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Generalized Centralizing and Skew-Centralizing Mappings on Rings

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Abstract

The main purpose of this paper is to introduce generalized centralizing and generalized skew$-$centralizing mappings which are generalizations of centralizing and skew$-$centralizing. We also describe the structure of a pair of endomorphism mappings that are generalized centralizing and generalized skew$-$centralizing mappings on a suitable ring.

1 Introduction

Throughout this paper, R will represent a ring with center $Z(R)$. An element $x \in R$ is called $n-$ torsion free if $nx = 0$ implies $x = 0$. Further a ring R is called an $n-$torsion free ring if every element in R is $n-$torsion free. A ring R is called a prime ring if $aRb = \{0\}$ with $a, b \in R$ implies $a = 0$ or $b = 0$. R is called a semiprime ring if $aRa = \{0\}$ with $a \in R$ implies $a = 0$. A prime ring is obviously semiprime. We refer the reader to [1] for the definitions and related properties of these objects. For $x, y \in R$, we denote $[x, y] = xy - yx$, the commutator of x and y, and $x \circ y = xy + yx$, the skew-commutator of x and y. The following commutator identities hold : $[xy, z] = x[y, z] + [x, z]y$

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and $[x, yz] = y[x, z] + [x, y]z$ for all x, y, z in R.
A mapping $f : R \to R$ is called centralizing if $[f(x), x] \in Z(R)$ for all $x \in R$; in particular, if $[f(x), x] = 0$ for all $x \in R$, then it is called commuting. A commuting map is centralizing but the converse is not true, in general. Analogously, a mapping $f : R \to R$ is called skew−centralizing on R if $f(x) \circ x \in Z(R)$ for all $x \in R$ and is called skew−commuting on R if $f(x) \circ x = 0$ for all $x \in R$.

In 1955, Divinsky [10] initiated the study of commuting and centralizing mappings and proved that a simple Artinian ring is commutative if it has a commuting automorphism different from the identity mapping. In 1970, Luh [12] generalized Divinsky's result to prime rings. In [15] Posner proved that a prime ring must be commutative if it possesses a nonzero centralizing derivation. In 1976, Mayne [13] obtained the analogous result to Posner's result for nonidentity centralizing automorphism. This result of Posner was generalized in many directions by serveral authors and they studied the relationship between the structure of prime or semiprime ring and the behaviour of additive maps satisfying various conditions. Some authors have studied derivations with annihilator conditions in prime and semiprime rings. For example, we refer the reader to [3], [7], [8], [9] and [11].

In 1993, Bresar [4] initiated the study of functional identities and obtained a characterization of commuting additive mappings on prime rings. In [5], Bresar proved that there are no nonzero skew−commuting additive mappings on a 2−torsion free semiprime rings. In other words, if R is a 2−torsion free semiprime ring, and $f : R \to R$ is an additive mapping such that $f(x) \circ x = 0$ for all $x \in R$, then $f(x) = 0$ for all $x \in R$.

Bell and Deng [2] extended the notion of commuting to n−commuting, where n is an arbitrary positive integer, by defining a mapping $f : R \to R$ to be n−commuting on R if $[x^n, f(x)] = 0$ for all $x \in R$. Park and Jung [14] introduced the concept of n− skew commuting (resp. n−skew centralizing) on R if $f(x) \circ x^n = 0$ (resp. $f(x) \circ x^n \in Z(R)$) for all $x \in R$. These inspire us to introduce the notion of generalizations of centralizing / commuting and skew−centralizing / skew−commuting. The aim of this research work is to investigate some results on the notion of generalizations of commuting / centralizing and skew−commuting / skew centralizing in suitable rings.

2 Preliminaries

In this section, we define the notion of generalizations of centralizing / commuting and skew−centralizing / skew−commuting. Throughout this paper,

R will represent a ring with center $Z(R)$.

Definition 2.1. Let f and g be mapping from R into itself.
(1) We say that f and g are generalized centralizing on R if $[f(x), g(x)] \in Z(R)$ for all x in R and we say that f and g are generalized commuting on R if $[f(x), g(x)] = 0$ for all x in R.
(2) We say that f and g are generalized skew−centralizing on R if $f(x) \circ g(x) \in Z(R)$ for all x in R and we say that f and g are generalized skew−commuting on R if $f(x) \circ g(x) = 0$ for all x in R.

Example 2.2. *Let* $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Big| a, b, c, d \in \mathbb{Z} \right\}$ *Then R is a noncommutative ring under the usual matrix addition and multiplication.*
Let f_1, f_2, g_1, g_2, g_3: $R \to R$ be mappings defined by

$$f_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$f_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & 0 \\ c & a \end{pmatrix}$$

$$g_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}$$

$$g_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$$g_3 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for all $a, b, c, d \in \mathbb{Z}$.
We can easily check that
(1) f_1 and f_2 are generalized commuting on R but f_1 and g_2 are not generalized commuting on R,
(2) f_2 and g_3 are generalized skew−centralizing on R.

3 Main Results

Theorem 3.1. *Let R be a 2− torsion free ring, and let f and g be endomorphisms of R. Suppose that $F : R \to R$ is a mapping $x \longmapsto f(x) \circ g(x)$ for all x in R. If F and g are generalized commuting on R then $[f(x), g(x^3)] = 0$ for all x in R.*

Proof. Assume that F and g are generalized commuting on R.
Then $[F(x), g(x)] = 0$ for all x in R.
By linearization of $[F(x), g(x)] = 0$ gives

$$\begin{aligned}&[F(x), g(y)] + [F(y), g(x)] + [f(x)g(y), g(x)]\\ &+ [f(x)g(y), g(y)] + [g(y)f(x), g(x)] + [g(y)f(x), g(y)]\\ &+ [f(y)g(x), g(x)] + [f(y)g(x), g(y)] + [g(x)f(y), g(x)]\\ &+ [g(x)f(y), g(y)] = 0 \text{ for all } x, y \text{ in } R. \end{aligned} \tag{3.1.1}$$

Replacing x by $-x$ in (3.1.1) and using $F(-x) = F(x)$, we get

$$\begin{aligned}&[F(x), g(y)] - [F(y), g(x)] + [f(x)g(y), g(x)]\\ &- [f(x)g(y), g(y)] + [g(y)f(x), g(x)] - [g(y)f(x), g(y)]\\ &+ [f(y)g(x), g(x)] - [f(y)g(x), g(y)] + [g(x)f(y), g(x)]\\ &- [g(x)f(y), g(y)] = 0 \text{ for all } x, y \text{ in } R. \end{aligned} \tag{3.1.2}$$

Adding (3.1.1) and (3.1.2), we get

$$\begin{aligned}&2([F(x), g(y)] + [f(x)g(y), g(x)] + [g(y)f(x), g(x)]\\ &+ [f(y)g(x), g(x)] + [g(x)f(y), g(x)]) = 0 \text{ for all } x, y \text{ in } R. \end{aligned} \tag{3.1.3}$$

As R is $2-$ torsion free and replacing y by x^2, so last relation becomes

$$\begin{aligned}&[F(x), g(x^2)] + [f(x)g(x^2), g(x)] + [g(x^2)f(x), g(x)]\\ &+ [f(x^2)g(x), g(x)] + [g(x)f(x^2), g(x)] = 0 \text{ for all } x, y \text{ in } R. \end{aligned} \tag{3.1.4}$$

It is easy to verify that $[F(x), g(x^2)] = 0$ for all x in R.
Since $[F(x), g(x)] = 0$ for all $x \in R$, $f(x)g(x^2) = g(x^2)f(x)$ for all $x \in R$.
Then $[f(x^2)g(x), g(x)] + [g(x)f(x^2), g(x)] = [f(x^2), g(x^2)] = 0$ for all $x \in R$.
It follows from (3.1.4) that

$$2[f(x)g(x^2), g(x)] = 0 \text{ for all } x \in R.$$

Since R is $2-$torsion free, $[f(x)g(x^2), g(x)] = 0$ for all $x \in R$.
By using $f(x)g(x^2) = g(x^2)f(x)$ for all $x \in R$, hence we have

$$[f(x), g(x^3)] = 0 \text{ for all } x \in R.$$

This completes the proof. □

Theorem 3.2. *Let R be a $2-$torsion free ring, and let f and g be endomorphisms of R. Suppose that $F : R \to R$ is a mapping $x \longmapsto f(x) \circ g(x)$ for all x in R. If F and f are generalized commuting on R then $[f(x^3), g(x)] = 0$ for all x in R.*

Proof. Using similar techniques as used in the proof of Theorem 3.1.
□

Theorem 3.3. *Let R be a 2−torsion free prime ring, and let f and g be endomorphisms of R and $[f(x), g(x)] \in Z(R) - \{0\}$ for all $x \in R - \{0\}$. Suppose that $F : R \to R$ is a mapping $x \longmapsto f(x) \circ g(x)$ for all $x \in R$. If F and g are generalized skew−centralizing on R then $g(x^2) = 0$ for all $x \in R$.*

Proof. Assume that F and g are generalized skew−centralizing on R.
Then $[F(x)g(x) + g(x)F(x), g(x)] = 0$ for all $x \in R$.
As $F(x) = f(x) \circ g(x)$ for all $x \in R$ and $[f(x), g(x)] \in Z(R) - \{0\}$ for all $x \in R - \{0\}$, so last relation becomes

$$4g(x^2)[f(x), g(x)] = 0 \text{ for all } x \in R.$$

Since R is 2−torsion free, $g(x^2)[f(x), g(x)] = 0$
Since $[f(x), g(x)] \in Z(R) - \{0\}$, $g(x^2)R[f(x), g(x)] = \{0\}$ for all $x \in R$.
Since R is prime and $[f(x), g(x)] \neq 0$ for all $x \in R - \{0\}$,

$$g(x^2) = 0 \text{ for all } x \in R.$$

This completes the proof. □

Theorem 3.4. *Let R be a 2−torsion free prime ring, and let f and g be endomorphisms of R and $[f(x), g(x)] \in Z(R) - \{0\}$ for all $x \in R - \{0\}$. Suppose that $F : R \to R$ is a mapping $x \longmapsto f(x) \circ g(x)$ for all $x \in R$. If F and f are generalized skew−centralizing on R then $f(x^2) = 0$ for all $x \in R$.*

Proof. Using similar techniques as used in the proof of Theorem 3.3.

□

Theorem 3.5. *Let R be a 2−torsion free ring and let f and g be endomorphisms of R. Suppose that $F : R \to R$ is a mapping $x \longmapsto [f(x), g(x)]$ for all x in R. If F and g are generalized skew−commuting on R then $[f(x), g(x^3)] = 0$ for all x in R.*

Proof. Assume that F and g are generalized skew−commuting on R.
Then $F(x) \circ g(x) = 0$ for all $x \in R$.
By linearization of $F(x) \circ g(x) = 0$ gives

$$\begin{aligned} &F(x)g(y) + F(y)g(x) + [f(x), g(y)]g(x) \\ &+ [f(x), g(y)]g(y) + [f(y), g(x)]g(x) + [f(y), g(x)]g(y) \\ &\quad + g(y)F(x) + g(x)F(y) + g(x)[f(x), g(y)] + g(y)[f(x), g(y)] \\ &\qquad + g(x)[f(y), g(x)] + g(y)[f(y), g(x)] = 0 \text{ for all } x, y \text{ in } R. \end{aligned} \tag{3.5.1}$$

Replacing x by $-x$ in (3.5.1) and using $F(-x) = F(x)$, we get

$$\begin{aligned} &F(x)g(y) - F(y)g(x) + [f(x), g(y)]g(x) \\ &- [f(x), g(y)]g(y) + [f(y), g(x)]g(x) - [f(y), g(x)]g(y) \\ &\quad + g(y)F(x) - g(x)F(y) + g(x)[f(x), g(y)] - g(y)[f(x), g(y)] \\ &\qquad + g(x)[f(y), g(x)] - g(y)[f(y), g(x)] = 0 \text{ for all } x, y \text{ in } R. \end{aligned} \tag{3.5.2}$$

for all $x, y \in R$.
Adding (3.5.1) and (3.5.2), and using R is $2-$torsion free, we get

$$F(x)g(y) + [f(x), g(y)]g(x) + [f(y), g(x)]g(x) + g(y)F(x) + g(x)[f(x), g(y)] + g(x)[f(y), g(x)] = 0 \text{ for all } x, y \text{ in } R. \tag{3.5.3}$$

Replacing y by x^2 in (3.5.3), we have

$$F(x)g(x^2) + [f(x), g(x^2)]g(x) + [f(x^2), g(x)]g(x) + g(x^2)F(x) + g(x)[f(x), g(x^2)] + g(x)[f(x^2), g(x)] = 0 \text{ for all } x \text{ in } R. \tag{3.5.4}$$

Since $F(x) \circ g(x) = 0$ for all $x \in R$, $[f(x), g(x^2)] = 0$ for all $x \in R$.
It follows from (3.5.4) that

$$F(x)g(x^2) + g(x^2)F(x) + [f(x^2), g(x)]g(x) + g(x)[f(x^2), g(x)] = 0 \text{ for all } x \text{ in } R. \tag{3.5.5}$$

Since $[f(x), g(x^2)] = 0$ for all $x \in R$,

$$[f(x^2), g(x)]g(x) + g(x)[f(x^2), g(x)] = 0 \text{ for all } x \in R.$$

Hence (3.5.5) reduce to

$$F(x)g(x^2) + g(x^2)F(x) = 0 \text{ for all } x \in R.$$

Then $2[f(x), g(x^3)] = 0$ for all $x \in R$.
Since R is $2-$torsion free, $[f(x), g(x^3)] = 0$ for all $x \in R$.

□

Theorem 3.6. *Let R be a $2-$torsion free ring, and let f and g be endomorphisms of R. Suppose that $F : R \to R$ is a mapping $x \longmapsto [f(x), g(x)]$ for all x in R. If F and f are generalized skew$-$commuting on R then $[f(x^3), g(x)] = 0$ for all x in R.*

Proof. Using similar techniques as used in the proof of Theorem 3.5.

□

Theorem 3.7. *Let R be a $2-$torsion free prime ring and let f and g be endomorphisms of R Suppose that $F : R \to R$ is a mapping $x \longmapsto [f(x), g(x)]$ for all x in R and satisfying $F(x) \circ g(x) \in Z(R) - \{0\}$ for all $x \in R - \{0\}$. Then $[f(x), g(x)] = 0$ for all $x \in R$.*

Proof. By assumption, $F(x) = [f(x), g(x)]$ for all $x \in R$.
Clearly, $F(0) = [f(0), g(0)] = 0$ and $F(-x) = F(x)$ for all $x \in R$.
Since $F(x) \circ g(x) \in Z(R)$ for all $x \in R$, $F(x+y) \circ g(x+y) \in Z(R)$ for all $x, y \in R$.
Hence $[F(x+y) \circ g(x+y), z] = 0$ for all $x, y, z \in R$.
This implies that for all $x, y, z \in R$, we have

$$\begin{aligned}&[F(x)g(y) + g(y)F(x), z] + [F(y)g(x) + g(x)F(y), z]\\ &+ [[f(x), g(y)]g(x), z] + [[f(x), g(y)]g(y), z]\\ &+ [[f(y), g(x)]g(x), z] + [[f(y), g(x)]g(y), z]\\ &+ [g(x)[f(x), g(y)], z] + [g(y)[f(x), g(y)], z]\\ &+ [g(x)[f(y), g(x)], z] + [g(y)[f(y), g(x)], z] = 0.\end{aligned} \tag{3.7.1}$$

Replacing x by $-x$ in (3.7.1), we get

$$\begin{aligned}&[F(x)g(y) + g(y)F(x), z] - [F(y)g(x) + g(x)F(y), z]\\ &+ [[f(x), g(y)]g(x), z] - [[f(x), g(y)]g(y), z]\\ &+ [[f(y), g(x)]g(x), z] - [[f(y), g(x)]g(y), z]\\ &+ [g(x)[f(x), g(y)], z] - [g(y)[f(x), g(y)], z]\\ &+ [g(x)[f(y), g(x)], z] - [g(y)[f(y), g(x)], z] = 0.\end{aligned} \tag{3.7.2}$$

Adding (3.7.1) and (3.7.2) and using R is $2-$torsion free, we get

$$\begin{aligned}&[F(x)g(y) + g(y)F(x), z] + [[f(x), g(y)]g(x), z]\\ &+ [[f(y), g(x)]g(x), z] + [g(x)[f(x), g(y)], z]\\ &+ [g(x)[f(y), g(x)], z] = 0 \ \text{ for all } x, y, z \text{ in } R.\end{aligned} \tag{3.7.3}$$

Replacing y and z by x^2 and $g(x)$, resctively in (3.7.3), we have

$$\begin{aligned}&[F(x)g(x^2) + g(x^2)F(x), g(x)] + [[f(x), g(x^2)]g(x), g(x)]\\ &+ [[f(x^2), g(x)]g(x), g(x)] + [g(x)[f(x), g(x^2)], g(x)]\\ &+ [g(x)[f(x^2), g(x)], g(x)] = 0 \ \text{ for all } x \text{ in } R.\end{aligned} \tag{3.7.4}$$

By assumption $F(x) \circ g(x) \in Z(R)$, we obtain, for all $x \in R$,

$$[f(x), g(x^2)] \in Z(R). \tag{3.7.5}$$

Since $[[f(x), g(x^2)], g(x)] = 0$,

$$F(x)g(x^2) = g(x^2)F(x) = 0. \tag{3.7.6}$$

It is easy to verify that, for all $x \in R$,

$$[[f(x^2), g(x)]g(x), g(x)] + [g(x)[f(x^2), g(x)], g(x)] = 2[f(x), g(x^2)]F(x) = 0. \tag{3.7.7}$$

and

$$[[f(x), g(x^2)]g(x), g(x)] + [g(x)[f(x), g(x^2)], g(x)] = 0. \tag{3.7.8}$$

From (3.7.4), (3.7.7) and (3.7.8), we get

$$2([F(x)g(x^2), g(x)] + [f(x), g(x^2)]F(x)) = 0 \text{ for all } x \text{ in } R. \tag{3.7.9}$$

By using (3.7.6) and R is 2-torsion free, we have
$[f(x), g(x^2)]F(x) = 0$ for all $x \in R$.
Since $[f(x), g(x^2)] \in Z(R) - \{0\}$ for all $x \in R - \{0\}$,

$$[f(x), g(x^2)]RF(x) = \{0\} \text{ for all } x \in R.$$

Since R is prime and $[f(x), g(x^2)] \neq 0$ for all $x \in R - \{0\}$,

$$F(x) = 0 \text{ for all } x \in R.$$

This completes the proof. □

Theorem 3.8. *Let R be a $2-$torsion free prime ring and let f and g be endomorphisms of R. Suppose that $F : R \to R$ is a mapping $x \longmapsto [f(x), g(x)]$ for all x in R and satisfying $F(x) \circ f(x) \in Z(R) - 0$ for all $x \in R - \{0.\}$ Then $[f(x), g(x)] = 0$ for all x in R.*

Proof. Using similar techniques as used in the proof of Theorem 3.7.

□

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References

[1] K. I. Beidar, W. S. Martindale, A. V. Mikhalev, *Rings with generalized identities*, Marcel Dekker, Inc., New York - Basel - Hong Kong, (1996), 51–95.

[2] H. E. Bell, Q. Deng, *On derivations and commutativity in semiprime rings*, Comm. Algebra, **23,** (1995), 3705–3713.

[3] H. E. Bell, W. S. Martindale, *Centralizing mappings of semiprime rings*, Canad. Math. Bull., **30,** (1987), 92–101.

[4] M. Bresar, *Centralizing mappings and derivations in prime rings*, J. Algebra, **156,** (1993), 385–394.

[5] M. Bresar, *On skew commuting mappings of rings* , Bull. Austral. Math. Soc., **47,** (1993), 291–296.

[6] M. Bresar, *Functional identities: a survey*, Contemp. Math., **259,** (2000), 93–109.

[7] V. De Filippis, *Lie ideals and annihilator conditions on power values of commutators with derivation*, Indian J. Pure Appl. Math., **32,** (2001), 649-656.

[8] B. Dhara, *Power values of derivations with annihilator conditions on Lie ideals in prime rings,* Comm. Algebra, **37,** (2009), 2159–2167.

[9] B. Dhara, S. Kar, S. Mondal, *Derivations with annihilator conditions in prime rings,* Bull. Korean Math. Soc., **50,** (2013), 1651–1657.

[10] N. Divinsky, *On commuting automorphisms of rings,* Trans. Roy. Soc. Canada. Sect. III. (3), **49,** (1955), 19–22.

[11] C. Lanski, *Left ideals and derivations in semiprime ring,* J. Algebra, **277,** (2004), 658–667.

[12] J. Luh, *A note on commuting automorphisms of rings,* Amer. Math. Monthly, **77,** (1970), 61–62.

[13] J. H. Mayne, *Centralizing automorphisms of prime ring,* Canad. Math. Bull., **19,** (1976), 113–115.

[14] K. H. Park, Y. S. Jung, *Skew-commuting and commuting mappings in rings,* Aequationes Math., **64,** (2002), 136–144 ; Erratum **70,** (2005), 199–200.

[15] E. C. Posner, *Derivations in prime rings,* Proc. Amer. Math. Soc., **8,** (1957), 1093–1100.