

Generalizing Abundancy Index to Gaussian Integers

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Abstract

Abundancy index refers to the ratio of the sum of the divisors of a number to the number itself. It is a concept of great importance in defining friendly and perfect numbers. Here, we describe a suitable generalization of abundancy index to the ring of Gaussian Integers ($\mathbb{Z}[i]$). We first show that this generalization possesses many of the useful properties of the traditional abundancy index in \mathbb{Z} . We then investigate k -powerful τ -perfect numbers and prove results regarding their existence in $\mathbb{Z}[i]$.

1 Introduction

The idea of abundancy index dates back to around 2000 years ago with work by the great mathematicians of Europe such as Fermat, Mersenne, Legendre, and Euler, and, more recently, by Paul Erdős. It is of great importance in the study of perfect and friendly numbers. In this paper, we seek to find a suitable generalization of a function, abundancy index $I(\eta)$ for complex numbers, specifically the Gaussian Integers ($\mathbb{Z}[i]$), study its properties, and apply it to the problem of the existence of different kinds of perfect numbers.

We first begin with definitions of notation. We refer to \mathbb{N} as the set of natural numbers, \mathbb{Z} as the set of integers, and $\mathbb{Z}[i]$ as the ring of Gaussian

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Integers. We also define $||\eta|| = \eta\bar{\eta} = a^2 + b^2$ to be the norm of a complex number $\eta = a + bi$ and $|\eta| = \sqrt{||\eta||} = \sqrt{a^2 + b^2}$ the absolute value. In $\mathbb{Z}[i]$, we give the following definition in order to differentiate between nearly identical factors of numbers:

Definition 1.1. We define two Gaussian Integers $\alpha, \beta \in \mathbb{Z}[i]$ to be associated; i.e., $\alpha \sim \beta$ if there exists a unit $u \in \mathbb{Z}[i]$ ($|u| = 1$) such that $\alpha = u\beta$.

For non-zero complex number η , let $\arg(\eta)$ represent the angle of η . Clearly, $\arg(\eta) \in [0, 2\pi)$. It is easy to show that for any Gaussian Integer, we have an associate in the first quadrant. Henceforth, when we refer to primes in $\mathbb{Z}[i]$, we only deal with those in the first quadrant; i.e., $\pi \in \mathbb{Z}[i]$ with $\arg(\pi) \in [0, \frac{\pi}{2})$.

With the above definitions of notation in mind, we proceed to discuss the traditional sum of divisors function, $\sigma_k(n)$ (where $n = p_1^{e_1} p_2^{e_2} \dots p_l^{e_l} > 0$ is the prime factorization of n), given by:

$$\sigma_k(n) = \sum_{c|n} c^k = \prod_{p|n} \frac{p_i^{k(e_i+1)} - 1}{p^k - 1} \quad (1.1)$$

We write $\sigma_1(n) = \sigma(n)$. From the above definition, $\sigma_k(n)$ is a weakly multiplicative function: $\sigma_k(mn) = \sigma_k(m)\sigma_k(n)$ for $(m, n) = 1$, i.e., m and n are co-prime.

Abundancy index of $n \in \mathbb{N}$ is defined as the following:

$$I_k(n) = \frac{\sigma_k(n)}{n^k} = \sigma_{-k}(n) \quad (1.2)$$

With $I_1(n) = I(n)$. $I_k(n)$ is referred to as the k -powerful abundancy in the rest of the paper.

Abundancy index is of great importance in friendly and solitary numbers. Two numbers are said to be ‘friendly’ if they have the same abundancy; i.e., $I(n_1) = I(n_2) \iff n_1$ and n_2 are friendly. For example, all perfect numbers form a set of friendly numbers with abundancy 2. Conversely, a number that has no friends is said to be ‘solitary’. For example, 1, 18, 48, and so on are solitary numbers as no other numbers share the same abundancy [5]. Friendly and solitary numbers still constitute a number of unsolved problems in modern number theory. For example, the question whether 10 is solitary remains an unsolved problem. More generally, we have the concept of k -powerful friendly numbers where the k -powerful abundancy of two numbers

are equal; i.e., $I_k(n_1) = I_k(n_2) \iff n_1$ and n_2 are k -powerful friendly. Conversely, we have k -powerful solitary numbers: numbers whose k -powerful abundancy is not shared by any other number.

We now discuss the Gaussian Integers $\mathbb{Z}[i]$. Forming an integral domain, the Gaussian Integers have been used to prove innumerable important results and novel uses have even been found in coding theory [6]. Due to its importance in various spheres of modern number theory, the ring $\mathbb{Z}[i]$ and its properties is of prime importance. A complex-to-integer generalization of abundancy index is described by Defant [3]. However, such an approach loses the important information of factorization in $\mathbb{Z}[i]$ (of the imaginary component). Here, we endeavor to describe a complex-to-complex definition of abundancy.

Perfect numbers have been an important object of mathematical studies for millennia. Numbers whose sum of divisors is equal to twice the number itself are said to be perfect; i.e., they have an abundancy of 2. Close relationships between perfect numbers and Mersenne Primes have been found in the set of integers as well as that of Gaussian Integers [1]. Several generalizations of these numbers are known of which one of the most popular is the multiply-perfect number, defined to be numbers of integral abundancy. Generalizing further, we have k -powerful multiply-perfect numbers with $I_k(n) = t \in \mathbb{N}$. In this paper, we take this concept, extend it to $\mathbb{Z}[i]$: the k -powerful τ -perfect numbers.

The paper is structured as follows. We begin by describing a suitable generalization of abundancy to $\mathbb{Z}[i]$. We then proceed to study and prove several important properties of this generalization. Finally, we prove some interesting results regarding the bounds on the function and the existence of k -powerful τ -perfect numbers for various k .

2 Extension to the $\mathbb{Z}[i]$

There are innumerable possible generalizations of abundancy from \mathbb{Z} to $\mathbb{Z}[i]$. We propose the following complex-to-complex definition of abundancy index:

Definition 2.1. *Given $\eta = u\pi_1^{e_1}\pi_2^{e_2} \dots \pi_k^{e_k}$ is the factorization of $\eta \in \mathbb{Z}[i]$ where u is a unit ($|u| = 1$) and each π lies in the first quadrant. Then:*

$$I(\eta) = \prod \frac{\pi_i^{e_i+1} - 1}{\pi_i^{e_i}(\pi_i - 1)} = \frac{u\sigma(\eta)}{\eta} \tag{2.3}$$

where $\sigma(\eta)$ is the complex sum of divisors (as defined in [2]) as:

$$\sigma(\eta) = \prod \frac{\pi_i^{e_i+1} - 1}{\pi_i - 1} \quad (2.4)$$

Similarly, we define $I_k(\eta)$ as:

$$I_k(\eta) = \prod \frac{\pi_i^{k(e_i+1)} - 1}{\pi_i^{ke_i}(\pi_i^k - 1)} = \frac{u^k \sigma_k(\eta)}{\eta^k} \quad (2.5)$$

with the general complex sum of divisors $\sigma_k(\eta)$ as:

$$\sigma_k(\eta) = \prod \frac{\pi_i^{k(e_i+1)} - 1}{\pi_i^k - 1} \quad (2.6)$$

We also observe that:

$$I_k(\eta) = \prod \frac{\pi_i^k - \frac{1}{\pi_i^{ke_i}}}{\pi_i^k - 1} = \prod \frac{\frac{1}{\pi_i^{k(e_i+1)}} - 1}{\frac{1}{\pi_i^k} - 1} = \sigma_{-k}(\eta) \quad (2.7)$$

Finally, we also note that $\sigma_k(\eta) \in \mathbb{Z}[i] \implies I_k(\eta) \in \mathbb{Q}[i]$

We now proceed to list and prove some important properties of abundancy index in the ring of Gaussian Integers (many of which are observed in the case of integers as well).

2.1 Important properties of $I(\eta)$

1. $I(\eta)$ is weakly multiplicative; i.e., for any $\alpha, \beta \in \mathbb{Z}[i]$ where $(\alpha, \beta) = 1$, $I_k(\alpha\beta) = I_k(\alpha)I_k(\beta)$.
2. For $\alpha, \beta \in \mathbb{Z}[i]$ such that $\alpha|\beta$, $|I_k(\alpha)| \leq |I_k(\beta)|$ with equality only when $\alpha \sim \beta$.
3. For all primes $\pi \in \mathbb{Z}[i]$, $I_k(\pi^n) \neq I_k(z)$ for all possible $z \in \mathbb{Z}[i]$; i.e., all powers of primes are solitary.
4. For all primes $\pi \in \mathbb{Z}[i]$, $\left| \frac{\pi^k+1}{\pi^k} \right| \leq |I_k(\pi^n)| < \frac{|\pi|^k}{|\pi_i|^{k-1}}$

2.2 Lemmas

Here, we provide some lemmas used in subsequent sections.

Lemma 2.2. *Given a complex integer $\eta \in \mathbb{Z}[i]$ where $\Re(\eta) > 0$ and $|\eta| > 1$, for an integer $n > 0$, we have the following inequality:*

$$|1 + \eta + \eta^2 + \dots + \eta^n| > |\eta^n| \tag{2.8}$$

Proof. The proof of the given lemma is described in detail by Spira [2]. \square

Lemma 2.3. *Given a complex integer $\eta \in \mathbb{Z}[i]$ where $\Re(\eta) > 0$ and $|\eta| > 1$, for an integer $n > 0$, we have the following inequality:*

$$|\eta^n - 1| > |\eta^n - \eta| \tag{2.9}$$

Proof. We first prove the case $n > 2$ and then take the cases $n = 2$ and $n = 1$ individually.

We prove the given lemma for the norm of each side of the inequality which also holds true for absolute values. Consider the following expression:

$$\left\| \frac{\eta^n - 1}{\eta^{n-1} - 1} \right\| = \left\| \eta - \frac{1 - \eta}{\eta^{n-1} - 1} \right\| \geq \|\eta\| - \left\| \frac{1 - \eta}{\eta^{n-1} - 1} \right\| \tag{2.10}$$

However, we observe that the left hand side is an integer and $\|\eta\|$ is an integer. Further, making use of Lemma 2.2 and the bound $n > 2$, we observe that:

$$\left\| \frac{1 - \eta}{\eta^{n-1} - 1} \right\| = \left\| \frac{1 - \eta}{(\eta - 1)(1 + \eta + \eta^2 + \dots + \eta^{n-2})} \right\| \tag{2.11}$$

$$= \left\| \frac{1}{1 + \eta + \eta^2 + \dots + \eta^{n-2}} \right\| \leq \left\| \frac{1}{\eta^{n-2}} \right\| < 1 \tag{2.12}$$

$$\implies \left\| \frac{1 - \eta}{\eta^{n-1} - 1} \right\| < 1 \tag{2.13}$$

Hence, we can neglect this term as it is less than 1 and the other terms are integers. Thus, the previous inequality simplifies to:

$$\left\| \frac{\eta^n - 1}{\eta^{n-1} - 1} \right\| \geq \|\eta\| \implies \|\eta^n - 1\| \geq \|\eta^n - \eta\| \tag{2.14}$$

This proves the first case of the inequality. We now show that the given inequality holds for $n = 2$ and $n = 1$.

For $n = 2$:

$$\left| \frac{\eta^2 - 1}{\eta - 1} \right| = |\eta + 1| > |\eta| \implies |\eta^2 - 1| > |\eta^2 - \eta| \quad (2.15)$$

Which follows from $\Re(\eta) > 0$.

For $n = 1$:

$$|\eta - 1| > |\eta - \eta| = 0 \quad (2.16)$$

Hence, the given inequality holds for $n > 1$. \square

Lemma 2.4. *Given a complex integer $\eta \in \mathbb{Z}[i]$ where $\Re(\eta) > 0$ and $|\eta| > 1$, for $n \geq m \geq 0$, we have the following inequality:*

$$|\eta^n - 1| \geq |\eta^n - \eta^m| \quad (2.17)$$

With equality when $m = 0$

Proof. If $m = 0$, then it results in simple equality.

In the case of $m > 0$, the given result can be proved by repeated application of Lemma 2.3:

$$|\eta^n - \eta^m| = |\eta^{m-1}| |\eta^{n-m+1} - \eta| \leq |\eta^{m-1}| |\eta^{n-m+1} - 1| \quad (2.18)$$

$$\leq |\eta^n - \eta^{m-1}| \quad (2.19)$$

Repeating the above procedure m times, we obtain the desired result. \square

3 Proofs of Properties of $I_k(\eta)$

Property 1. *For $\alpha, \beta \in \mathbb{Z}[i]$ where $(\alpha, \beta) = 1$, $I_k(\alpha\beta) = I_k(\alpha)I_k(\beta)$; i.e., $I_k(\alpha)$ is weakly multiplicative.*

Proof. From Definition 2.1, we have

$$I_k(\alpha\beta) = \frac{u^k \sigma_k(\alpha\beta)}{(\alpha\beta)^k} = u^k \frac{\sigma_k(\alpha)}{\alpha} \frac{\sigma_k(\beta)}{\beta} \quad (3.20)$$

Where u is the unit corresponding to $\alpha\beta$. Let v, w be the units corresponding to α , and β respectively. Clearly, $u = vw$ (as if α, β , and $\alpha\beta$ are factorized into

primes in the first quadrant, then the factorization of $\alpha\beta$ will give vw as the corresponding unit of $\alpha\beta$ which we have defined to be u). Thus, $u^k = v^k w^k$. Substituting:

$$I_k(\alpha\beta) = \frac{v^k \sigma_k(\alpha) w^k \sigma_k(\beta)}{\alpha^k \beta^k} = I_k(\alpha)I_k(\beta) \tag{3.21}$$

□

Property 2. For $\alpha, \beta \in \mathbb{Z}[i]$ such that $\beta|\alpha$, $|I_k(\alpha)| \geq |I_k(\beta)|$ with equality only when $\alpha \sim \beta$.

Proof. Let the prime factorization of α be $u \prod \pi_i^{e_i}$ and that of β be $v \prod \rho_j^{f_j}$. Since $\beta|\alpha$, $\{\rho_j\} \subseteq \{\pi_i\}$ and for any $\pi_i = \rho_j \implies e_i \geq f_j$.

Now, we consider the following expression:

$$\left| \frac{I_k(\alpha)}{I_k(\beta)} \right| = \left| \left(\prod \frac{\pi_i^{k(e_i+1)} - 1}{\pi_i^{ke_i}(\pi_i^k - 1)} \right) / \left(\prod \frac{\rho_j^{k(f_j+1)} - 1}{\rho_j^{kf_j}(\rho_j^k - 1)} \right) \right| \tag{3.22}$$

Since we aim to prove that this expression is greater than one, we observe the terms corresponding to primes not included in $\{\rho_j\}$ increase the value of the result by a factor greater than one. Thus, they can be neglected and the equality can be replaced with a greater than or equal to. In the remaining terms, if $e_i = f_j$ for some pair of primes in the two sets, we observe that the terms in the numerator and denominator cancel. Hence, we need only consider the cases where $e_i \geq f_j$ in the set $\{\rho_j\}$ with $\pi_i = \rho_j$.

Hence, our original expression reduces to the following:

$$\left| \frac{I_k(\alpha)}{I_k(\beta)} \right| \geq \left| \prod \frac{\rho_j^{k(e_i+1)} - 1}{\rho_i^{k(e_i-f_j)}(\rho_i^{k(f_j+1)} - 1)} \right| \tag{3.23}$$

However, using Lemma 2.4 with $n = k(e_i + 1)$ and $m = k(e_i - f_j)$, we have the following:

$$\left| \frac{\rho_j^{k(e_i+1)} - 1}{\rho_i^{k(e_i-f_j)}(\rho_i^{k(f_j+1)} - 1)} \right| = \frac{|\rho_j^{k(e_i+1)} - 1|}{|\rho_i^{k(e_i+1)} - \rho^{k(e_i-f_j)}|} \geq 1 \tag{3.24}$$

With equality when $m = k(e_i - f_j) = 0 \implies e_i = f_j$. Substituting into the previous equation:

$$\left| \frac{I_k(\alpha)}{I_k(\beta)} \right| \geq \left| \prod \frac{\rho_j^{k(e_i+1)} - 1}{\rho_i^{k(e_i-f_j)}(\rho_i^{k(f_j+1)} - 1)} \right| \geq 1 \tag{3.25}$$

With equality when $e_i = f_j$ for all the primes ρ_j . Thus, equality occurs only when α and β have the same factorization; i.e., when $\alpha \sim \beta$.

Hence, $|I_k(\alpha)| \geq |I_k(\beta)|$ with equality when $\alpha \sim \beta$. □

We prove the third property as a corollary of a more general theorem:

Theorem 3.1 (Greening’s Criterion). *For a given integer k and $\eta \in \mathbb{Z}[i]$ such that $(\eta^k, \sigma_k(\eta)) = 1$, η is k -powerful solitary; i.e., there are no values of $z \in \mathbb{Z}[i]$ and $z \asymp \eta$ such that $I_k(\eta) = I_k(z)$.*

Proof. We shall prove the above result by contradiction. Let there, in fact, exist $z \in \mathbb{Z}[i]$, $z \asymp \eta$, such that $I_k(\eta) = I_k(z)$. Then:

$$z^k u^k \sigma_k(\eta^n) = \eta^k v^k \sigma_k(z) \tag{3.26}$$

Where u and v are the units corresponding to η and z respectively. We know that $(\eta^k, \sigma_k(\eta)) = 1$. Since, $\eta^k | z^k \sigma_k(\eta)$ and $(\eta^k, \sigma_k(\eta)) = 1$, $\eta^k | z^k \implies \eta | z$ (by comparing the power of every prime π in the factorization of z and η). Since $\pi \asymp z$, we have (from Property 2), $I_k(\eta) < I_k(z)$. This contradicts our initial assumption. Thus, η is k -powerful solitary. □

Property 3. *For all primes $\pi \in \mathbb{Z}[i]$, $I_k(\pi^n) \neq I_k(z)$ for all possible $z \in \mathbb{Z}[i]$ and $z \asymp \pi^n$; i.e., all powers of primes are solitary.*

Proof. Clearly, $\sigma_k(\pi^n) = 1 + \pi^k + \pi^{2k} \dots + \pi^{kn} = 1 + \pi^k l$ for $l = 1 + \pi^k + \pi^{2k} \dots + \pi^{k(n-1)} \in \mathbb{Z}[i]$. Hence, $\pi^k \nmid \sigma_k(\pi^n)$. Thus, $(\pi^{kn}, \sigma_k(\pi^n)) = 1$. Hence, by Greening’s Criterion (Theorem 3.1), all powers of primes are solitary. □

Corollary 3.2. *For all primes $\pi \in \mathbb{Z}[i]$, $\pi^{n_1} \bar{\pi}^{n_2}$ is k -powerful solitary for all $n_1, n_2, k \in \mathbb{Z}$; i.e., there are no values of $z \in \mathbb{Z}[i]$ such that $I_k(\pi^{n_1} \bar{\pi}^{n_2}) = I_k(z)$.*

Proof. In the case that $\pi \sim \bar{\pi}$, the above result is equivalent to Property 3. In the case that $\pi \not\sim \bar{\pi}$, we shall prove the above result by contradiction.

Let $|((\pi^{n_1} \bar{\pi}^{n_2})^k, \sigma_k(\pi^{n_1} \bar{\pi}^{n_2}))| > 1$. We observe that:

$$\sigma_k(\pi^{n_1} \bar{\pi}^{n_2}) = \sigma_k(\pi^{n_1}) \sigma_k(\bar{\pi}^{n_2}) \tag{3.27}$$

As proved in Property 3, $(\pi^{kn_1}, \sigma_k(\pi^{n_1})) = (\bar{\pi}^{kn_2}, \sigma_k(\bar{\pi}^{n_2})) = 1$. Hence, $|(\bar{\pi}^{kn_2}, \sigma_k(\pi^{n_1}))| > 1$ or $|(\pi^{kn_1}, \sigma_k(\bar{\pi}^{n_2}))| > 1$. We take the first case: the other case follows in the same manner. This means that $\bar{\pi} | \sigma_k(\pi^{n_1})$. Expanding, we have:

$$\bar{\pi} \text{ divides } (1 + \pi^k + \pi^{2k} + \dots + \pi^{kn_1}) \implies \|\bar{\pi}\| \text{ divides } \|1 + \pi^k + \pi^{2k} + \dots + \pi^{kn_1}\| \tag{3.28}$$

$$\implies \frac{\|1 + \pi^k + \pi^{2k} + \dots + \pi^{kn_1}\|}{\|\pi\|} \in \mathbb{Z} \tag{3.29}$$

$$\implies \left\| \frac{1}{\pi} + \pi^{k-1} + \pi^{2k-1} + \dots + \pi^{kn_1-1} \right\| \in \mathbb{Z} \tag{3.30}$$

However, since $|\pi| > 1$ and $k \geq 1$, we have:

$$\left\| \frac{1}{\pi} + \pi^{k-1} + \pi^{2k-1} + \dots + \pi^{kn_1-1} \right\| \notin \mathbb{Z} \tag{3.31}$$

Hence, $|((\pi^{n_1} \bar{\pi}^{n_2})^k, \sigma_k(\pi^{n_1} \bar{\pi}^{n_2}))| = 1$. Thus, by applying Greening's Criterion (Theorem 3.1), we have the desired result. \square

Property 4. For all primes $\pi \in \mathbb{Z}[i]$, $\left| \frac{\pi^k + 1}{\pi^k} \right| \leq |I_k(\pi^n)| < \frac{|\pi|^k}{|\pi_i|^{k-1}}$

Proof. From Property 2, we have:

$$\pi | \pi^n \implies |I_k(\pi)| \leq |I_k(\pi^n)| \implies \left| \frac{\pi^k + 1}{\pi^k} \right| \leq |I_k(\pi^n)| \tag{3.32}$$

This proves the left side of the inequality.

From Equation 2.7, we observe:

$$|I_k(\pi^n)| = |\sigma_{-k}(\pi^n)| = \left| \sum_{i=0}^n \frac{1}{\pi^{ik}} \right| \leq \sum_{i=0}^n \frac{1}{|\pi|^{ik}} \tag{3.33}$$

$$< \sum_{i=0}^{\infty} \frac{1}{|\pi|^{ik}} = \frac{|\pi|^k}{|\pi|^k - 1} \tag{3.34}$$

This proves the right side of the inequality.

Note that, since we have proved the above inequality for absolute values, it must hold for the norms as well. \square

We now proceed to prove results on a direct application of the above properties: perfect numbers.

4 Perfect Numbers

We begin this section with a formal definition of k -powerful τ -perfect numbers. This is followed by a number of lemmas and, finally, a theorem in regarding the existence of such numbers for $k \in \mathbb{N}$.

Definition 4.1. We define a complex number $\eta \in \mathbb{Z}[i]$ to be k -powerful τ -perfect, if, for $k \in \mathbb{N}$, $\tau \in \mathbb{Z}[i]$, and $|\tau| > 1$:

$$I_k(\eta) = \tau \tag{4.35}$$

Along the same line, we also define k -powerful t -norm-perfect numbers for $k \in \mathbb{N}$, $t \in \mathbb{Z}$, and $t > 1$ to be complex numbers $\eta \in \mathbb{Z}[i]$ such that:

$$||I_k(\eta)|| = t \tag{4.36}$$

We now proceed to prove that such numbers exist for only $k = 1$. We first prove the following lemmas.

Lemma 4.2. For $s > 1$,

$$\zeta_{\mathbb{Q}[i]}(s) = \zeta(s)L(s, \chi_1) = \zeta(s)\beta(s) \tag{4.37}$$

where $\zeta_{\mathbb{Q}[i]}(s)$ is the Dedekind Zeta function on the field $\mathbb{Q}[i]$, $\zeta(s)$ is the Riemann Zeta function, and $L(s, \chi_1) = \beta(s)$ is the Dirichlet L -function with character $\chi_1(n)$ modulo 4 (equivalent to the Dirichlet Beta function $\beta(s)$).

Proof. Let the set of prime ideals in $\mathbb{Z}[i]$ (in the first quadrant) be represented by \mathcal{P} . For each prime ideal $\pi \in \mathbb{Z}[i]$, we have the following 3 cases:

- $\pi \sim p$ and $||\pi|| = p^2$ for some prime $p \in \mathbb{Z}$ such that $p \equiv 3 \pmod{4}$.
- $\pi \sim \bar{\pi}$ and $||\pi|| = p$ for some prime $p \in \mathbb{Z}$ such that $p \equiv 2 \pmod{4}$ (this case only holds for $\pi \sim 1 + i$).
- $\pi \approx \bar{\pi}$ and $||\pi|| = ||\bar{\pi}|| = p$ for some prime $p \in \mathbb{Z}$ such that $p \equiv 1 \pmod{4}$

Thus, from the definition of $\zeta_{\mathbb{Q}[i]}(s)$:

$$\zeta_{\mathbb{Q}[i]}(s) = \prod_{\pi \in \mathcal{P}} \frac{1}{1 - \|\pi\|^{-s}} = \prod_{\substack{\pi \in \mathcal{P} \\ p = \|\pi\|}} \frac{1}{1 - p^{-s}} \tag{4.38}$$

$$= \frac{1}{1 - 2^{-s}} \prod_{p \equiv 1 \pmod{4}} \frac{1}{(1 - p^{-s})^2} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 - p^{-2s}} \tag{4.39}$$

$$= \prod_p \frac{1}{1 - p^{-s}} \left(\prod_{p \equiv 1 \pmod{4}} \frac{1}{1 - p^{-s}} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 + p^{-s}} \right) \tag{4.40}$$

$$= \zeta(s) \left(\prod_{p \equiv 1 \pmod{4}} \frac{1}{1 + \chi_1(p)p^{-s}} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 + \chi_1(p)p^{-s}} \right) \tag{4.41}$$

$$= \zeta(s) \prod_{p \neq 2} \frac{1}{1 + \chi_1(p)p^{-s}} = \zeta(s)L(s, \chi_1) = \zeta(s)\beta(s) \tag{4.42}$$

As follows from the definitions of $\zeta(s)$, $L(s, \chi_1)$, and $\beta(s)$. □

Lemma 4.3. For $s > 1$, $\beta(s) \leq \zeta(s)$

Proof. By definition:

$$\beta(s) = \prod_{p \equiv 1 \pmod{4}} \frac{1}{1 - p^{-s}} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 + p^{-s}} \tag{4.43}$$

$$\leq \frac{1}{1 - 2^{-s}} \prod_{p \equiv 1 \pmod{4}} \frac{1}{1 - p^{-s}} \prod_{p \equiv 3 \pmod{4}} \frac{1}{1 - p^{-s}} \tag{4.44}$$

$$= \prod_p \frac{1}{1 - p^{-s}} = \zeta(s) \tag{4.45}$$

□

Lemma 4.4. For $\eta \in \mathbb{Z}[i]$ and $k \geq 2$:

$$\|I_k(\eta)\| < \zeta_{\mathbb{Q}[i]}(k) \tag{4.46}$$

Proof. Let \mathcal{P} represent the set of prime ideals in $\mathbb{Z}[i]$ (in the first quadrant). Further, let $\mathcal{D}(\eta) = \{\pi_i\}$ be the set of primes dividing η with $e_i = \text{ord}_{\pi_i}(\eta)$. Using Property 4, we have:

$$\|I_k(\eta)\| = \prod_{\pi_i \in \mathcal{D}(\eta)} \|I_k(\pi_i^{e_i})\| \leq \prod \frac{\|\pi_i\|^k}{\|\pi_i\|^k - 1} = \prod \frac{\|\pi_i\|^k}{\|\pi_i\|^k - 1} \tag{4.47}$$

$$< \prod_{\pi \in \mathcal{P}} \frac{\|\pi\|^k}{\|\pi\|^k - 1} = \zeta_{\mathbb{Q}[i]}(k) \tag{4.48}$$

□

Now we prove the main theorem:

Theorem 4.5. *There exist no k -powerful τ -perfect numbers or k -powerful t -norm-perfect numbers for all $k \geq 2$.*

Proof. We take two cases: $k \geq 3$ and $k = 2$.

For the first case, from Lemmas 4.2, 4.3, 4.4, we have the following:

$$\|I_k(\eta)\| < \zeta_{\mathbb{Q}[i]}(k) = \zeta(k)\beta(k) \leq \zeta(k)^2 \quad (4.49)$$

Thus, since $\zeta(s)$ is monotone decreasing for $s > 1$, we have:

$$\|I_k(\eta)\| < \zeta(k)^2 \leq \zeta(3)^2 \approx 1.20206^2 < 2 \quad (4.50)$$

For the case $k = 2$, from Lemma 4.2, we have:

$$\|I_2(\eta)\| < \zeta_{\mathbb{Q}[i]}(2) = \zeta(2)\beta(2) \quad (4.51)$$

Using $\zeta(2) = \frac{\pi^2}{6}$ and $\beta(2) = G = 0.916$ (Catalan's constant):

$$\|I_2(\eta)\| < \zeta(2)\beta(2) = \frac{\pi^2 G}{6} \approx 1.645 \times 0.916 = 1.50682 < 2 \quad (4.52)$$

Thus, it is impossible to find any $\tau \in \mathbb{Z}[i]$ with $|\tau| \geq 2$ to satisfy $I_k(\eta) = \tau$ for any Gaussian Integer η . Hence, it is not possible for any Gaussian Integer to be k -powerful τ -perfect for $k \geq 2$. Similarly, by the above result, the existence of k -powerful t -norm-perfect numbers is impossible in $\mathbb{Z}[i]$ for $k \geq 2$.

□

5 Conclusion

In this paper, we have proposed a complex-to-complex definition of abundancy index for the Gaussian Integers that possesses many of the properties of the traditional abundancy index in integers. We have further proved that the existence of k -powerful τ -perfect numbers in $\mathbb{Z}[i]$ only holds for $k \geq 2$.

We note that many of the results can be easily generalized to other imaginary quadratic rings which are unique factorization domains; i.e., we are considering rings of the form $\mathbb{Z}[\sqrt{d}]$ (if $d \equiv 1 \pmod{4}$) and $\mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ (if $d \equiv 3 \pmod{4}$) for $d \in \{-2, -3, -7, -11, -19, -43, -67, -163\}$. This is because, in most of the proofs, we only make use of the fact that the norm is an integer. Since the the given rings have integral norm, it should not be very difficult to further generalize the above results to these rings.

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