\beta)^2(s_1^2 + t_1^2) \tag{2.35}$$

Also, from Equations (2.29), (2.32) and (2.35), we obtain

$$3(1+\lambda)a_3 - 4\lambda a_2^2 - 3(1+\lambda)a_3 - 2(3+\lambda)a_2^2 = (1-\beta)(s_2+t_2)$$

which is equivalent to

$$a_2^2 = \frac{(1-\beta)^2(s_2+t_2)}{2[\alpha(1-\lambda)+2]}$$

Now, by applying Lemma 1.1 , for the coefficient s_2 and t_2 , we have Equations (2.23).

Next, to find an estimate for $|a_3|$, we subtract Equation (2.30) from Equation (2.27) and use Equation (2.33) to obtain

$$3(1+\lambda)a_3 - 4\lambda a_2^2 + 3(1+\lambda)a_3 + 2(3+\lambda)a_2^2 = (1-\beta)(s_2-t_2).$$

Equivalently

$$6(1+\lambda)a_3 = \frac{(1-\beta)(s_2-t_2)}{6(1+\lambda)} + 2(\lambda-3)a_2^2$$

So, we have

$$a_3 = \frac{(1-\beta)(s_2 - t_2)}{6(1+\lambda)} + \frac{(1-\beta)^2(s_1^2 + t_1^2)2(\lambda - 3)}{48(1+\lambda)} \tag{2.36}$$

Applying Lemma 1.1 once again for the coefficient s_1, s_2, t_1 and t_2 , we immediately have Equation (2.24).
Now, to find an estimate for $|a_4|$ by subtracting Equation (2.33) from Equation (2.30), we obtain

$$4(1+2\lambda)a_4 - 18\lambda a_2 a_3 + 8\lambda a_2^3 + (4+8\lambda)a_4 - 2(10+11\lambda)a_2 a_3 + 4(5+3\lambda)a_2^3$$
$$= (1-\beta)(s_3 - t_3)$$

Equivalently

$$8(1+2\lambda)a_4 - a_2[20(1+2\lambda)a_3 - 20(1+\lambda)a_2^2] = (1-\beta)(s_3 - t_3)$$

Then, by using Equations (2.35) and (2.36), we have

$$a_4 = \frac{(1-\beta)(s_3 - t_3)}{8(1+2\lambda)} +$$

$$a_2[\frac{5(1-\beta)(s_2 - t_2)}{12(1+\lambda)} + \frac{5(1-\beta)^2(s_1^2 + t_1^2)(\lambda - 3)}{48(1+\lambda)} - \frac{5(1-\beta)^2(s_1^2 + t_1^2)(1+\lambda)}{16}]$$

Now, applying Lemma 1.1 once again for the coefficient s_1, s_2, s_3, t_1, t_2 and t_3, we have Equation (2.25).
This completes the proof of Theorem 2.4. □

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