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Pseudo Symmetric Ideals and Pseudo Symmetric Near-rings

T. Manikantan, S. Ramkumar

Post Graduate and Research Department of Mathematics
Thiruvalluvar Government Arts College (Affiliated to Periyar University)
Rasipuram - 637 401, Namakkal District
Tamilnadu, India

email: manikantan@tgac.ac.in, ramkumars_slm@rediffmail.com

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Abstract

In this paper, we first introduce the notions of pseudo symmetric subset and pseudo symmetric ideal of a near-ring and define pseudo symmetric near-ring. We prove that every completely prime (resp. completely semiprime) ideal of a near-ring is a pseudo symmetric ideal. We also show that a near-ring satisfying the strong insertion factor property (resp. the weak commutative near-ring) is a pseudo symmetric near-ring. Moreover, we investigate the relations among the radicals of type 1 prime, pseudo symmetric, completely prime, completely semiprime, primary and semiprimary ideals of a near-ring.

1 Introduction

In 1964, Van der walt [17] introduced the notion of prime ideal of a near-ring. Groenewald [9] defined the concepts of completely prime ideal, completely semi prime ideal and completely prime radical of a near-ring. In [10], he studied the notion of completely prime radical of a near-ring. Ramakotaiah and Koteswara Rao [14] introduced different types of prime ideals and the radicals associated with these prime ideals of near-rings. Birkenmeier et

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al. [6] studied the interconnections between prime ideals and type 1 prime ideals of near-rings. Satyanarayana Bhavanari [16] discussed the notions of prime ideal, completely prime ideal, semi-prime ideal and completely semi-prime ideal of a near-ring. Argac and Groenewald [5] studied the concepts of weakly and strongly regular near-rings, and proved some basic properties of left weakly regular near-rings. Abbass and Obaid [1] provided some properties of primary and semiprimary ideals of near-rings. In 1980, Anjaneyulu [3, 4] initiated the notions of pseudo symmetric ideal in a semigroup and pseudo symmetric semigroup. Nochefranca and Shum [12] studied some properties of pseudo symmetric ideals of semigroups and pseudo symmetric semigroups, and they characterized the radicals of pseudo symmetric ideals of semigroups. Motivated by the above theories, we first introduce the notions of pseudo symmetric subset and pseudo symmetric ideal of a near-ring and define pseudo symmetric near-ring. We prove that every completely prime (resp. completely semiprime) ideal of a near-ring is a pseudo symmetric ideal. We also show that a near-ring satisfying the strong insertion factor property (resp. the weak commutative near-ring) is a pseudo symmetric near-ring. Moreover, we investigate the relations among the radicals of type 1 prime, pseudo symmetric, completely prime, completely semiprime, primary and semiprimary ideals of a near-ring.

2 Preliminaries

In this section, we state some known and useful definitions and results which will be used in the next section.

Definition 2.1. *A near-ring [13] is a non-empty set $\mathcal{N}$ with two binary operations " $+$ " and " $\cdot$ " satisfying the following axioms: (i) $(\mathcal{N}, +)$ is a group, (ii) $(\mathcal{N}, \cdot)$ is a semigroup and (iii) $(p+r)\cdot i = p \cdot i + r \cdot i \;\; \forall\, p, r, i \in \mathcal{N}$. More precisely, it is a right near-ring because it satisfies the right distributive law. A near-ring $\mathcal{N}$ is called a zero symmetric near-ring (briefly, ZSN) if $k0 = 0 \;\forall\; k \in \mathcal{N}$.*

In what follows, $\mathcal{N}$ denotes the "right near-ring" unless otherwise specified.

Definition 2.2. *[15] Let $\mathfrak{T}$ and $\mathcal{K}$ be any two subsets of $\mathcal{N}$. Then the set $\{n \in \mathcal{N} | n\mathcal{K} \subseteq \mathfrak{T}\}$ is denoted by $(\mathfrak{T} : \mathcal{K})$ and we denote $(\mathfrak{T} : \{q\})$ by $(\mathfrak{T} : q)$.*

Definition 2.3. *[13] A subgroup $\mathfrak{T}$ of $\mathcal{N}$ is called a subnear-ring (briefly, SN) of $\mathcal{N}$ if $\mathfrak{T}\mathfrak{T} \subseteq \mathfrak{T}$.*

Definition 2.4. *[13] A subset $\mathfrak{T}$ of $\mathcal{N}$ is called an ideal of $\mathcal{N}$, written as $\mathfrak{T} \lhd \mathcal{N}$, if (i) $(\mathfrak{T}, +)$ is a normal subgroup (NSG, in short) of $(\mathcal{N}, +)$, (ii) $\mathfrak{T}\mathcal{N} \subseteq \mathfrak{T}$ and (iii) $u(v+i) - uv \in \mathfrak{T}\ \forall\ u, v \in \mathcal{N}$ and $i \in \mathfrak{T}$. A NSG $\mathfrak{T}$ of $(\mathcal{N}, +)$ with (ii) is called a right ideal of $\mathcal{N}$, written as $\mathfrak{T} \lhd_r \mathcal{N}$, while a NSG $\mathfrak{T}$ of $(\mathcal{N}, +)$ with (iii) is called a left ideal, written as $\mathfrak{T} \lhd_l \mathcal{N}$, of $\mathcal{N}$.*

Definition 2.5. *[13] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called a prime ideal (briefly, PI) of $\mathcal{N}$ if for any two ideals $\mathcal{I}, \mathcal{K}$ of $\mathcal{N}$, $\mathcal{I}\mathcal{K} \subseteq \mathfrak{T} \Rightarrow \mathcal{I} \subseteq \mathfrak{T}$ or $\mathcal{K} \subseteq \mathfrak{T}$.*

Definition 2.6. *[15] Let $\mathfrak{T} \lhd \mathcal{N}$. Then the prime radical $\mathfrak{R}(\mathfrak{T})$ in $\mathcal{N}$ is defined to be the intersection of all PIs of $\mathcal{N}$ containing $\mathfrak{T}$.*

Definition 2.7. *[14] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called a prime ideal of type 1 (in short, PI of type 1) if $\forall\ u, v \in \mathcal{N}$, $u\mathcal{N}v \subseteq \mathfrak{T} \Rightarrow u \in \mathfrak{T}$ or $v \in \mathfrak{T}$.*

Definition 2.8. *[9, 14] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called a completely prime ideal (in short, CPI) (or PI of type 2) of $\mathcal{N}$ if $\forall\ u, v \in \mathcal{N}, uv \in \mathfrak{T} \Rightarrow u \in \mathfrak{T}$ or $v \in \mathfrak{T}$.*

Remark 2.9. *[14] It is easy to check that a PI of type 2 in $\mathcal{N}$ is a PI of type 1.*

Definition 2.10. *[11] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called reflexive if $u, v \in \mathcal{N}$ such that $u\mathcal{N}v \subseteq \mathfrak{T}$, then $v\mathcal{N}u \subseteq \mathfrak{T}$.*

Definition 2.11. *[13] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called a semiprime ideal (briefly, SPI) of $\mathcal{N}$ if for any ideal $\mathcal{K}$ of $\mathcal{N}$, $\mathcal{K}^2 \subseteq \mathfrak{T} \Rightarrow \mathcal{K} \subseteq \mathfrak{T}$.*

Definition 2.12. *[10] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called a completely semiprime ideal (briefly, CSPI) of $\mathcal{N}$ if $\forall\ v \in \mathcal{N}, v^2 \in \mathfrak{T} \Rightarrow v \in \mathfrak{T}$.*

Definition 2.13. *[15] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called semi-symmetric (briefly, SSI) if $u^n \in \mathfrak{T}$ for some positive integer n implies $< u >^n \subseteq \mathfrak{T}$, where $u \in \mathcal{N}$ and $< u >$ stands for the principal ideal of $\mathcal{N}$ generated by u.*

Remark 2.14. *[15] It is evident that every CSPI of a zero symmetric near-ring $\mathcal{N}$ is a SSI of $\mathcal{N}$.*

Definition 2.15. *[8] Let $\mathfrak{T}$ be a subset of $\mathcal{N}$. Then the radical of $\mathfrak{T}$ is defined by $\sqrt{\mathfrak{T}} = \{v \in \mathcal{N} : v^n \in \mathfrak{T}$ for some $n \in Z^+\}$.*

If $\mathfrak{T} \lhd \mathcal{N}$, then $\sqrt{\mathfrak{T}}$ is also an ideal of $\mathcal{N}$ containing $\mathfrak{T}$.

Definition 2.16. *[1] Let $\mathfrak{T} \lhd \mathcal{N}$. Then $\mathfrak{T}$ is called a primary ideal (briefly, PRI) of $\mathcal{N}$ if $\forall\ u, v \in \mathcal{N}$, $uv \in \mathfrak{T} \Rightarrow u \in \mathfrak{T}$ or $v^m \in \mathfrak{T}$ for some $m \in Z^+$.*

Proposition 2.17. *[1] Let $\mathfrak{T}$ be a CPI of $\mathcal{N}$. Then $\mathfrak{T}$ is a PRI of $\mathcal{N}$.*

Theorem 2.18. *[1] Let $\mathfrak{T} \triangleleft \mathcal{N}$ such that $\sqrt{\mathfrak{T}} = \mathfrak{T}$. Then $\mathfrak{T}$ is a CPI of $\mathcal{N}$ if and only if $\mathfrak{T}$ is a PRI of $\mathcal{N}$.*

Definition 2.19. *[1] Let $\mathfrak{T} \triangleleft \mathcal{N}$. Then $\mathfrak{T}$ is called a semiprimary ideal (briefly, SPRI) of $\mathcal{N}$ if $\forall\ u, v \in \mathcal{N}$, $uv \in \mathfrak{T} \Rightarrow u^m \in \mathfrak{T}$ or $v^m \in \mathfrak{T}$ for some $m \in Z^+$.*

Definition 2.20. *[13] A near-ring $\mathcal{N}$ is said to fulfill the insertion-of-factors-property (IFP) provided that $u, v, n \in \mathcal{N}$ such that $uv = 0$ implies $unv = 0$. $\mathcal{N}$ has the strong IFP if every homomorphic image of $\mathcal{N}$ has the IFP.*

Definition 2.21. *[13] A near-ring $\mathcal{N}$ is called a weak commutative near-ring (briefly, WCN) if $uwv = uvw\ \forall\ u, v, w \in \mathcal{N}$.*

3 Pseudo Symmetric Ideals and Pseudo Symmetric Near-Rings

In this section, we introduce the notions of pseudo symmetric subset, pseudo symmetric ideal and pseudo symmetric near-ring. We obtain the conditions for a near-ring to be a pseudo symmetric near-ring. We investigate the relations among the radicals of type 1 prime, pseudo symmetric, completely prime, completely semiprime, primary and semiprimary ideals of a near-ring.

Definition 3.1. *A non-empty subset $\mathfrak{T}$ of $\mathcal{N}$ is called a pseudo symmetric subset (briefly, PSS) of $\mathcal{N}$ if $\forall\ p, k \in \mathcal{N}$, $pk \in \mathfrak{T}$ implies $prk \in \mathfrak{T}\ \forall\ r \in \mathcal{N}$.*

Definition 3.2. *A non-empty subset $\mathfrak{T}$ of $\mathcal{N}$ is called a pseudo symmetric ideal (briefly, PSI) of $\mathcal{N}$ if $\mathfrak{T}$ is both a pseudo symmetric subset and an ideal of $\mathcal{N}$. A pseudo symmetric near-ring (briefly, PSN) $\mathcal{N}$ is a near-ring $\mathcal{N}$ in which each ideal of $\mathcal{N}$ is pseudo symmetric.*

Example 3.3. *Let $\mathcal{N} = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the near-ring over the Dihedral group D_8 with the following operations tables:*
(Scheme 14: (10,1,10,5,10,5,10,1) See [13], p.415)

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

.	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	3	0	3	0	1
2	0	2	0	2	0	2	0	2
3	0	3	0	1	0	1	0	3
4	4	4	4	4	4	4	4	4
5	4	5	4	7	4	7	4	5
6	4	6	4	6	4	6	4	6
7	4	7	4	5	4	5	4	7

Then $\{0\}, \{0,2\}, \{0,4\}, \{0,2,4,6\}$ and $\mathcal{N}$ are PSSs of $\mathcal{N}$. It is routine to check that $\{0\}, \{0,2\}, \{0,2,4,6\}$ and $\mathcal{N}$ are the only ideals of $\mathcal{N}$ and so these ideals are PSIs of $\mathcal{N}$. Hence $\mathcal{N}$ is a PSN.

Proposition 3.4. *Let $\mathfrak{T}$ be a NSG of $\mathcal{N}$. Then, for each $k \in \mathcal{N}$, $(\mathfrak{T} : k) = \{p \in \mathcal{N} | pk \in \mathfrak{T}\}$ is a NSG of $\mathcal{N}$.*

Proof. Let $k \in \mathcal{N}$ and $p_1, p_2 \in (\mathfrak{T} : k)$. Then $p_1k \in \mathfrak{T}$ and $p_2k \in \mathfrak{T}$. Since $\mathfrak{T}$ is a subgroup of $\mathcal{N}$, $p_1k - p_2k \in \mathfrak{T}$ which implies $(p_1 - p_2)k \in \mathfrak{T}$. Thus $p_1 - p_2 \in (\mathfrak{T} : k)$. Since $\mathfrak{T}$ is a NSG of $\mathcal{N}$, $p_2k + p_1k - p_2k \in \mathfrak{T}$ which implies $(p_2 + p_1 - p_2)k \in \mathfrak{T}$. So $p_2 + p_1 - p_2 \in (\mathfrak{T} : k)$. Hence $(\mathfrak{T} : k)$ is a NSG of $\mathcal{N}$. $\square$

Theorem 3.5. *Let $\mathfrak{T} \lhd \mathcal{N}$. Then the following statements are equivalent:*

(i) *$\mathfrak{T}$ is a PSI of $\mathcal{N}$.*

(ii) $(\mathfrak{T} : k) = \{p \in \mathcal{N} | pk \in \mathfrak{T}\} \lhd_r \mathcal{N} \ \forall \ k \in \mathcal{N}$.

Proof. (i)$\Rightarrow$ (ii) Let $k \in \mathcal{N}$. Then, by Proposition 3.4, $(\mathfrak{T} : k)$ is a NSG of $\mathcal{N}$. Let $p \in (\mathfrak{T} : k)$. Then $pk \in \mathfrak{T}$. Since $\mathfrak{T}$ is a PSI of $\mathcal{N}$, we have $prk \in \mathfrak{T}$ $\forall \ r \in \mathcal{N}$. This implies that $pr \in (\mathfrak{T} : k) \ \forall \ p \in (\mathfrak{T} : k)$ and $r \in \mathcal{N}$. That is $(\mathfrak{T} : k)\mathcal{N} \subseteq (\mathfrak{T} : k)$. Hence $(\mathfrak{T} : k) \lhd_r \mathcal{N}$.
(ii)$\Rightarrow$(i) Let $p, k \in \mathcal{N}$ and $pk \in \mathfrak{T}$. Then $p \in (\mathfrak{T} : k)$. Since $(\mathfrak{T} : k) \lhd_r \mathcal{N}$, we have $pr \in (\mathfrak{T} : k) \ \forall \ r \in \mathcal{N}$. So $prk \in \mathfrak{T} \ \forall \ r \in \mathcal{N}$. Hence $\mathfrak{T}$ is a PSI of $\mathcal{N}$. $\square$

Theorem 3.6. *Let $\mathcal{N}$ be a ZSN and $\mathfrak{T} \lhd \mathcal{N}$. Then the following statements are equivalent:*

(i) *$\mathfrak{T}$ is a PSI of $\mathcal{N}$.*

(ii) $(\mathfrak{T} : k) = \{p \in \mathcal{N} | pk \in \mathfrak{T}\} \triangleleft \mathcal{N} \ \forall \ k \in \mathcal{N}$.

Proof. (i)⇒(ii) The proof is straightforward from Lemma 2.2 of [10].
(ii)⇒(i) Since $(\mathfrak{T} : k) \triangleleft_r \mathcal{N}$, by Theorem 3.5, $\mathfrak{T}$ is a PSI of $\mathcal{N}$. □

Theorem 3.7. *The intersection of any two PSIs of $\mathcal{N}$ is a PSI of $\mathcal{N}$.*

Proof. Let $\mathcal{K}$ and $\mathfrak{T}$ be any two PSIs of $\mathcal{N}$. Suppose $p, k \in \mathcal{K} \cap \mathfrak{T}$. Then $p, k \in \mathcal{K}$ and $p, k \in \mathfrak{T}$. Since $\mathcal{K}$ and $\mathfrak{T}$ are PSIs of $\mathcal{N}$, we have $prk \in \mathcal{K}$ and $prk \in \mathfrak{T} \ \forall \ r \in \mathcal{N}$. Thus $prk \in \mathcal{K} \cap \mathfrak{T} \ \forall \ r \in \mathcal{N}$. So $\mathcal{K} \cap \mathfrak{T}$ is a PSS of $\mathcal{N}$. Since the intersection of any two ideals of $\mathcal{N}$ is an ideal of $\mathcal{N}$, $\mathcal{K} \cap \mathfrak{T} \triangleleft \mathcal{N}$. Hence $\mathcal{K} \cap \mathfrak{T}$ is a PSI of $\mathcal{N}$. □

Remark 3.8. *Since the union of any two ideals of $\mathcal{N}$ need not be an ideal of $\mathcal{N}$, the union of any two PSIs of $\mathcal{N}$ is not necessarily a PSI of $\mathcal{N}$.*

To prove the union of two PSIs of $\mathcal{N}$ is a PSI of $\mathcal{N}$, we need an additional condition and so we have the following obvious theorem.

Theorem 3.9. *The union of any two PSIs of $\mathcal{N}$ is a PSI of $\mathcal{N}$ if one is contained in the other.*

Theorem 3.10. *For a near-ring $\mathcal{N}$, the following statements are equivalent:*

(i) $\mathcal{N}$ has the strong IFP.

(ii) $\mathcal{N}$ is a PSN.

Proof. $\mathcal{N}$ has the strong IFP
⇔ By Proposition 9.2 of [13], for every ideal $\mathfrak{T}$ of $\mathcal{N}$, $\forall \ p, k \in \mathcal{N}$ and $pk \in \mathfrak{T}$ implies $prk \in \mathfrak{T} \ \forall \ r \in \mathcal{N}$
⇔ Every ideal of $\mathcal{N}$ is a PSI of $\mathcal{N}$
⇔ $\mathcal{N}$ is a PSN. □

Proposition 3.11. *Every CPI of $\mathcal{N}$ is a CSPI of $\mathcal{N}$.*

Proof. Let $\mathfrak{T}$ be a CPI of $\mathcal{N}$. Suppose $p \in \mathcal{N}$ and $p^2 \in \mathfrak{T}$. Since $\mathfrak{T}$ is CPI of $\mathcal{N}$, we have $p \in \mathfrak{T}$. Hence $\mathfrak{T}$ is CSPI of $\mathcal{N}$. □

Lemma 3.12. (Lemma 2.1, [10]) *Let $\mathfrak{T}$ be a CSPI of $\mathcal{N}$ and let $p, k \in \mathcal{N}$. If $pk \in \mathfrak{T}$ and $r \in \mathcal{N}$, then $prk \in \mathfrak{T}$.*

From Lemma 3.12 and Definition 3.1, we have the following proposition.

Proposition 3.13. *Every CSPI of $\mathcal{N}$ is a PSI of $\mathcal{N}$.*

Combining Proposition 3.11 and Proposition 3.13, we have the following theorem.

Theorem 3.14. *Every CPI of $\mathcal{N}$ is a PSI of $\mathcal{N}$.*

The following example illustrates the above theorem.

Example 3.15. *Let $\mathcal{N} = \{0, a, b, c\}$ be the near-ring with the operations tables as follows: (Scheme 20 : (7,8,1,2) See [13], p. 408)*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

Then $\{0\}, \{0, a\}$ and $\mathcal{N}$ are only ideals and also CPIs of $\mathcal{N}$. We can check that $\{0\}, \{0, a\}$ and $\mathcal{N}$ are PSIs of $\mathcal{N}$.

The converse of Theorem 3.14 is not true in general. We can demonstrate it by the following example.

Example 3.16. *Consider the near-ring $\mathcal{N}$ as in Example 3.3. Then $\{0, 2\}$ is a PSI of $\mathcal{N}$. Since $3, 4 \in \mathcal{N}$ and $3 \cdot 4 = 0 \in \{0, 2\}$ but $3 \notin \{0, 2\}$ and $4 \notin \{0, 2\}$, $\{0, 2\}$ is not a CPI of $\mathcal{N}$,*

Theorem 3.17. (Corollary 2.3, [15]) *An ideal $\mathfrak{T}$ of a ZSN $\mathcal{N}$ is CPI if and only if $\mathfrak{T}$ is PI and SSI.*

Lemma 3.18. (Lemma 3.1, [7]) *If $\mathfrak{T}$ is a prime left ideal of a ZSN $\mathcal{N}$, then the largest two-sided ideal $(\mathfrak{T} : \mathcal{N}) = \{r \in \mathcal{N} | rn \subseteq \mathfrak{T}\} \subseteq \mathfrak{T}$ is a PI of $\mathcal{N}$.*

Definition 3.19. *[2] Let $\mathfrak{T} \triangleleft_l \mathcal{N}$. Then $\mathfrak{T}$ is called completely prime (resp. completely semiprime) if $\forall\ p, k \in \mathcal{N}$ with $pk \in \mathfrak{T}$ and $k \notin \mathfrak{T} \Rightarrow p \in (\mathfrak{T} : \mathcal{N})$ (resp. $p^2 \in \mathfrak{T} \Rightarrow p \in (\mathfrak{T} : \mathcal{N})$).*

Proposition 3.20. *If $\mathfrak{T}$ is a completely prime (resp. completely semiprime) left ideal of a ZSN $\mathcal{N}$, then $(\mathfrak{T} : \mathcal{N})$ is a PSI and SSI of $\mathcal{N}$.*

Proof. Let $\mathfrak{T}$ be a completely prime left ideal of $\mathcal{N}$ and let $p, k \in \mathcal{N}$ such that $pk \in (\mathfrak{T} : \mathcal{N})$ and $k \notin (\mathfrak{T} : \mathcal{N})$. Then there exists an element $r \in \mathcal{N}$ such that $pkr \in \mathfrak{T}$ and $br \notin \mathfrak{T}$. By Definition 3.19, we have $p \in (\mathfrak{T} : \mathcal{N})$. Thus $(\mathfrak{T} : \mathcal{N})$ is a CPI of $\mathcal{N}$. So, by Theorem 3.14 and Theorem 3.17, $(\mathfrak{T} : \mathcal{N})$

is a PSI and SSI of $\mathcal{N}$. Suppose that $\mathfrak{T}$ is a CSPI of $\mathcal{N}$. Let $p \in \mathcal{N}$ and $p^2 \in (\mathfrak{T} : \mathcal{N}) \subseteq \mathfrak{T}$. Then, by Definition 3.19, we have $p \in (\mathfrak{T} : \mathcal{N})$. Therefore $(\mathfrak{T} : \mathcal{N})$ is a CSPI of $\mathcal{N}$. So, by Proposition 3.13 and Remark 2.14, $(\mathfrak{T} : \mathcal{N})$ is a PSI and SSI of $\mathcal{N}$. □

Theorem 3.21. *If $\mathfrak{T}$ is a SSI of a ZSN $\mathcal{N}$, then $\mathfrak{R}(\mathfrak{T})$ is a PSI and SSI of $\mathcal{N}$.*

Proof. Let $\mathfrak{T}$ be a SSI of $\mathcal{N}$. Then, by Theorem 1.3 of [15], $\mathfrak{R}(\mathfrak{T})$ is a CSPI of $\mathcal{N}$. Since, every CSPI of $\mathcal{N}$ is a PSI and SSI of $\mathcal{N}$, $\mathfrak{R}(\mathfrak{T})$ is a PSI and SSI of $\mathcal{N}$. □

Theorem 3.22. *Every PI of type 1 in a PSN $\mathcal{N}$ is completely prime.*

Proof. Let $\mathfrak{T}$ be any PI of type 1 in $\mathcal{N}$. Suppose $p, k \in \mathcal{N}$ and $pk \in \mathfrak{T}$. Since $\mathcal{N}$ is PSN, $\mathfrak{T}$ is a PSI of $\mathcal{N}$. Therefore, $prk \in \mathfrak{T}\ \forall\ r \in \mathcal{N}$. That is, $p\mathcal{N}k \subseteq \mathfrak{T}$ which implies that $p \in \mathfrak{T}$ or $k \in \mathfrak{T}$. Hence $\mathfrak{T}$ is a CPI of $\mathcal{N}$. □

In the next theorem, we study the relations between CPI, PI of type 1 and PSI of a near-ring.

Theorem 3.23. *Let $\mathcal{N}$ be a near-ring. Then the following statements hold:*

(i) Every CPI $\mathfrak{T}$ of $\mathcal{N}$ is both a PI of type 1 and a PSI of $\mathcal{N}$.

(ii) Every PSI $\mathfrak{T}$ of $\mathcal{N}$ is a PI of type 1 if and only if $\mathfrak{T}$ is a CPI of $\mathcal{N}$.

(iii) Every PI of type 1 $\mathfrak{T}$ of $\mathcal{N}$ is a PSI of $\mathcal{N}$ if and only if $\mathfrak{T}$ is a CPI of $\mathcal{N}$.

Proof. (i) The proof is straightforward from Remark 2.9 and Lemma 3.14.
(ii)$\Longrightarrow$ The proof follows from Theorem 3.22.
(ii)$\Longleftarrow$ The converse follows from (i).
(iii)$\Longrightarrow$ Let $\mathfrak{T}$ be a PSI of $\mathcal{N}$ and let $p, k \in \mathcal{N}$ and $pk \in \mathfrak{T}$. Then $prk \in \mathfrak{T}\ \forall\ r \in \mathcal{N}$. That is $p\mathcal{N}k \subseteq \mathfrak{T}$. Since $\mathfrak{T}$ is a PI of type 1, we have $p \in \mathfrak{T}$ or $k \in \mathfrak{T}$. Hence $\mathfrak{T}$ is CPI of $\mathcal{N}$.
(iii)$\Longleftarrow$ The converse follows from (i). □

Theorem 3.24. *For a reflexive ideal $\mathfrak{T}$ of $\mathcal{N}$, the following statements are equivalent:*

(i) $\mathfrak{T}$ is a CPI of $\mathcal{N}$.

(ii) $\mathfrak{T}$ is both a PI and a CSPI of $\mathcal{N}$.

(iii) $\mathfrak{T}$ *is both a PI and a PSI of* $\mathcal{N}$.

Proof. $(i) \Rightarrow (ii)$ Since, every CPI of $\mathcal{N}$ is a PI of $\mathcal{N}$ and by Proposition 3.11, every CPI of $\mathcal{N}$ is a CSPI of $\mathcal{N}$. Hence $\mathfrak{T}$ is both a PI and a CSPI of $\mathcal{N}$.
$(ii) \Rightarrow (iii)$ The proof is straightforward from Proposition 3.13.
$(iii) \Rightarrow (i)$ By Lemma 2.10 of [6], every PI $\mathfrak{T}$ of $\mathcal{N}$ is a PI of type 1. Since $\mathfrak{T}$ is a PSI of $\mathcal{N}$, by Theorem 3.23 (iii), $\mathfrak{T}$ is a CPI of $\mathcal{N}$. □

Theorem 3.25. *Let* $\mathfrak{T}$ *be a PRI of* $\mathcal{N}$ *such that* $\sqrt{\mathfrak{T}} = \mathfrak{T}$. *Then* $\mathfrak{T}$ *is a PSI of* $\mathcal{N}$.

Proof. Let $\mathfrak{T}$ be a PRI of $\mathcal{N}$ such that $\sqrt{\mathfrak{T}} = \mathfrak{T}$. Then, by Theorem 2.18, $\mathfrak{T}$ is a CPI of $\mathcal{N}$. Since every CPI of $\mathcal{N}$ is a PSI of $\mathcal{N}$, $\mathfrak{T}$ is a PSI of $\mathcal{N}$. □

Theorem 3.26. *Let* $\mathfrak{T}$ *be a subset of* $\mathcal{N}$ *and* $\sqrt{\mathfrak{T}}$ *be a PI of type 1 of* $\mathcal{N}$. *Then the following statements are equivalent:*

1. $\sqrt{\mathfrak{T}}$ *is a PRI of* $\mathcal{N}$.
2. $\sqrt{\mathfrak{T}}$ *is a PSI of* $\mathcal{N}$.

Proof. $(i) \Rightarrow (ii)$ Let $\sqrt{\mathfrak{T}}$ be a PRI of $\mathcal{N}$ and let $p, k \in \mathcal{N}$ such that $pk \in \sqrt{\mathfrak{T}}$. Then $p \in \sqrt{\mathfrak{T}}$ or $k^m \in \sqrt{\mathfrak{T}}$ for some $m \in Z^+$ and by Definition 2.15, we have $p \in \sqrt{\mathfrak{T}}$ or $(k^m)^r \in \mathfrak{T}$ for some $r \in Z^+$. That is, $p \in \sqrt{\mathfrak{T}}$ or $k^l \in \mathfrak{T}$ for some $l = mr \in Z^+$. This implies that $p \in \sqrt{\mathfrak{T}}$ or $k \in \sqrt{\mathfrak{T}}$. Thus $\sqrt{\mathfrak{T}}$ is a CPI of $\mathcal{N}$. Hence, by Theorem 3.14, $\sqrt{\mathfrak{T}}$ is a PSI of $\mathcal{N}$.
$(ii) \Rightarrow (i)$ Let $\sqrt{\mathfrak{T}}$ be a PSI of $\mathcal{N}$. Since $\sqrt{\mathfrak{T}}$ is a PI of type 1, by Theorem 3.23, $\sqrt{\mathfrak{T}}$ is a CPI of $\mathcal{N}$. Hence, by Proposition 2.17, $\sqrt{\mathfrak{T}}$ is a PRI of $\mathcal{N}$. □

Theorem 3.27. *Let* $\mathfrak{T}$ *be a subset of* $\mathcal{N}$ *and* $\sqrt{\mathfrak{T}}$ *be a PI of type 1 of* $\mathcal{N}$. *Then the following statements are equivalent.*

1. $\sqrt{\mathfrak{T}}$ *is a SPRI of* $\mathcal{N}$.
2. $\sqrt{\mathfrak{T}}$ *is a PSI of* $\mathcal{N}$.

Proof. $(i) \Rightarrow (ii)$ Let $\sqrt{\mathfrak{T}}$ be a SPRI of $\mathcal{N}$ and let $p, k \in \mathcal{N}$ such that $pk \in \sqrt{\mathfrak{T}}$. Then $p^m \in \sqrt{\mathfrak{T}}$ or $k^m \in \sqrt{\mathfrak{T}}$ for some $m \in Z^+$ and by Definition 2.15, we have $(p^m)^r \in \mathfrak{T}$ or $(k^m)^r \in \mathfrak{T}$ for some $r \in Z^+$. That is $p^l \in \mathfrak{T}$ or $k^l \in \mathfrak{T}$ for some $l = mr \in Z^+$ Again, by Definition 2.15, we have $p \in \sqrt{\mathfrak{T}}$ or $k \in \sqrt{\mathfrak{T}}$. This shows that $\sqrt{\mathfrak{T}}$ is a CPI of $\mathcal{N}$. Hence, by Theorem 3.14, $\sqrt{\mathfrak{T}}$ is a PSI of $\mathcal{N}$.
$(ii) \Rightarrow (i)$ Let $\sqrt{\mathfrak{T}}$ be a PSI of $\mathcal{N}$. Then, by Theorem 3.26, $\sqrt{\mathfrak{T}}$ is a primary idea of $\mathcal{N}$. Since every PRI of $\mathcal{N}$ is a SPRI of $\mathcal{N}$, $\sqrt{\mathfrak{T}}$ is a SPRI of $\mathcal{N}$. □

Remark 3.28. *[1] It is clear that, every PRI of $\mathcal{N}$ is a SPRI of $\mathcal{N}$.*

Theorem 3.29. *(Theorem 3.26, [1]) Let $\mathfrak{T} \triangleleft \mathcal{N}$. Then $\sqrt{\mathfrak{T}}$ is a CPI of $\mathcal{N}$ if and only if $\mathfrak{T}$ is a SPRI of $\mathcal{N}$.*

In the following theorem, we discuss the relationship among the radicals of type 1 prime, pseudo symmetric, completely prime, completely semiprime, primary and semiprimary ideals of a near-ring.

Theorem 3.30. *Let $\mathfrak{T} \triangleleft \mathcal{N}$ and $\sqrt{\mathfrak{T}}$ be a PI of type 1 of $\mathcal{N}$. Then the following statements are equivalent.*

1. *$\sqrt{\mathfrak{T}}$ is a PRI of $\mathcal{N}$.*
2. *$\sqrt{\mathfrak{T}}$ is a SPRI of $\mathcal{N}$.*
3. *$\sqrt{\mathfrak{T}}$ is a PSI of $\mathcal{N}$.*
4. *$\sqrt{\mathfrak{T}}$ is a CPI of $\mathcal{N}$.*
5. *$\mathfrak{T}$ is a SPRI of $\mathcal{N}$.*
6. *$\sqrt{\mathfrak{T}}$ is a CSPI of $\mathcal{N}$.*

Proof. From Remark 3.28, Theorem 3.27 and Theorem 3.26, we have $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (i)$, from Theorem 3.23 and Theorem 3.29, we have $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$, and from Proposition 3.11 and Proposition 3.13, we have $(iv) \Rightarrow (vi) \Rightarrow (iii)$. Hence the proof is complete. □

Theorem 3.31. *If $\mathfrak{T}$ is both a PI of type 1 and a PSI of $\mathcal{N}$, then $\mathfrak{T}$ is a SPRI of $\mathcal{N}$.*

Proof. Assume that $\mathfrak{T}$ is a PSI of $\mathcal{N}$. Since $\mathfrak{T}$ is a PI of type 1 of $\mathcal{N}$, by Theorem 3.23, $\mathfrak{T}$ is a CPI of $\mathcal{N}$. Then, by Proposition 2.17, $\mathfrak{T}$ is a PRI of $\mathcal{N}$. Since every PRI of $\mathcal{N}$ is a SPRI of $\mathcal{N}$, by Remark 3.22 of [1], $\mathfrak{T}$ is a SPRI of $\mathcal{N}$. □

Theorem 3.32. *Every commutative near-ring is a PSN.*

Proof. Let $\mathcal{N}$ be a commutative near-ring and $\mathfrak{T} \triangleleft \mathcal{N}$. Suppose $p, k \in \mathcal{N}$ and $pk \in \mathfrak{T}$. Then $(pk)\mathcal{N} \subseteq \mathfrak{T}$. Since $\mathcal{N}$ is commutative, $p\mathcal{N}k \subseteq \mathfrak{T}$. Thus $\mathfrak{T}$ is a PSI of $\mathcal{N}$. Hence $\mathcal{N}$ is a PSN. □

Definition 3.33. *An element r of $\mathcal{N}$ is called a mid unit provided $prk = pk$ $\forall\ p, k \in \mathcal{N}$.*

Theorem 3.34. *If $\mathcal{N}$ is a near-ring in which every element is a mid unit, then $\mathcal{N}$ is a PSN.*

Proof. Let $\mathfrak{T} \triangleleft \mathcal{N}$ and $p, k \in \mathcal{N}$ such that $pk \in \mathfrak{T}$. Since every element of $\mathcal{N}$ is a mid unit, $\forall\ r \in \mathcal{N}$, $prk = pk$. So $prk \in \mathfrak{T}\ \forall\ r \in \mathcal{N}$. Thus $\mathfrak{T}$ is a PSI of $\mathcal{N}$. Hence $\mathcal{N}$ is a PSN. □

Example 3.35. *Let $\mathcal{N} = \{0, 1, 2\}$ be the additive group of integers modulo 3 and $(\mathcal{N}, \cdot)$ as follows: (Scheme 4: (1,1,1) See [13], p. 407)*

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

We can verify that every element of $\mathcal{N}$ is a mid unit. Then, by Theorem 3.34, $\mathcal{N}$ is a PSN.

Theorem 3.36. *Every WCN is a PSN.*

Proof. Let $\mathcal{N}$ be a WCN and $\mathfrak{T} \triangleleft \mathcal{N}$. Suppose $p, k, r \in \mathcal{N}$ and $pk \in \mathfrak{T}$. Since $\mathcal{N}$ is WCN, $prk = pkr = (pk)r \in \mathfrak{T}\ \forall\ r \in \mathcal{N}$. Thus $\mathfrak{T}$ is a PSI of $\mathcal{N}$. Hence $\mathcal{N}$ is a PSN. □

Next, the following example illustrates the above theorem.

Example 3.37. *Let $\mathcal{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be the near-ring with the operations tables as follows:*
(Scheme 3: (1,9,9,9,1,1,9,9,1,9,9,9) See [13], p. 421)

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	3	2	7	6	5	4	10	11	8	9
2	2	3	0	1	5	4	7	6	11	10	9	8
3	3	2	1	0	6	7	4	5	9	8	11	10
4	4	5	6	7	8	9	10	11	0	1	2	3
5	5	4	7	6	11	10	9	8	2	3	0	1
6	6	7	4	5	9	8	11	10	3	2	1	0
7	7	6	5	4	10	11	8	9	1	0	3	2
8	8	9	10	11	0	1	2	3	4	5	6	7
9	9	8	11	10	3	2	1	0	6	7	4	5
10	10	11	8	9	1	0	3	2	7	6	5	4
11	11	10	9	8	2	3	0	1	5	4	7	6

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	0	0	1	1	0	1	1	1
2	0	2	2	2	0	0	2	2	0	2	2	2
3	0	3	3	3	0	0	3	3	0	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4
5	4	5	5	5	4	4	5	5	4	5	5	5
6	4	6	6	6	4	4	6	6	4	6	6	6
7	4	7	7	7	4	4	7	7	4	7	7	7
8	8	8	8	8	8	8	8	8	8	8	8	8
9	8	9	9	9	8	8	9	9	8	9	9	9
10	8	10	10	10	8	8	10	10	8	10	10	10
11	8	11	11	11	8	8	11	11	8	11	11	11

Then $(\mathcal{N}, +, \cdot)$ is a WCN. We can check that $\{0\}, \{0, 1, 2, 3\}$ and $\mathcal{N}$ are the only ideals and PSIs of $\mathcal{N}$. Hence $\mathcal{N}$ is a PSN.

4 Future work

We will apply the notions of soft sets and fuzzy sets to pseudo symmetric ideal (near-ring), and introduce the notions of soft pseudo symmetric ideal (near-ring), fuzzy pseudo symmetric ideal (near-ring) and fuzzy soft pseudo symmetric ideal (near-ring).

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