International Journal of Mathematics and
Computer Science, **15**(2020), no. 2, 671–682

Congruences and homomorphisms on n-ary semigroups

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(Received December 26, 2019, Accepted February 8, 2020)

Abstract

In this paper, we introduce the concept of congruences on an n-ary semigroups and investigate their related properties. Moreover, we introduce quotient n-ary semigroups via congruence relations. Furthermore, we establish fundamental theorem of homomorphism for n-ary semigroup and related properties with respect to congruence relations.

1 Introduction

An n-ary system, a generalization of algebraic structures which has many applications in different branches of mathematics, was initiated by Kasner [4] in 1904. For example, in the theory of automata (see [5]) n-ary semigroup and n-ary groups are used, some other n-ary systems are applied in the theory of quantum groups (see [3]) and combinatorics (see [18], [19]). Different applications of ternary structures in physics are described in Kerner [7]. In physics, such structures as n-ary Filippov algebras (see [1]) and n-Lie algebras (see [6]) are also used. Some n-ary structures induced by hypercubes have application in error-correcting and error-detecting coding theory, cryptology, as well as in the theory of (t, m, s)-nets (see [2]).

Key words and phrases: n-ary semigroups, n-ary operation, congruence, homomorphism, isomorphism.
AMS (MOS) Subject Classifications: 08A02, 17A42, 08A30, 20N15, 16Y99.
Corresponding author email: fsciutl@ku.ac.th
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

In 1928, W. Dörnte introduced the notion of an n-ary group, which is a generalization of the notion of a group to a nonempty set G with an n-ary operation instead of a binary operation. By an n-ary operation is meant any mapping $f : G^n \to G$ from the n-th Cartesian power of G to G. The important study of n-ary groups and n-ary semigroups was done by Dudek (see [10]-[17]).

The notion of congruence was introduced by Karl Fredrich Gauss in the beginning of the nineteenth century. It is well known that congruences always play an important role in the study of algebraic structures. In particular, congruences are a special type of equivalence relations that play a vital role in the study of quotient structures of different algebraic structures. In 1997, V. N. Dixit and S. Dewan [9] represented the concept of congruences on a ternary semigroups and they studied some interesting properties of them. In 2007, S. Kar and B. K. Maity [8] introduced some concepts such as cancellation congruences, group congruences and Rees congruences and investigated these congruences in ternary semigroups.

In this paper, we introduce the concept of congruences on an n-ary semigroups and investigate their related properties. Moreover, we establish the quotient n-ary semigroups via congruences. Furthermore, some homomorphisms and related properties with respect to congruence relations are proposed.

2 Preliminaries

Throughout this paper, n is a positive integer greater than one. In this section, we start with some elementary notions that will be used in the sequel.

Definition 2.1. A nonempty set G is called an *n-ary semigroup* if there exists an n-ary operation $G^n \to G$, written as $(x_1, x_2, \ldots, x_n) \mapsto x_1x_2\cdots x_n$, satisfying the following condition:

$$x_1x_2\cdots x_{i-1}\left(\prod_{k=i}^{n+i-1} x_k\right) x_{n+i}x_{n+i+1}\cdots x_{2n-1} = x_1x_2\cdots x_{j-1}\left(\prod_{k=j}^{n+j-1} x_k\right) x_{n+j}x_{n+j+1}\cdots x_{2n-1},$$

for all $x_1, x_2, \ldots, x_{2n-1} \in G$ and $1 \le i < j \le n$.

Remark 2.2 Let G be an ordinary semigroup under the binary operation

$$(x_1, x_2) \mapsto x_1 * x_2.$$

Then G, with the n-ary operation

$$(x_1, x_2, \ldots, x_n) \mapsto x_1 * x_2 * \cdots * ((x_i * x_{i+1}) * x_{i+2}) * \cdots * x_n,$$

is an n-ary semigroup.

Example 2.3

(1) Let $G = \{-i, 0, i\}$ be a subset of complex numbers. Then G is a 3-ary semigroup under the usual multiplication. However, G is not a 4-ary semigroup under the usual multiplication, because $(i, -i, i, i) = -1 \notin G$.

(2) Let $G = \{2, 4, 6, \ldots\}$ be the set of all positive even numbers. Then G is an n-ary semigroup under the usual multiplication.

Definition 2.4 A nonempty set S of an n-ary semigroup G is called an *n-ary subsemigroup* if $\prod_{i=1}^{n} x_i \in S$ for all $x_1, x_2, \ldots, x_n \in S$.

Example 2.5 The set of all integers, $\mathbb{Z}$, forms an n-ary semigroup under the operation $x_1 x_2 \ldots x_n = \min\{x_1, x_2, \ldots, x_n\}$ for all $x_1, x_2, \ldots, x_n \in \mathbb{Z}$. As a result, the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of all natural numbers gives an n-ary subsemigroup.

Theorem 2.6 *The nonempty intersection of any two n-ary subsemigroups of an n-ary semigroup G is an n-ary subsemigroup of G.*

Proof. The proof is straightforward. □

The following corollary follows by induction:

Corollary 2.7 *The nonempty intersection of any family of n-ary subsemigroups of an n-ary semigroup G is an n-ary subsemigroup.*

Definition 2.8 Let G_1 and G_2 be two n-ary semigroups. The mapping $f : G_1 \to G_2$ is called (*an n-ary semigroup*) *homomorphism* from G_1 into G_2 if

$$f(a_1 a_2 \ldots a_n) = f(a_1) f(a_2) \ldots f(a_n),$$

for all $a_1, a_2, \dots, a_n \in G_1$.

A homomorphism $f : G_1 \to G_2$ is called *an isomorphism* if it is both one-one and onto and in this case we say that the n-ary semigroups G_1 and G_2 are isomorphic and write $G_1 \cong G_2$.

Theorem 2.9 *Let G_1, G_2 and G_3 be any n-ary semigroups and the map $g_1 : G_1 \to G_2$, $g_2 : G_2 \to G_3$ be homomorphisms. Then $g_2 \circ g_1$ is a homomorphism from G_1 into G_3.*

Proof. The proof is straightforward. □

3 Congruence relations and homomorphism

In this section, we introduce the concept of congruence relations and establish quotient n-ary semigroups via congruence relations. Moreover, some homomorphisms and related properties with respect to congruence relations are provided.

Definition 3.1 An equivalence relation ρ on an n-ary semigroup G is said to be

1. *a left congruence relation* if $(a, b) \in \rho$ implies

$$\left(\left(\prod_{i=1}^{n-1} t_i\right) a, \left(\prod_{i=1}^{n-1} t_i\right) b\right) \in \rho,$$

for all $t_1, \dots, t_{n-1}, a, b \in G$.

2. *an intra congruence relation* if $(a, b) \in \rho$ implies

$$(t_1 a t_2 \dots t_{n-1}, t_1 b t_2 \dots t_{n-1}) \in \rho,$$

$$(t_1 t_2 a t_3 \dots t_{n-1}, t_1 t_2 b t_3 \dots t_{n-1}) \in \rho,$$

$$\vdots$$

$$(t_1 t_2 \dots t_{n-2} a t_{n-1}, t_1 t_2 \dots t_{n-2} b t_{n-1}) \in \rho,$$

for all $t_1, \dots, t_{n-1}, a, b \in G$.

3. *a right congruence relation* if $(a, b) \in \rho$ implies

$$\left(a\left(\prod_{i=1}^{n-1} t_i\right), b\left(\prod_{i=1}^{n-1} t_i\right)\right) \in \rho,$$

for all $t_1, \ldots, t_{n-1}, a, b \in G$.

4. *a congruence relation* if $(a_1, b_1), \ldots, (a_n, b_n) \in \rho$ implies

$$(a_1 \cdots a_n, b_1 \cdots b_n) \in \rho,$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in G$.

Proposition 3.2 *An equivalence relation ρ on an n-ary semigroup G is a congruence if and only if it is a left, an intra and a right congruence relations on G.*

Proof. Assume that ρ is a congruence relation on G. Let $a, b \in G$ be such that $(a, b) \in \rho$ and let $t_1, \ldots, t_{n-1} \in G$. Since ρ is an equivalence relation on G, we have $(t_1, t_1), \ldots, (t_{n-1}, t_{n-1}) \in \rho$. Then $(t_1 \ldots t_{n-1}a, t_1 \ldots t_{n-1}b) \in \rho$. So ρ is a left congruence relation on G. Similarly, ρ is an intra and a right congruence relation on G.

Conversely, assume that ρ is a left, an intra and a right congruence relation on G. Let $a_1, \ldots, a_n, b_1, b_2, \ldots, b_n \in G$ be such that $(a_1, b_1), \ldots, (a_n, b_n) \in \rho$. Since ρ is a right congruence relation, we have $(a_1 a_2 \cdots a_n, b_1 a_2 \cdots a_n) \in \rho$. Since ρ is an intra congruence relation, we have

$$(b_1 a_2 a_3 \cdots a_n, b_1 b_2 a_3 \cdots a_n) \in \rho,$$

$$(b_1 b_2 a_3 a_4 \cdots a_n, b_1 b_2 b_3 a_4 \cdots a_n) \in \rho,$$

$$(b_1 b_2 b_3 a_4 a_5 \cdots a_n, b_1 b_2 b_3 b_4 a_5 \cdots a_n) \in \rho,$$

$$\vdots$$

$$(b_1 \cdots b_{n-2} a_{n-1} a_n, b_1 \cdots b_{n-2} b_{n-1} a_n) \in \rho.$$

Since ρ is a left congruence relation, we have $(b_1 \cdots b_{n-1} a_n, b_1 \cdots b_{n-1} b_n) \in \rho$. By transitive relation, $(a_1 \cdots a_n, b_1 \cdots b_n) \in \rho$. Hence ρ is a congruence relation on G. □

Proposition 3.3 *Let ρ be a congruence relation on an n-ary semigroup G. Then $\rho \circ \rho$ is a congruence relation on G.*

Proof. Clearly, $\rho \circ \rho$ is a congruence relation on G.
Let $a_1, a_2, \ldots, a_n,\ b_1, b_2, \ldots, b_n \in G$ be such that $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n) \in \rho \circ \rho$. Then there are $c_1, c_2, \ldots, c_n \in G$ such that

$$(a_1, c_1) \in \rho \quad \text{and} \quad (c_1, b_1) \in \rho,$$

$$(a_2, c_2) \in \rho \quad \text{and} \quad (c_2, b_2) \in \rho,$$

$$\vdots$$

$$(a_n, c_n) \in \rho \quad \text{and} \quad (c_n, b_n) \in \rho.$$

Since ρ is a congruence relation, we have $(a_1 a_2 \cdots a_n,\ c_1 c_2 \cdots c_n) \in \rho$ and $(c_1 c_2 \cdots c_n,\ b_1 b_2 \cdots b_n) \in \rho$. Thus $(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n) \in \rho \circ \rho$. Hence $\rho \circ \rho$ is a congruence relation on G. □

Next, we will consider the relation between congruence relations and homomorphisms on n-ary semigroups.

Theorem 3.4 *Let G_1 and G_2 be two n-ary semigroups and $f : G_1 \to G_2$ be a homomorphism. Let ρ be a congruence relation on G_1. Then*

$$f(\rho) = \{(f(x), f(y)) \in G_2 \times G_2 \mid (x, y) \in \rho\}$$

is a congruence relation on G_2.

Proof. Clearly, $f(\rho)$ is an equivalence relation.
Let $x_1, x_2, \ldots, x_n,\ y_1, y_2, \ldots, y_n \in G_1$ be such that

$$(f(x_1), f(y_1)), (f(x_2), f(y_2)), \ldots, (f(x_n), f(y_n)) \in f(\rho).$$

Then $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in \rho$. Since ρ is a congruence relation, we have $(x_1 x_2 \cdots x_n,\ y_1 y_2 \cdots y_n) \in \rho$. So $(f(x_1 x_2 \cdots x_n),\ f(y_1 y_2 \cdots y_n)) \in f(\rho)$. Since f is a homomorphism, $(f(x_1) f(x_2) \cdots f(x_n), f(y_1) f(y_2) \cdots f(y_n)) \in f(\rho)$. Thus $f(\rho)$ is a congruence relation on G_2. □

Definition 3.5 Let ρ be a congruence relation on an n-ary semigroup G.
The equivalence class of $a \in G$ is defined to be the set

$$[a]_\rho = \{b \in G | (a,b) \in \rho\}.$$

The quotient set ${}^{G}/_{\rho}$ is the set of all equivalence class of G with respect to ρ; That is, ${}^{G}/_{\rho} = \{[a]_\rho \,|\, a \in G\}$.

Note that, for any $a, b \in G$, we have $(a,b) \in \rho$ if and only if $[a]_\rho = [b]_\rho$.

Theorem 3.6 *Let G be an n-ary semigroup and let ρ be a congruence relation on G. Define multiplicative operation on ${}^{G}/_{\rho}$ by $[a_1]_\rho[a_2]_\rho \cdots [a_n]_\rho = [a_1a_2\ldots a_n]_\rho$ for all $[a_1]_\rho, [a_2]_\rho, \ldots, [a_n]_\rho \in {}^{G}/_{\rho}.$ Then ${}^{G}/_{\rho}$ is an n-ary semigroup. We call it the quotient n-ary semigroup.*

Proof. We first show that the multiplication operation is well-defined.
Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in G$ be such that $[a_1]_\rho = [b_1]_\rho, \ldots, [a_n]_\rho = [b_n]_\rho$. Then $(a_1, b_1), \ldots, (a_n, b_n) \in \rho$. Since ρ is a congruence relation, we have $(a_1 \cdots a_n,\ b_1 \cdots b_n) \in \rho$. Thus $[a_1a_2\ldots a_n]_\rho = [b_1b_2\ldots b_n]_\rho$. Therefore, the multiplicative operation is well-defined.

Next we will show that ${}^{G}/_{\rho}$ is an n-ary semigroup.
Let $a_1, \ldots, a_n,\ a_{n+1}, \ldots, a_{2n-1} \in G$. Then

$$\begin{aligned}
\Big([a_1]_\rho[a_2]_\rho \cdots [a_n]_\rho\Big)[a_{n+1}]_\rho \ldots [a_{2n-1}]_\rho &= [a_1 \ldots a_n]_\rho[a_{n+1}]_\rho \ldots [a_{2n-1}]_\rho \\
&= [(a_1 \ldots a_n)a_{n+1} \ldots a_{2n-1}]_\rho \\
&= [a_1(a_2 \ldots a_{n+1})a_{n+2} \ldots a_{2n-1}]_\rho \\
&= [a_1]_\rho[a_2 \ldots a_{n+1}]_\rho[a_{n+2}]_\rho \ldots [a_{2n-1}]_\rho \\
&= [a_1]_\rho \left(\prod_{i=2}^{n+1} [a_i]_\rho\right) \prod_{j=n+2}^{2n-1} [a_j]_\rho.
\end{aligned}$$

Similarly, $[a_1]_\rho \left(\prod_{i=2}^{n+1} [a_i]_\rho\right) \prod_{j=n+2}^{2n-1} [a_j]_\rho = \cdots = \prod_{i=1}^{n-1} [a_i]_\rho \left(\prod_{j=n}^{2n-1} [a_j]_\rho\right)$.

Thus ${}^{G}/_{\rho}$ is an n-ary semigroup. □

Theorem 3.7 *Let G be an n-ary semigroup and ρ be a congruence relation on G. Then the mapping $\pi : G \to {}^{G}/_{\rho}$ given by $\pi(a) = [a]_\rho$ is a surjective homomorphism, called the natural homomorphism from G onto ${}^{G}/_{\rho}$.*

Proof. For all $a_1, \dots, a_n \in G$, we have

$$\pi(a_1 a_2 \dots a_n) = [a_1 \dots a_n]_\rho = [a_1]_\rho [a_2]_\rho \dots [a_n]_\rho = \pi(a_1)\,\pi(a_2)\dots\pi(a_n).$$

Thus π is a homomorphism. Next, let $[a]_\rho \in {}^{G}\!/_{\rho}$, then $\pi(a) = [a]_\rho$, which shows that π is surjective. □

Definition 3.8 Let G_1 and G_2 be two n-ary semigroups and $f : G_1 \to G_2$ be a mapping. Define a relation $\ker f$ on G_1 by the rule

$$(a, b) \in \ker f \quad \text{if and only if} \quad f(a) = f(b).$$

Theorem 3.9 *Let G_1 and G_2 be two n-ary semigroups and the map $f : G_1 \to G_2$ be a homomorphism. Then $\ker f$ is a congruence relation on G_1.*

Proof. First, we have to show $\ker f$ is an equivalence relation on G_1. Let $a, b, c \in G_1$. Then $(a, a) \in \ker f$. Hence $\ker f$ is reflexive. If $(a, b) \in \ker f$, then $f(a) = f(b)$. So $f(b) = f(a)$, that is $(b, a) \in \ker f$. Thus $\ker f$ is symmetric. Suppose that $(a, b) \in \ker f$ and $(b, c) \in \ker f$. Then $f(a) = f(b)$ and $f(b) = f(c)$. So $f(a) = f(c)$, that is $(a, c) \in \ker f$. Thus $\ker f$ is transitive. Therefore $\ker f$ is an equivalence relation.

Next, we will show that $\ker f$ is a congruence. Let

$$(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in \ker f.$$

Then $f(a_1) = f(b_1)$, $f(a_2) = f(b_2)$, $\dots$, $f(a_n) = f(b_n)$. So

$$f(a_1 a_2 \dots a_n) = f(a_1) f(a_2) \dots f(a_n) = f(b_1) f(b_2) \dots f(b_n) = f(b_1 b_2 \dots b_n).$$

Thus $(a_1 a_2 \dots a_n, b_1 b_2 \dots b_n) \in \ker f$. Hence $\ker f$ is a congruence on G_1. □

Theorem 3.10 (The Fundamental Theorem of Homomorphism for n-ary semigroups) *Let G_1 and G_2 be two n-ary semigroups and the map $f : G_1 \to G_2$ be a homomorphism. Then $f(G_1)$ is an n-ary subsemigroup of G_2 and ${}^{G_1}\!/_{\ker f} \cong f(G_1)$.*

Proof. Let $x_1, x_2, \ldots, x_n \in f(G_1)$. Then there exist $a_1, a_2, \ldots, a_n \in G_1$ such that $x_1 = f(a_1), x_2 = f(a_2), \ldots, x_n = f(a_n)$. So

$$x_1, x_2, \ldots, x_n = f(a_1)f(a_2)\ldots f(a_n) = f(a_1a_2\ldots a_n) \in f(G_1).$$

Thus $f(G_1)$ is an n-ary subsemigroup of G_2.

Define $\alpha : {G_1}/_{\ker f} \to f(G_1)$ by $\alpha([a]_{\ker f}) = f(a)$, for all $a \in G_1$. Let $a, b \in G_1$. $[a]_{\ker f} = [b]_{\ker f} \Leftrightarrow (a, b) \in \ker f \Leftrightarrow f(a) = f(b) \Leftrightarrow \alpha([a]_{\ker f}) = \alpha([b]_{\ker f}))$. Hence α is well-defined and one-to one. For any $y \in f(G_1)$. Then $y = f(x)$ for some $x \in G_1$. We have $y = f(x) = \alpha([x]_{\ker f})$, and so α is onto.

Finally, let $[a_1]_{\ker f}, \ldots, [a_n]_{\ker f} \in {G}/_{\ker f}$. Then $\alpha([a_1]_{\ker f} \cdots [a_n]_{\ker f})$ $= \alpha([a_1 \cdots a_n]_{\ker f}) = f(a_1 \cdots a_n) = f(a_1) \cdots f(a_n) = \alpha([a_1]_{\ker f}) \cdots \alpha([a_n]_{\ker f}))$. Therefore, α is an isomorphism and ${G_1}/_{\ker f} \cong f(G_1)$. □

Theorem 3.11 *Let G be an n-ary semigroup and $f : G \to G$ be a homomorphism. If ρ is a congruence relation on G such that $\rho \subseteq \ker f$, then there is a surjective homomorphism $g : {G}/_{\rho} \to {G}/_{\ker f}$ such that $g \circ \pi_1 = \pi_2$ where $\pi_1 : G \to {G}/_{\rho}$ and $\pi_2 : G \to {G}/_{\ker f}$ are natural homomorphisms.*

Proof. Define $g : {G}/_{\rho} \to {G}/_{\ker f}$ by $g([a]_\rho) = [a]_{\ker f}$ for all $a \in G$. Let $a, b \in G$. Suppose that $[a]_\rho = [b]_\rho$. Then $(a, b) \in \rho$. Since $\rho \in \ker f$, we have $(a, b) \in \ker f$. Thus $[a]_{\ker f} = [b]_{\ker f}$, that is $g([a]_\rho) = g([b]_\rho)$. Therefore, g is well-defined. For any $y \in {G}/_{\ker f}$, $y = [a]_{\ker f}$ for some $a \in G$. We have $y = [a]_{\ker f} = g([a]_\rho)$, and so g is onto. Next, let $[a_1]_\rho, \ldots, [a_n]_\rho \in {G}/_{\rho}$. Then

$$g([a_1]_\rho \cdots [a_n]_\rho) = g([a_1 \cdots a_n]_\rho) = [a_1 \cdots a_n]_{\ker f} =$$

$$[a_1]_{\ker f} \cdots [a_n]_{\ker f} = g([a_1]_\rho) \cdots g([a_n]_\rho).$$

Thus g is a surjective homomorphism. Finally, let $a \in G$. Then $g \circ \pi_1(a) = g(\pi_1(a)) = g([a]_\rho) = [a]_{\ker f} = \pi_2(a)$. Consequently, $g \circ \pi_1 = \pi_2$. □

Theorem 3.12 *Let G be an n-ary semigroup and let ρ and σ be congruence relations on G such that $\rho \subseteq \sigma$. Then there exists a surjective homomorphism $g : {G}/_{\rho} \to {G}/_{\sigma}$ such that $g \circ \pi_1 = \pi_2$, where $\pi_1 : G \to {G}/_{\rho}$ and $\pi_2 : G \to {G}/_{\sigma}$ are natural homomorphisms.*

Proof. The proof is similar to that of Theorem 3.11. □

As an immediate consequence of Theorems 3.12 and 2.9, we have the following:

Corollary 3.13 *Let G be an n-ary semigroup and $\sigma_1, \sigma_2, \ldots, \sigma_n (n \geqslant 2)$ be congruence relations on G such that $\sigma_1 \subseteq \sigma_2 \subseteq \ldots \subseteq \sigma_n$. Then there exists a surjective homomorphism $g : {}^{G}/_{\sigma_1} \to {}^{G}/_{\sigma_n}$.*

Theorem 3.14 *Let ρ and σ be congruence relations on an n-ary semigroup G such that $\rho \subseteq \sigma$. Then ${}^{\sigma}/_{\rho} = \left\{ ([a]_\rho, [b]_\rho) \in {}^{G}/_{\rho} \times {}^{G}/_{\rho} \mid (a, b) \in \sigma \right\}$ is a congruence relation on ${}^{G}/_{\rho}$ and $\left({}^{G}/_{\rho}\right) / \left({}^{\sigma}/_{\rho}\right) \cong {}^{G}/_{\sigma}$.*

Proof. First, we have to show ${}^{\sigma}/_{\rho}$ is an equivalence relation on ${}^{G}/_{\rho}$. Let $[a]_\rho, [b]_\rho, [c]_\rho \in {}^{G}/_{\rho}$. Clearly, $([a]_\rho, [a]_\rho) \in {}^{\sigma}/_{\rho}$. So ${}^{\sigma}/_{\rho}$ is reflexive. If $([a]_\rho, [b]_\rho) \in {}^{\sigma}/_{\rho}$, then $(a, b) \in \sigma$. Since σ is symmetric, we have $(b, a) \in \sigma$. Thus $([b]_\rho, [a]_\rho) \in {}^{\sigma}/_{\rho}$, and so ${}^{\sigma}/_{\rho}$ is symmetric. Suppose that $([a]_\rho, [b]_\rho), ([b]_\rho, [c]_\rho) \in {}^{\sigma}/_{\rho}$. Then $(a, b) \in \sigma$ and $(b, c) \in \sigma$. Since σ is transitive, we have $(a, c) \in \sigma$. Thus $([a]_\rho, [c]_\rho) \in {}^{\sigma}/_{\rho}$, and so ${}^{\sigma}/_{\rho}$ is transitive. Therefore ${}^{\sigma}/_{\rho}$ is an equivalence relation.

Next, we will show that ${}^{\sigma}/_{\rho}$ is a congruence relation on ${}^{G}/_{\rho}$. Let $([a_1]_\rho, [b_1]_\rho)$, $([a_2]_\rho, [b_2]_\rho), \ldots, ([a_n]_\rho, [b_n]_\rho) \in {}^{\sigma}/_{\rho}$. Then $(a_1, b_1), (a_2, b_2), \ldots (a_n, b_n) \in \sigma$. Since σ is a congruence relation on G, $(a_1 a_2 \ldots a_n,\ b_1 b_2 \ldots b_n) \in \sigma$. This implies that $([a_1 a_2 \ldots a_n]_\rho, [b_1 b_2 \ldots b_n]_\rho) \in {}^{\sigma}/_{\rho}$, and hence ${}^{\sigma}/_{\rho}$ is a congruence relation.

Finally, let $g : {}^{G}/_{\rho} \to {}^{G}/_{\sigma}$ be a mapping defined by $g([a]_\rho) = \pi_\sigma(a)$, for all $a \in G$, where $\pi_\sigma : G \to {}^{G}/_{\sigma}$ is a natural homomorphism. Let $([a]_\rho, [b]_\rho) \in {}^{\sigma}/_{\rho}$. Then

$(a, b) \in \sigma \Leftrightarrow [a]_\rho = [b]_\rho \Leftrightarrow \pi_\sigma(a) = \pi_\sigma(b) \Leftrightarrow g([a]_\rho) = g([b]_\rho) \Leftrightarrow ([a]_\rho, [b]_\rho) \in \ker g$. Thus $\ker g = {}^{\sigma}/_{\rho}$. By Theorem 3.10, we have $\left({}^{G}/_{\rho}\right) /_{\ker g} \cong g\left({}^{G}/_{\rho}\right) = {}^{G}/_{\sigma}$. Since $\ker g = {}^{\sigma}/_{\rho}$, $\left({}^{G}/_{\rho}\right) / \left({}^{\sigma}/_{\rho}\right) \cong {}^{G}/_{\sigma}$. □

Corollary 3.15 *Let G be an n-ary semigroup and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be congruence relations on G such that $\sigma_1 \subseteq \sigma_2 \subseteq \ldots \subseteq \sigma_n$. Then for each*

$i = 1, \ldots, n-1$,
$\sigma_{i+1}/_{\sigma_i} = \left\{([a]_{\sigma_i}, [b]_{\sigma_i}) \in G/_{\sigma_i} \times G/_{\sigma_i} \mid (a,b) \in \sigma_{i+1}\right\}$ *is a congruence relation on $G/_{\sigma_i}$ and*

$$\left(G/_{\sigma_i}\right)\Big/\left(\sigma_{i+1}/_{\sigma_i}\right) \cong G/_{\sigma_{i+1}}.$$

Moreover, for each $i = 1, \ldots, n-2$, the mapping

$$\varphi_i : \left(G/_{\sigma_i}\right)\Big/\left(\sigma_{i+1}/_{\sigma_i}\right) \to \left(G/_{\sigma_{i+1}}\right)\Big/\left(\sigma_{i+2}/_{\sigma_{i+1}}\right).$$

is a surjective homomorphism.

Proof. This follows from Theorems 3.14 and 3.7. □

4 Acknowledgement

This research is supported by the Faculty of Science, Kasetsart University, Bangkok, Thailand (International SciKU Branding, ISB).

References

[1] A. P. Pojidaev, Enveloping algebras of Fillipov algebras, Comm. Algebra, **31,** (2003), 883–900.

[2] C. F. Laywine, G. L. Mullen, Discrete Mathematics Using Latin Squares, Wiley, New York, 1998.

[3] D. Nikshych, L. Vainerman, Finite Quantum Groupoids and Their Applications, New Directions in Hopf Algebras, Math. Sci. Res. Inst. Publ., **43,** (2002), 211–216.

[4] E. Kasner, An extension of the group concept, Bull. Amer. Math. Soc. **10,** (1904), 290-291.

[5] J. W. Grzymala-Busse, Automorphisms of polyadic automata, J. Assoc. Comput. Mach., **16,** (1969), 208–219.

[6] L. Vainerman, R. Kerner, On special classes of n-algebras, J. Math. Phys., **37,** (1996),2553–2565.

[7] R. Kerner, Ternary Algebraic Structures and Their Applications in Physics. Univ. P. and M. Curie, Paris, 2000.

[8] S. Kar and K. Maity . Congruences on ternary semigroups, J. Chungcheong Math. Soc., **20,** (2007), 191–201.

[9] V. N. Dixit, S. Dewan, Congruence and Green's equivalence relation on ternary semigroup, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **46,** (1997), 103–117.

[10] W. A. Dudek, Remarks on n-groups, Demonstratio Math., **13,** (1980), 165–181.

[11] W. A. Dudek, Autodistributive n-groups, Comment. Math. Ann. Soc. Math. Polon. Prace Mat., **23,** (1983), 1–11.

[12] W. A. Dudek, On (i, j)-associative n-groupoids with the non-empty center, Ricer. Mat., (Napoli), **35,** (1986), 105–111.

[13] W. A. Dudek . Idempotents in n-ary semigroups, Southeast Asian Bull. Math., **25,** (2001), 97–104.

[14] W. A. Dudek, Remarks to Glazek's results on n-ary groups, Disc. Math. Gen. Alg. Appl., **27,** (2007), 199–233.

[15] W. A. Dudek, I. Grozdinska, On ideals in regular n-semigroups, Mat. Bilten Skopje, **4,** (1980), 25–44.

[16] W. A. Dudek, J. Michalski, On retracts of polyadic groups, Demonstratio Math., **15,** (1982), 783–805.

[17] W. Dörnte, Unterschungen über einen verallgemeinerten Gruppenbegriff, Math. Z., **29,** (1929), 1–19.

[18] Z. Stojakovic, W. A. Dudek, Single identities for varieties equivalent to quadruple systems, Discrete Math., **183,** (1998), 277-284.

[19] Z. Stojakovic, W. A. Dudek , Conjugate invariant quasigroups, Quasigroups and Related Systems, **13,** (2005), 157–174.