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Partial A_b-metric space and some fixed point theorems

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Abstract

In this note, we introduce the concept of partial A_b-metric space by generalising the concepts of partial metric space, b-metric space and partial A-metric spaces. Further, we prove some fixed point theorems in partial A_b-metric space. By giving examples we also show that partial A_b-metric space is generalization of partial A-metric space.

1 Introduction and Preliminaries

Motivated by experience from computer science and possible examples from finite and infinite sequences, Steve Matthews [1] in the year 1992, led to the introduction of a generalisation of metric space called partial metric space. The main characteristic of partial metric space is that self distance may not be zero. The definition of partial metric space by S. Matthews [1] is as follows.

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Definition 1. *[1] Let X be a nonempty set and $p : X \times X \longrightarrow [0, \infty)$, then (X, p) is said to be a partial metric space if for all $x, y, z \in X$ we have:*

(1) $x = y$ *if and only if* $p(x, y) = p(x, x) = p(y, y)$*;*

(2) $p(x, x) \leq p(x, y)$*;*

(3) $p(x, y) = p(y, x)$*;*

(4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The pair (X, p) is called partial metric space.

The concept of b-metric space is given by Bakhtin [2] and Czerwik [3] made further study of b-metric space.

Definition 2. *[2] Let X be a nonempty set. A b-metric on X is a function $d : X^2 \to [0, \infty)$ if there exists a real number $b \geq 1$ such that the following conditions hold for all $x, y, z \in X$:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*,*

(ii) $d(x, y) = d(y, x)$*,*

(iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$

The pair (X, d) is called a b-metric space.

Sedghi et. al. [4] introduce the concept of S-metric space. The concept is further generalised to partial S-metric space by N. Mlaiki [7]. As an extension to S-metric space, the concept of S_b-metric space is introduced and this can be found in [8, 9, 10, 11, 15, 17, 19, 20, 22]. The concept of partial S_b-metric space is introduced by Nizar Souayah [12]. In an another extension to metric space Abbas et. al. [5] introduce the concept of A-metric space. This definition is further generalised and studied by different authors in different directions [6, 13, 14, 16, 18, 21]. Following are definitions of A-metric and A_b-metric spaces.

Definition 3. *[5] Let X be a nonempty set. A function $A : X^n \to [0, \infty)$ is said to be an A-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions hold:*

(A1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0$,

(A2) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ *if and only if* $x_1 = x_2 = x_3 = ... = x_{n-1} = x_n$,

(A3)

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \leq & [A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) \\ & +A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a) \\ & +A(x_3, x_3, x_3, ..., (x_3)_{n-1}, a) \\ \vdots & \\ & +A(x_{n-1}, x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) \\ & +A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a).] \end{aligned}$$

The pair (X, A) is called an A-metric space.

Definition 4. *[13] Let X be a nonempty set and $b \geq 1$ be a given real number. A function $A : X^n \to [0, \infty)$ is said to be an A_b-metric on X if for any $x_i, a \in X, i = 1, 2, ..., n$, the following conditions hold:*

(A_b1) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0$,

(A_b2) $A(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ *if and only if* $x_1 = x_2 = x_3 = ... = x_{n-1} = x_n$,

(A_b3)

$$\begin{aligned} A(x_1, x_2, x_3, ..., x_{n-1}, x_n) \leq & b[A(x_1, x_1, x_1, ..., (x_1)_{n-1}, a) \\ & +A(x_2, x_2, x_2, ..., (x_2)_{n-1}, a) \\ & +A(x_3, x_3, x_3, ..., (x_3)_{n-1}, a) \\ \vdots & \\ & +A(x_{n-1}, x_{n-1}, x_{n-1}, ..., (x_{n-1})_{n-1}, a) \\ & +A(x_n, x_n, x_n, ..., (x_n)_{n-1}, a)]. \end{aligned}$$

The pair (X, A_b) is called an A_b-metric space.

In this note, we introduce the concept of partial A_b-metric space by generalising the concepts of partial metric space, b-metric space and partial A-metric space.

Now we present the following definition of partial A_b-metric space.

Definition 5. *Let X be a non empty set and $b \geq 1$ be a real number. A partial A_b-metric on a non-empty set X is a function $A_b : X^n \to [0, \infty)$ such that for all $x_1, x_2, \ldots, x_n, t \in X$;*

(i) $x_1 = x_2 = \cdots = x_n$ *if and only if* $A_b(x_1, x_1, \ldots, x_1) = A_b(x_2, x_2, \ldots, x_2) = \cdots = A_b(x_n, x_n, \ldots, x_n) = A_b(x_1, x_2, \ldots, x_n)$,

(ii) $A_b(x_1, x_1, \ldots, x_1) \leq A_b(x_1, x_2, \ldots, x_n)$,

(iii) $A_b(x_1, x_1, \ldots, x_1, x_2) = A_b(x_2, x_2, \ldots, x_2, x_1)$,

(iv) *there exists $s \geq 1$ such that*

$$\begin{aligned} A_b(x_1, x_2, \ldots, x_n) \;\; \leq \;\; & b[A_b(x_1, x_1, \ldots, x_1, t) + A_b(x_2, x_2, \ldots, x_2, t) \\ & + \cdots + A_b(x_n, x_n, \ldots, x_n, t)] - A_b(t, t, \ldots, t) \end{aligned}$$

(X, A_b) is then called a partial A_b-metric space.

Definition 6. *Let (X, A_b) be a partial A_b-metric space and $\{x_q\}$ be a sequence in X. Then*

1. *$\{x_q\}$ is called convergent if and only if there exists $z \in X$ such that $A_b(x_q, x_q, \ldots, x_q, z) \to A_b(z, z, \ldots, z)$ as $q \to \infty$.*

2. *$\{x_q\}$ is said to be a Cauchy sequence in (X, A_b) if $\lim_{q,m\to\infty} A_b(x_q, x_q, \ldots, x_q, x_m)$ exists and finite.*

3. *(X, A_b) is a complete partial A_b-metric space if for every Cauchy sequence $\{x_q\}$ there exists $x \in X$ such that*

$$\lim_{q,m\to\infty} A_b(x_q, x_q, \ldots, x_q, x_m) = \lim_{q\to\infty} A_b(x_q, x_q, \ldots, x_q, x) = A_b(x, x, \ldots, x).$$

Now, we give an example of a partial A_b-metric space that is not a partial A-metric space.

Example 1. *Let $X = [0, \infty)$ and $p \geq n - 1$ be a constant and $A_b : X^n \to [0, \infty)$ defined by*

$$A_b(x_1, x_2, \ldots, x_n) = [\max\{x_1, x_2, \ldots, x_{n-1}\}]^p + |\max\{x_1, x_2, \ldots, x_{n-1}\} - x_n|^p$$

for all $x_1, x_2, \ldots, x_n \in X$. Then (X, A_b) is a partial A_b-metric space with coefficient $b = (n-1)p > 1$, but it is not a partial A-metric space. Indeed, for $x_1 = 5$, $x_2 = x_3 = \cdots = x_{n-1} = 3$, $x_n = 1$, $t = 4$, we have $A_b(x_1, x_2, \ldots, x_n) = 5^p + 4^p$ and

$$\begin{aligned} & A_b(x_1, x_1, \ldots, x_1, t) + A_b(x_2, x_2, \ldots, x_2, t) + \cdots + A_b(x_n, x_n, \ldots, x_n, t)] - A_b(t, t, \ldots, t) \\ = & (5^p + 1) + (3^p + 1)(n-1) - (4^p - 0) \\ = & 5^p + (n-1)3^p + n - 4^p \end{aligned}$$

hence $A_b(x_1, x_2, \ldots, x_n) > A_b(x_1, x_1, \ldots, x_1, t) + A_b(x_2, x_2, \ldots, x_2, t) + \cdots + A_b(x_n, x_n, \ldots, x_n, t)] - A_b(t, t, \ldots, t)$ for all $p > 1$. Therefore, A_b is not a partial A-metric space on X.

2 Main Result

Now, we establish the following theorem.

Theorem 2.1. *Let (X, A_b) be a complete partial A_b-metric space with coefficient $b \geq 1$ and $T : X \to X$ be a mapping satisfying the following condition*

$$A_b(Tx_1, Tx_2, \ldots, Tx_n) \leq \lambda A_b(x_1, x_2, \ldots, x_n) \tag{2.1}$$

for all $x_1, x_2, \ldots, x_n \in X$, $\lambda \in [0, 1)$. Then T has a unique fixed point $u \in X$ and $A_b(u, u, \ldots, u) = 0$.

Proof. First, let us prove the uniqueness. Let $u, v \in X$ be two distinct fixed points of T, that is, $Tu = u$ and $Tv = v$.
We have

$$\begin{aligned} A_b(u, u, \ldots, u, v) &= A_b(Tu, Tu, \ldots, Tu, Tv) \\ &\leq \lambda A_b(u, u, \ldots, u, v) \\ &< A_b(u, u, \ldots, u, v). \end{aligned}$$

Thus, $A_b(u, u, \ldots, u, v) = 0 \Rightarrow u = v$. This shows that fixed point is unique.

Now, we prove $A_b(u, u, \ldots, u, v) = 0$. Let $A_b(u, u, \ldots, u, v) > 0$. By (1),

$$\begin{aligned} A_b(u, u, \ldots, u) &= A_b(Tu, Tu, \ldots, Tu) \\ &\leq \lambda A_b(u, u, \ldots, u) \\ &< A_b(u, u, \ldots, u) \ \ \textit{a contradiction.} \end{aligned}$$

Therefore, $A_b(u, u, \ldots, u) = 0$.

Let $0 < \varepsilon < 1$. Then, for every $q \in \mathbb{N}$ there exists $n_0 = n_0(\varepsilon, q)$ such that

$$\lambda^{n_0} < \frac{\varepsilon}{4(n-1)b}. \tag{2.2}$$

Let $T^{n_0} \equiv F$ and $F^k x_0 = x_k \forall k \in \mathbb{N}$, where $x_0 \in X$ is arbitrary. Then, for all $x_1, x_2, \ldots, x_n \in X$ we have

$$\begin{aligned} A_b(Fx_1, Fx_2, \ldots, Fx_n) &= A_b(T^{n_0}x_1, T^{n_0}x_2, \ldots, T^{n_0}x_n) \\ &\leq \lambda^{n_0} A_b(x_1, x_2, \ldots, x_n). \end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$\begin{aligned} A_b(x_{k+1}, x_{k+1}, \ldots, x_{k+1}, x_k) &= A_b(Fx_k, Fx_k, \ldots, Fx_k, Fx_{k-1}) \\ &\leq \lambda^{n_0} A_b(x_k, x_k, \ldots, x_k, x_{k-1}) \\ &\leq \lambda^{n_0+k} A_b(x_1, x_1, \ldots, x_1, x_0) \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

Let $l \in \mathbb{N}$ such that

$$A_b(x_{l+1}, x_{l+1}, \ldots, x_{l+1}, x_l) < \frac{\varepsilon}{4(n-1)b}. \tag{2.3}$$

Let

$$B_b(x_l, \frac{\varepsilon}{2}) := \{y \in X | A_b(x_l, x_l, \ldots, x_l, y) < \frac{\varepsilon}{2} + A_b(x_l, x_l, \ldots, x_l)\} \tag{2.4}$$

Let us show that F maps $B_b(x_l, \frac{\varepsilon}{2})$ into itself. We have $B_b(x_l, \frac{\varepsilon}{2}) \neq 0$ since $x_l \in B_b(x_l, \frac{\varepsilon}{2})$. Let $x_z \in B_b(x_z, \frac{\varepsilon}{2})$, then

$$\begin{aligned} A_b(Fx_z, Fx_z, \ldots, Fx_z, Fx_l) &\leq \lambda^{n_0} A_b(x_z, x_z, \ldots, x_z, x_l) \\ &\leq \frac{\varepsilon}{4(n-1)b} A_b(x_z, x_z, \ldots, x_z, x_l) \\ &\leq \frac{\varepsilon}{4(n-1)b}[\frac{\varepsilon}{2} + A_b(x_l, x_l, \ldots, x_l)] \\ &\leq \frac{\varepsilon}{4(n-1)b}[1 + A_b(x_l, x_l, \ldots, x_l)] \end{aligned} \tag{2.5}$$

We have

$$\begin{aligned}
A_b(Fx_z, Fx_l, \ldots, Fx_l, Fx_l) &\leq b[A_b(Fx_z, Fx_z, \ldots, Fx_z, Fx_l) + A_b(Fx_l, Fx_l, \ldots, Fx_l) \\
&\quad +A_b(Fx_l, Fx_l, \ldots, Fx_l) + \cdots + A_b(Fx_l, Fx_l, \ldots, Fx_l)] \\
&\quad -A_b(Fx_l, Fx_l, \ldots, Fx_l) \\
&\leq b\big[\frac{\varepsilon}{4(n-1)b}\{1 + A_b(x_l, x_l, \ldots, x_l)\} + (n-1)A_b(x_l, x_l, \ldots, x_l, Fx_l)\big] \\
&\leq b\big[\frac{\varepsilon}{4(n-1)b}\{1 + A_b(x_l, x_l, \ldots, x_l)\} + (n-1)A_b(x_l, x_l, \ldots, x_l, x_{l+1})\big] \\
&\leq b\big[\frac{\varepsilon}{4(n-1)b}\{1 + A_b(x_l, x_l, \ldots, x_l)\} + (n-1)\frac{\varepsilon}{4(n-1)b}\big] \\
&\leq \frac{\varepsilon}{4(n-1)} + \frac{\varepsilon}{4(n-1)}A_b(x_l, x_l, \ldots, x_l) + \frac{\varepsilon}{4} \\
&\leq \frac{n\varepsilon}{4(n-1)} + \frac{\varepsilon}{4(n-1)}A_b(x_l, x_l, \ldots, x_l) \\
&\leq \frac{\varepsilon}{2} + A_b(x_l, x_l, \ldots, x_l).
\end{aligned}$$

Then, $Fx_z \in B_b(x_l, \frac{\varepsilon}{2})$. Thus F maps $B_b(x_l, \frac{\varepsilon}{2})$ to itself.
So, if $x_l \in B_b(x_l, \frac{\varepsilon}{2})$, then $Fx_l \in B_b(x_l, \frac{\varepsilon}{2})$.
Continuing we have

$$F^q x_l \in B_l(x_l, \frac{\varepsilon}{2}) \forall q \in \mathbb{N},$$

that is, $x_m \in B_l(x_l, \frac{\varepsilon}{2}) \forall m \geq l$.
Thus, for all $m > q \geq l$, let $q = l + i \Rightarrow i = q - l$

$$\begin{aligned}
A_b(x_q, x_q, \ldots, x_q, x_m) &= A_b(Fx_{n-1}, Fx_{n-1}, \ldots, Fx_{n-1}, Fx_{m-1}) \\
&\leq \lambda A_b(x_{n-1}, x_{n-1}, \ldots, x_{n-1}, x_{m-1}) \\
&\leq \lambda^2 A_b(x_{n-2}, x_{n-2}, \ldots, x_{n-2}, x_{m-2}) \\
&\vdots \\
&\leq \lambda^i A_b(x_l, x_l, \ldots, x_l, x_{m-i}) \\
&< A_b(x_l, x_l, \ldots, x_l, x_{m-i}) \\
&< \frac{\varepsilon}{2} + A_b(x_l, x_l, \ldots, x_l)
\end{aligned}$$

But, $A_b(x_l, x_l, \ldots, x_l) < A_b(x_l, x_l, \ldots, x_{l+1}) < \frac{\varepsilon}{4(n-1)b}$.
Hence

$$A_b(x_q, x_q, \ldots, x_q, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4(n-1)b} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\{x_q\}$ is a Cauchy sequence.
Since X is a complete partial A_b-metric space, there exists $u \in X$ such that;

$$\begin{aligned}\lim_{q\to\infty} A_b(x_q, x_q, \ldots, x_q, u) &= \lim_{q\to\infty} A_b(x_q, x_q, \ldots, x_q, x_m) \\ &= A_b(u, u, \ldots, u) = 0.\end{aligned}$$

Let us prove that u is a fixed point of T. For all $q \in \mathbb{N}$, we have

$$\begin{aligned}A_b(u, u, \ldots, u, Tu) &\leq b[A_b(u, u, \ldots, u, x_{q+1}) + A_b(u, u, \ldots, u, x_{q+1}) \\ &\quad + \cdots + A_b(u, u, \ldots, u, x_{q+1}) + A_b(Tu, Tu, \ldots, Tu, x_{q+1})] \\ &\quad - A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}) \\ &\leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + A_b(Tu, Tu, \ldots, Tu, Tx_q)] \\ &\leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + \lambda A_b(u, u, \ldots, u, x_q)] \\ &\leq (n-1)bA_b(u, u, \ldots, u, x_{q+1}) + b\lambda A_b(u, u, \ldots, u, x_q) \to 0 \text{ as } q \to \infty.\end{aligned}$$

Thus, $A_b(u, u, \ldots, u, Tu) = 0$, that is $Tu = u$. Hence, u is a unique fixed point of T. □

Theorem 2.2. *Let (X, A_b) be a complete partial A_b-metric space with coefficient $b \geq 1$ and $T : X \to X$ be a mapping satisfying the following condition*

$$\begin{aligned}A_b(Tx_1, Tx_2, \ldots, Tx_n) \leq \lambda[A_b(x_1, x_1, \ldots, x_1, Tx_1) + A_b(x_2, x_2, \ldots, x_2, Tx_2) \\ + \cdots + b(x_n, x_n, \ldots, x_n, Tx_n)] \quad (2.6)\end{aligned}$$

$\forall x_1, x_2, \ldots, x_n \in X$, where $\lambda \in [0, \frac{1}{n})$, $\lambda \neq \frac{1}{nb}$. Then, T has a unique fixed point $u \in X$ and $A_b(u, u, \ldots, u) = 0$.

Proof. We first prove the uniqueness of the fixed point of T.
From (6), we obtain

$$\begin{aligned}A_b(u, u, \ldots, u) &= A_b(Tu, Tu, \ldots, Tu) \\ &\leq \lambda[A_b(u, u, \ldots, u, Tu) + A_b(u, u, \ldots, u, Tu) + \cdots + A_b(u, u, \ldots, u, Tu)] \\ &= n\lambda[A_b(u, u, \ldots, u, Tu)] \text{ Since } \lambda \in [0, \frac{1}{n}), \text{ we have} \\ &< A_b(u, u, \ldots, u),\end{aligned}$$

Therefore $A_b(u, u, \ldots, u) = 0$.
If possible let $u, v \in X$ be two distinct fixed points then

$$A_b(u, u, \ldots, u) = A_b(v, v, \ldots, v) = 0.$$

From (6)

$$\begin{aligned} A_b(u,u,\ldots,v) &= A_b(Tu,Tu,\ldots,Tu,Tv) \\ &\leq \lambda[A_b(u,u,\ldots,u,Tu)+A_b(u,u,\ldots,u,Tu) \\ &\quad +\cdots+A_b(u,u,\ldots,u,Tu)+A_b(v,v,\ldots,v,Tv)] \\ &\leq (n-1)\lambda A_b(u,u,\ldots,u)+\lambda A_b(v,v,\ldots,v) \\ &= 0. \end{aligned}$$

Therefore, $u=v$ and hence fixed point is unique.

Let $x_0 \in X$ be arbitrary such that $x_q = T^q x_0$ and $A_{b_q} = A_b(x_q, x_q, \ldots, x_q, x_{q+1})$.

Let $A_{b_q} > 0$ for all $q \in \mathbb{N}$, otherwise x_q is a fixed point of T for at least one $q \geq 0$.
For all q, we obtain from (6)

$$\begin{aligned} A_{b_q} &= A_b(x_q,x_q,\ldots,x_q,x_{q+1}) \\ &= A_b(Tx_{q-1},Tx_{q-1},\ldots,Tx_{q-1},Tx_q) \\ &\leq \lambda[(n-1)A_b(x_{q-1},x_{q-1},\ldots,x_{q-1},Tx_{q-1})+A_b(x_q,x_q,\ldots,x_q,Tx_q)] \\ &= \lambda[(n-1)A_b(x_{q-1},x_{q-1},\ldots,x_{q-1},x_q)+A_b(x_q,x_q,\ldots,x_q,x_{q+1})] \\ &= \lambda[(n-1)A_{b_{q-1}}+A_{b_q}] \\ \Rightarrow (1-\lambda)A_{b_q} &\leq (n-1)\lambda A_{b_{q-1}} \\ \Rightarrow A_{b_q} &\leq \frac{(n-1)\lambda}{1-\lambda}A_{b_{q-1}}, \lambda \in [0,\frac{1}{n}). \end{aligned} \tag{2.7}$$

Let $\beta = \frac{(n-1)\lambda}{1-\lambda} < 1$. Continuing

$$A_{b_q} \leq \beta^q A_{b_0}.$$

Therefore, $\lim_{q\to\infty} A_{b_q} = 0$. For $q, m \in \mathbb{N}$

$$\begin{aligned} &A_b(x_q,x_q,\ldots,x_q,x_m) \\ =\ & A_b(T^q x_0,T^q x_0,\ldots,T^q x_0,T^m x_0) \\ =\ & A_b(Tx_{q-1},Tx_{q-1},\ldots,Tx_{q-1},Tx_{m-1}) \\ \leq\ & \lambda[(n-1)A_b(x_{q-1},x_{q-1},\ldots,x_{q-1},Tx_{q-1})+A_b(x_{m-1},x_{m-1},\ldots,x_{m-1},Tx_{m-1})] \\ =\ & \lambda[(n-1)A_b(x_{q-1},x_{q-1},\ldots,x_{q-1},x_q)+A_b(x_{m-1},x_{m-1},\ldots,x_{m-1},x_m)] \\ =\ & \lambda[(n-1)A_{b_{q-1}}+A_{b_{m-1}}]. \end{aligned}$$

So, for every $\varepsilon > 0$, as $\lim_{n\to\infty} A_{b_q} = 0$, we can find $n_0 \in \mathbb{N}$ such that $A_{b_{q-1}} \leq \frac{\varepsilon}{2(n-1)}$ and $A_{b_{m-1}} < \frac{\varepsilon}{2}$, for all $q, m \geq n_0$. Then, we obtain

$$(n-1)A_{b_{q-1}} + A_{b_{m-1}} \leq (n-1)\frac{\varepsilon}{2(n-1)} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

As $\lambda < 1$, it follows that $A_b(x_q, x_q, \ldots, x_q, x_m) < \varepsilon \; \forall q, m \geq n_0$.

Hence $\{x_q\}$ is Cauchy in X and $\lim_{q,m\to\infty} A_b(x_q, x_q, \ldots, x_q, x_m) = 0$.

Since X is complete, there exists $u \in X$ such that

$$\begin{aligned}\lim_{q\to\infty} A_b(x_q, x_q, \ldots, x_q, u) &= \lim_{q\to\infty} A_b(x_q, x_q, \ldots, x_q, u) \\ &= A_b(u, u, \ldots, u) = 0. \qquad (2.8)\end{aligned}$$

For any $q \in \mathbb{N}$, we have

$$A_b(u, u, \ldots, u, Tu) \leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + A_b(Tu, Tu, \ldots, Tu, x_{q+1})]$$

$$-A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}) \leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + A_b(Tu, Tu, \ldots, Tu, Tx_q)]$$
$$\leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + \lambda\{(n-1)A_b(u, u, \ldots, u, Tu) + A_b(x_q, x_q, \ldots, x_q, Tx_q)\}]$$
$$\Rightarrow [1-(n-1)\lambda b]A_b(u, u, \ldots, u, Tu) \leq (n-1)bA_b(u, u, \ldots, u, x_{q+1}) + b\lambda A_b(x_q, x_q, \ldots, x_q, Tx_q)$$
$$\Rightarrow A_b(u, u, \ldots, u, Tu) \leq \frac{(n-1)b}{1-(n-1)\lambda b} A_b(u, u, \ldots, u, x_{q+1})$$
$$+ \frac{\lambda b}{1-(n-1)\lambda b} A_b(x_q, x_q, \ldots, x_q, Tx_q).$$

Since $A_b(x_q, x_q, \ldots, x_q, Tx_q) \to A_b(u, u, \ldots, u, Tu)$, $q \to \infty$, we obtain

$$\begin{aligned}A_b(u, u, \ldots, u, Tu) &\leq \frac{(n-1)b}{1-(n-1)\lambda b} A_b(u, u, \ldots, u, x_{q+1}) \\ &\quad + \frac{\lambda b}{1-(n-1)\lambda b} A_b(u, u, \ldots, u, Tu) \\ \Rightarrow \Big(1 - \frac{\lambda b}{1-(n-1)\lambda b}\Big) A_b(u, u, \ldots, u, Tu) &\leq \frac{(n-1)b}{1-(n-1)\lambda b} A_b(u, u, \ldots, u, x_{q+1}) \\ \Rightarrow A_b(u, u, \ldots, u, Tu) &\leq \frac{(n-1)b}{1-n\lambda b} A_b(u, u, \ldots, u, x_{q+1}).\end{aligned}$$

As $\lambda \neq \frac{1}{nb}$ and from (8), we obtain $A_b(u, u, \ldots, u, Tu) = 0$ and $Tu = u$. $\square$

Theorem 2.3. *Let (X, A_b) be complete partial A_b-metric space with coefficient $b \geq 1$ and $T : X \to X$ be a mapping satisfying following conditions*

$$A_b(Tx_1, Tx_2, \ldots, Tx_n) \leq \lambda \max\{A_b(x_1, x_2, \ldots, x_n), A_b(x_1, x_1, \ldots, x_1, Tx_1), A_b(x_2, x_2, \ldots, x_2, Tx_2), \ldots, A_b(x_n, x_n, \ldots, x_n, Tx_n)\} \quad (2.9)$$

for all $x_1, x_2, \ldots, x_n \in X$, where $\lambda \in [0, \frac{1}{(n-1)b})$. Then T has a unique fixed point $u \in X$ and $A_b(u, u, \ldots, u) = 0$.

Proof. Let us prove that if a fixed point of T exists, then it is unique. Let $u, v \in X$ be two fixed points of T and $u \neq v$; that is, $Tu = u$ and $Tv = v$. It follows from (9)

$$\begin{aligned} A_b(u, u, \ldots, u, v) &= A_b(Tu, Tu, \ldots, Tu, Tv) \\ &\leq \lambda \max\{A_b(u, u, \ldots, u, v), A_b(u, u, \ldots, u, Tu), A_b(v, v, \ldots, v, Tv)\} \\ &= \lambda \max\{A_b(u, u, \ldots, u, v), A_b(u, u, \ldots, u), A_b(v, v, \ldots, v)\} \\ &= \lambda A_b(u, u, \ldots, u, v) \\ &< A_b(u, u, \ldots, u, v) \qquad [\text{because } \lambda < 1]. \end{aligned}$$

We obtain $A_b(u, u, \ldots, u, v) < A_b(u, u, \ldots, u, v)$ which gives $A_b(u, u, \ldots, u, v) = 0$, then $u = v$. Therefore, if a fixed point of T exists, then it is unique.

Let $x_0 \in X$ and define a sequence $\{x_q\}$ by $x_{q+1} = Tx_q$ $\forall q \geq 0$. For any q, we obtain from (9)

$$\begin{aligned} A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) &= A_b(Tx_q, Tx_q, \ldots, Tx_q, Tx_{q-1}) \\ &\leq \lambda \max\{A_b(x_q, x_q, \ldots, x_q, x_{q-1}), A_b(x_q, x_q, \ldots, x_q, Tx_q), \\ &\quad A_b(x_{q-1}, x_{q-1}, \ldots, x_{q-1}, Tx_{q-1})\}. \end{aligned}$$

Since $A_b(x_{q-1}, x_{q-1}, \ldots, x_{q-1}, Tx_{q-1}) = A_b(x_{q-1}, x_{q-1}, \ldots, x_{q-1}, x_q)$ and by symmetry we have

$$A_b(x_{q-1}, x_{q-1}, \ldots, x_{q-1}, Tx_{q-1}) = A_b(x_q, x_q, \ldots, x_q, x_{q-1}),$$

thus

$$A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) \leq \lambda \max\{A_b(x_q, x_q, \ldots, x_q, x_{q-1}), A_b(x_q, x_q, \ldots, x_q, x_{q+1})\}.$$

If

$$\max\{A_b(x_q, x_q, \ldots, x_q, x_{q-1}), A_b(x_q, x_q, \ldots, x_q, x_{q+1})\} = A_b(x_q, x_q, \ldots, x_q, x_{q+1}),$$

then we obtain

$$\begin{aligned} A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) &\leq \lambda A_b(x_q, x_q, \ldots, x_q, x_{q+1}) \\ &= \lambda A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) \\ &< A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) \end{aligned}$$

which is meaningless.
Therefore,

$$\max\{A_b(x_q, x_q, \ldots, x_q, x_{q-1}), A_b(x_q, x_q, \ldots, x_q, x_{q+1})\} = A_b(x_q, x_q, \ldots, x_q, x_{q-1})$$

and

$$A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) \leq \lambda A_b(x_q, x_q, \ldots, x_q, x_{q-1}) \tag{2.10}$$

that is,

$$A_b(Tx_q, Tx_q, \ldots, Tx_q, Tx_{q-1}) \leq \lambda A_b(x_q, x_q, \ldots, x_q, x_{q-1}). \tag{2.11}$$

By repeating this process, we obtain

$$A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) \leq \lambda^q A_b(x_1, x_1, \ldots, x_1, x_0). \tag{2.12}$$

For $q, m \in \mathbb{N}$, $m > q$, we obtain

$$\begin{aligned} A_b(x_q, x_q, \ldots, x_q, x_m) &= b[(n-1)A_b(x_q, x_q, \ldots, x_q, x_{q+1}) + A_b(x_m, x_m, \ldots, x_m, x_{q+1})] \\ &\quad - A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}) \\ &\leq (n-1)bA_b(x_q, x_q, \ldots, x_q, x_{q+1}) + bA_b(x_m, x_m, \ldots, x_m, x_{q+1}) \\ &\leq (n-1)bA_b(x_q, x_q, \ldots, x_q, x_{q+1}) + b[(n-1)bA_b(x_m, x_m, \ldots, x_m, x_{q+2}) \\ &\quad + bA_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_{q+2}) - A_b(x_{q+2}, x_{q+2}, \ldots, x_{q+2})] \\ &\leq (n-1)bA_b(x_q, x_q, \ldots, x_q, x_{q+1}) + b^2A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_{q+2}) \\ &\quad + (n-1)b^2A_b(x_m, x_m, \ldots, x_m, x_{q+2}) \\ &\leq (n-1)bA_b(x_q, x_q, \ldots, x_q, x_{q+1}) + b^2A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_{q+2}) \\ &\quad + (n-1)b^2[b\{(n-1)A_b(x_m, x_m, \ldots, x_m, x_{q+3}) \\ &\quad + A_b(x_{q+2}, x_{q+2}, \ldots, x_{q+2}, x_{q+3})\} - A_b(x_{q+3}, x_{q+3}, \ldots, x_{q+3})] \\ &\leq (n-1)bA_b(x_q, x_q, \ldots, x_q, x_{q+1}) + b^2A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_{q+2}) \\ &\quad + (n-1)b^3A_b(x_{q+2}, x_{q+2}, \ldots, x_{q+2}, x_{q+3}) \\ &\quad + (n-1)^2b^3A_b(x_m, x_m, \ldots, x_m, x_{q+3}) \\ &\leq (n-1)bA_b(x_q, x_q, \ldots, x_q, x_{q+1}) + b^2A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_{q+2}) \\ &\quad + \cdots + (n-1)^{m-q-2}b^{m-q}A_b(x_m, x_m, \ldots, x_m, x_{m-1}) \\ &= (n-1)bA_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, x_q) + b^2A_b(x_{q+2}, x_{q+2}, \ldots, x_{q+2}, x_{q+1}) \\ &\quad + \cdots + (n-1)^{m-q-2}b^{m-q}A_b(x_m, x_m, \ldots, x_m, x_{m-1}). \end{aligned}$$

Now, using (12) we obtain

$$\begin{aligned}
A_b(x_q, x_q, \ldots, x_q, x_m) &\leq (n-1)b\lambda^q A_b(x_1, x_1, \ldots, x_1, x_0) + b^2\lambda^{q+1}A_b(x_1, x_1, \ldots, x_1, x_0) \\
&\quad +(n-1)b^3\lambda^{q+2}A_b(x_1, x_1, \ldots, x_1, x_0) + \ldots \\
&\quad +(n-1)^{m-q-2}b^{m-q}\lambda^{m-1}A_b(x_1, x_1, \ldots, x_1, x_0) \\
&\leq b\lambda^q[(n-1) + b\lambda + (n-1)b^2\lambda^2 + (n-1)^2b^3\lambda^3 \\
&\quad + \cdots + (n-1)^{m-q-2}b^{m-q-1}\lambda^{m-q-1}]A_b(x_1, x_1, \ldots, x_1, x_0) \\
&\leq (n-1)b\lambda^q[1 + (n-1)b\lambda + (n-1)b^2\lambda^2 + (n-1)^2b^3\lambda^3 \\
&\quad + \cdots + (n-1)^{m-q-3}b^{m-q-1}\lambda^{m-q-1}]A_b(x_1, x_1, \ldots, x_1, x_0) \\
&< (n-1)b\lambda^q[1 + (n-1)b\lambda + \{(n-1)b\lambda\}^2 + \{(n-1)b\lambda\}^3 \\
&\quad + \cdots + \{(n-1)b\lambda\}^{m-q-1}]A_b(x_1, x_1, \ldots, x_1, x_0) \\
&\leq (n-1)b\lambda^q\frac{1-\{(n-1)b\lambda\}^{m-q}}{1-(n-1)b\lambda}A_b(x_1, x_1, \ldots, x_1, x_0) \\
&< (n-1)b\lambda^q\frac{1}{1-(n-1)b\lambda}A_b(x_1, x_1, \ldots, x_1, x_0) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Hence $\lim_{q,m\to\infty} A_b(x_q, x_q, \ldots, x_q, x_m) = 0$.
Thus, $\{x_q\}$ is a Cauchy sequence in X. Since X is a complete partial A_b-metric space, there exists $u \in X$ such that

$$\begin{aligned}
\lim_{q\to\infty} A_b(x_q, x_q, \ldots, x_q, u) &= \lim_{q,m\to\infty} A_b(x_q, x_q, \ldots, x_q, x_m) \\
&= A_b(u, u, \ldots, u) = 0. \qquad (2.13)
\end{aligned}$$

Let us prove that u is a fixed point of T, $\forall q \in \mathbb{N}$, we have

$$\begin{aligned}
A_b(u, u, \ldots, u, Tu) &\leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + A_b(Tu, Tu, \ldots, Tu, x_{q+1}) \\
&\quad -A_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1})] \\
&\leq b[(n-1)A_b(u, u, \ldots, u, x_{q+1}) + A_b(Tu, Tu, \ldots, Tu, Tx_q)].
\end{aligned}$$

Using (11), we obtain $A_b(Tu, Tu, \ldots, Tu, Tx_q) \leq \lambda A_b(u, u, \ldots, u, x_q)$, then

$$\begin{aligned}
A_b(u, u, \ldots, u, Tu) &\leq (n-1)bA_b(u, u, \ldots, u, x_{q+1}) + b\lambda A_b(u, u, \ldots, u, x_q) \\
&= (n-1)bA_b(x_{q+1}, x_{q+1}, \ldots, x_{q+1}, u) + b\lambda A_b(x_q, x_q, \ldots, x_q).
\end{aligned}$$

Using (13) in the above inequality, we obtain $A_b(u, u, \ldots, u, Tu) = 0$, then $Tu = u$. Therefore, u is a fixed point of T and is unique. □

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