International Journal of Mathematics and
Computer Science, **15**(2020), no. 2, 559–575

On $p-$monotone T_2-spaces

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(Received October 12, 2019, Accepted December 27, 2019)

Abstract

In this paper, we generalize monotonically T_2-spaces and strongly monotonically T_2-spaces to bitopological spaces namely pairwise monotonically T_2-spaces (pMT_2-space) and pairwise strongly monotonically T_2-spaces (strongly$-pMT_2-$space), respectively. We study the relations between pMT_2-space, strongly $-pMT_2-$space and $p-M_k-(m_k)-$spaces for $k = 1, 2, 3$.

1 Introduction

In [1996], Buck [6] defined the concept of monotonically T_2-space and strongly monotonically T_2-space, and he studied the relations among monotonically T_2-space and m_k-spaces, $k = 1, 2, 3$. Al-Bsoul [3] and Bin [4] studied these further.

In [10], Kelly introduced the concept of bitopological space $X = (X, \tau_1, \tau_2)$, as a set X with two topologies τ_1, τ_2. Several papers were written on this

Key words and phrases: pMT_2-space, strongly$-pMT_2-$space, $p - M_k-$spaces, $p - m_k-$spaces for $k = 1, 2, 3$.
AMS (MOS) Subject Classifications: 54B10, 54D10, 54E55.
ISSN 1814-0432, 2020, http://ijmcs.future-in-tech.net

topic as generalizations of a single topology (X, τ), ([5], [7], [1], [2].) Let $X = (X, \tau_1, \tau_2)$ be a bitopological space and $A \subseteq X$, $int_{\tau_i}(A)$, $\overline{A}^{\tau_i}$ denote the interior and the closure of A in τ_i, respectively for $i = 1, 2$. When $X = (X, \tau_1, \tau_2)$ has a topological property Q, both τ_1 and τ_2 have this property. In addition, when we write i and j we have in mind that $i \neq j$; $i, j = 1, 2$. For the concepts not defined here, we refer the reader to [10] and [8].

2 Basic Properties

Definition 2.1. *A bitopological space $X = (X, \tau_1, \tau_2)$ is called a pairwise monotonically T_2-space (pMT_2-space), if there is a function:*

$$U(x, y) = \begin{cases} U^i(x, y) : X \times X \longrightarrow \tau_i \\ U^j(x, y) : X \times X \longrightarrow \tau_j \end{cases}$$

such that for each pair (x, y) of distinct points in X and for a τ_i- open set $U^i(x, y)$ and a τ_j- open set $U^j(x, y)$ containing x the following conditions are satisfied :

(i) $U^i(x, y) \cap U^j(y, x) = \phi$ and $U^j(x, y) \cap U^i(y, x) = \phi$,

(ii) For each $M \subseteq X$, if $x \in \overline{\bigcup \{U^i(y, x) | y \in M\}}^{\tau_j}$, then $x \in \overline{M}^{\tau_j}$. In addition, if

(iii) For $z \in U^i(x, y)$, then $U^i(z, y) \subseteq U^i(x, y)$ or if $w \in U^j(x, y)$, then $U^j(w, y) \subseteq U^j(x, y)$,

then $X = (X, \tau_1, \tau_2)$ is pairwise strongly monotonically T_2-spaces (strongly$-pMT_2-$space).

Clearly, if $X = (X, \tau_1, \tau_2)$ is strongly$-pMT_2-$space, then it is a pMT_2-space and hence $p - T_2-$space. In Definition 2.1 if $U^i(x, y) \cap U^i(y, x) = \phi$, then X is τ_i- are monotonically T_2- spaces for $i = 1, 2$.

Theorem 2.2. *A subspace of a pMT_2-space is a pMT_2-space.*

Proof. Let A be a subspace of a pMT_2-space $X = (X, \tau_1, \tau_2)$. Then there exists a function $U(x, y)$ satisfying the conditions of Definition 2.1. Consider the function

$$U_A(x, y) = \begin{cases} U^i_A(x, y) = A \cap U^i(x, y) \\ U^j_A(x, y)) = A \cap U^j(x, y) \end{cases}.$$

Then $U_A(x, y)$ is the function needed and, as a result, A is pMT_2-space. □

The following theorem illustrates that while a $p - T_2-$space is weaker than a pMT_2- space, a pMT_2- space is stronger than a $p - T_3-$ space.

Theorem 2.3. *Every pMT_2-space $X = (X, \tau_1, \tau_2)$ is a $p - T_3-$ space.*

Proof. Let A be a τ_j-closed set and $x \notin A = \overline{A}^{\tau_j}$, for $y \in A$. There is a function

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

such that $U^i(x,y) \cap U^j(y,x) = \phi$, $U^j(x,y) \cap U^i(y,x) = \phi$, and

$$x \notin \overline{\bigcup \{U^i(y,x) | y \in A\}}^{\tau_j}.$$

So there exists a τ_j-open set U_x containing x such that $U_x \cap \big(\bigcup \{U^i(y,x)|y \in A\}\big) = \phi$. Hence $x \in U_x$ and $A \subseteq \bigcup \{U^i(y,x)|y \in A\}$. Therefore, X is a $p - T_3-$ space. □

Corollary 2.4. *Every strongly−pMT_2-space $X = (X, \tau_1, \tau_2)$ is $p - T_3-$ space.*

As an application of Theorem 2.3, consider the following example.

Example 2.5. Let $X = (\mathbb{R}, \tau_{cof}, \tau_u)$, where τ_u denotes the usual topology and τ_{cof} denotes the cofinite topology. Then X is not a $p - T_3-$ space and hence it is not a pMT_2-space. Notice that X is not $p - T_2-$ space.

Theorem 2.6. *Let $X = (X, \tau_1, \tau_2)$ be a first countable bitopological space. Then X is $p - T_3-$ space if and only if X is pMT_2-space.*

Proof. (⇐) Theorem 2.3.
(⇒) Let $x \neq y \in X$, since X is first countable , there exists countable nested local bases $\{V_n(x)\}_{n=1}^{\infty}$ of x, and $\{V_n(y)\}_{n=1}^{\infty}$ of y. Let $\kappa(x)$ be the minimum index such that $x \notin \overline{V_{\kappa(x)}(y)}^{\tau_j}$, where $V_{\kappa(x)}(y)$ is a τ_i-open set, and let $\lambda(x)$ be the minimum index such that $x \notin \overline{V_{\lambda(x)}(y)}^{\tau_i}$, where $V_{\lambda(x)}(y)$ is a τ_j-open set, also let $i(x,y), j(x,y)$ be the smallest integers such that

$$V_{j(x,y)}(x) \cap V_{\kappa(x)}(y) = \phi, \quad \text{and} \quad V_{i(x,y)}(x) \cap V_{\lambda(x)}(y) = \phi,$$

where $V_{i(x,y)}(x) \in \tau_i$ and $V_{j(x,y)}(x) \in \tau_j$. Define

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases},$$

by

$$U^i(x,y) = V_{i(x,y)}(x),$$

and

$$U^j(x,y) = V_{j(x,y)}(x),$$

since $y \notin U^i(x,y) \subseteq V_{\lambda(x)}(y)$ and $y \notin U^j(x,y) \subseteq V_{\kappa(x)}(y)$, so we have $U^i(x,y) \cap U^j(y,x) = \phi$, and $U^j(x,y) \cap U^i(y,x) = \phi$. For the second condition, let $M \subseteq X$ and $x \notin \overline{M}^{\tau_j}$, so $\overline{M}^{\tau_j} \subseteq X - \{x\}$; let j be minimum index such that $V_j(x) \cap M = \phi$, where $V_j(x)$ is a τ_j-open set . Let $y \in M$, so $V_j(x) \subseteq V_{\lambda(y)}(x) \subseteq X - U^i(y,x)$, so $x \in V_j(x) \subseteq X - \bigcup\{U^i(y,x)|y \in M\}$, therefore $x \notin \overline{\bigcup\left\{U^i(y,x)|y \in M\right\}}^{\tau_j}$, hence the result. □

Example 2.7. Let $X = (\mathbb{R}, \tau_u, \tau_s)$,where τ_s is Sorgenfrey line, since X is first countable and $p - T_3-$ space, then X is pMT_2-space.

Definition 2.8 (9). *A cover $\mathcal{U}$ of a bitopological space $X = (X, \tau_1, \tau_2)$ is called $\tau_1\tau_2-$ open cover (base), if $\mathcal{U} \subseteq \tau_1 \cup \tau_2$, and it is called $p-$ open cover (base) ,if it is $\tau_1\tau_2-$ open cover and contains at least one nonempty member of τ_1 and one nonempty member of τ_2.*

Definition 2.9 (1). *A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p-[a,b]-$ compact if every $p-$open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq b$ has a subcover with cardinality $< a$.*

If $X = (X, \tau_1, \tau_2)$ is $p - [a,b]-$ compact for all $b \geq a$, then it is called $p - [a,\infty]-$ compact, where a, b are infinite regular cardinals .

Definition 2.10 (1). *A space (X, τ) is called P_a- space if the intersection of a members of τ is open.*

Definition 2.11 (7). *Let τ_1 and τ_2 be two topologies on X. Then $\tau_1 \cup \tau_2$ forms a subbase for some topology on X; this topology is called the least upper bound (L.U.B) topology and is denoted by $(X, \langle\tau_1, \tau_2\rangle)$. Each basic open set B in $(X, \langle\tau_1, \tau_2\rangle)$ has the form $B = \bigcap_{i=1}^{n} B_i$ where $B_i \in \tau_1$ or $B_i \in \tau_2$ for all $i = 1, 2, \ldots, n$. The intersection of the B_i's which are in τ_1, is in τ_1 and the intersection of the B_i's which are in τ_2, is in τ_2. So $B = U \cap V$ where $U \in \tau_1$ and $V \in \tau_2$.*

Theorem 2.12. *If $X = (X, \tau_1, \tau_2)$ is pMT_2-space with P_a- intersection property, then for disjoint $p - [a,\infty]-$ compact subsets E, F of X, there exists $\langle\tau_1, \tau_2\rangle-$ open subsets U, V such that $E \subseteq U$ and $F \subseteq V$.*

Proof. Let E, F be two $p-[a,\infty]-$ compact subsets of X , and let $e \in E$ and $f \in F$, since X is pMT_2-space, there exists a function

$$U(x,y) = \begin{cases} U^1(x,y) : X \times X \longrightarrow \tau_1 \\ U^2(x,y) : X \times X \longrightarrow \tau_2 \end{cases}$$

such that $U^1(e,f) \cap U^2(f,e) = \phi$, and $U^2(e,f) \cap U^1(f,e) = \phi$. Now

$$\mathcal{U} = \{U^1(e,f)|e \in E\} \cup \{U^2(e,f)|e \in E\}$$

is a $p-$ open cover of E, but E is a $p-[a,\infty]-$ compact subspace,then there exists a subcover $\mathcal{U}^*$ of E with cardinality $< a$, let $U = (\mathcal{U}^* \cap \tau_1) \cup (\mathcal{U}^* \cap \tau_2)$, and $\mathcal{U}^* \cap \tau_1 \subseteq \tau_1 \cup \tau_2 \subseteq \langle \tau_1, \tau_2 \rangle$ and $\mathcal{U}^* \cap \tau_2 \subseteq \tau_1 \cup \tau_2 \subseteq \langle \tau_1, \tau_2 \rangle$, and so U is a $\langle \tau_1, \tau_2 \rangle-$ open subset with $E \subseteq U$, in same technique we can define a $\langle \tau_1, \tau_2 \rangle-$ open subset V with $F \subseteq V$ with $U \cap V = \phi$, hence the result. □

In the end of this section, we discuss the known problem that (if every subspace of X has a property ,then the space X has this property) , in our notation pMT_2-space.

Theorem 2.13. *If every proper subspace of $X = (X, \tau_1, \tau_2)$ is pMT_2-space, then X is pMT_2-space.*

Proof. Let $z_1, z_2, z_3 \in X$ be three distinct points, let $Z_k = X - z_k$,$k = 1, 2, 3$, then Z_k are pMT_2-subspaces of X, so there exists functions

$$U_k(x,y) = \begin{cases} U_k^i(x,y) : X \times X \longrightarrow \tau_i \\ U_k^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

for $k = 1, 2, 3$; which satisfies the conditions of Definition 2.1. Define

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

as

$$U(x,y) = \begin{cases} U_1(x,y), & \text{if } x = z_3, y = z_2 \\ U_2(x,y), & \text{if } x = z_3, y \neq z_2 \\ U_2(x,y), & \text{if } x \neq z_2, y = z_3 \\ U_3(x,y), & \text{if} x \neq z_3 \neq y \end{cases},$$

for $x \neq y \in X$, clearly $U^i(x,y) \cap U^j(y,x) = \phi$, and $U^j(x,y) \cap U^i(y,x) = \phi$. For the second condition let $M \subseteq X$ and $x \notin \overline{M}^{\tau_j}$, so we have the following cases :

(i) For $x = z_3$ and $z_2 \notin M$, then $U(y,x) = U_2(y,z_3)$ for all $y \in M$, and hence $x = z_3 \notin \overline{\bigcup\left\{U^i(y,z_3)|y \in M\right\}}^{\tau_j}$.

(ii) For $x = z_3$ and $y = z_2 \in M$, so

$$\overline{\bigcup\left\{U^i(y,x)|y \in M\right\}}^{\tau_j} = \overline{\bigcup\left\{U_2^i(y,x)|y \in M - \{z_2\}\right\}}^{\tau_j} \cup \overline{U_1^i(z_2,z_3)}^{\tau_j},$$

therefore $z_3 \in \overline{\bigcup\left\{U^i(y,z_3)|y \in M\right\}}^{\tau_j}$.

(iii) For $x \neq z_3$, then if $z_3 \notin M$, $z_2 \in M$ or $z_3 \notin M$ and $z_2 \notin M$, then let $U(y,x) = U_3(y,x)$ for all $y \in M$.

(iv) For $x \neq z_3$ and $z_2, z_3 \in M$, then

$$\overline{\bigcup\left\{U^i(y,x)|y \in M\right\}}^{\tau_j} = \overline{\bigcup\left\{U_3^i(y,x)|y \in M - \{z_3\}\right\}}^{\tau_j} \cup \overline{U_2^i(z_3,x)}^{\tau_j},$$

therefore $x \notin \overline{\bigcup\left\{U^i(y,x)|y \in M\right\}}^{\tau_j}$.

(v) For $z_3 \in M$ and $z_2 \notin M$, then

$$U(y,x) = \begin{cases} U_1(z_3,z_2), & \text{if } x = z_2, y = z_3 \\ U_2(z_3,x), & \text{if } x \neq z_2, y = z_3 \\ U_3(y,z_2), & \text{if } x = z_2, y \neq z_3 \\ U_3(y,x), & \text{if} x \neq z_2, y \neq z_3 \end{cases},$$

hence $x \notin \overline{\bigcup\left\{U^i(y,x)|y \in M\right\}}^{\tau_j}$ and therefore X is pMT_2-space .

□

3 Characterizations of pMT_2-space and Strongly$-pMT_2-$space

In this section , we give characterizations for pMT_2-space and strongly$-pMT_2-$space, but before that we introduce the following definition and two lemmas.

Definition 3.1. *A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p-$Urysohn space is for $x \neq y \in X$ if there exists a τ_i-open set U_x contains x and a τ_j-open set U_y contains y such that $\overline{U_x}^{\tau_j} \cap \overline{U_y}^{\tau_i} = \phi$, and there exists a τ_j-open set V_x contains x and a τ_i-open set V_y contains y such that $\overline{V_x}^{\tau_i} \cap \overline{V_y}^{\tau_j} = \phi$.*

Lemma 3.2 (12). *A bitopological space $X=(X,\tau_1,\tau_2)$ is $p-T_1-$ space if and only if every singleton point is pairwise closed (i.e. closed in τ_i and τ_j).*

Lemma 3.3. *Every $p-T_1-$space ,$p-T_3-$space is $p-$Urysohn space.*

Proof. Let $x \neq y \in X$, by Lemma 3.2 $X-\{x\}$ is a τ_i-open contains x , by [11] there exists a τ_i-open set U_x such that $x \in U_x \subseteq \overline{U_x}^{\tau_j} \subseteq X-\{y\}$, but $X-\overline{U_x}^{\tau_j}$ is a τ_j-open contains x, so there exists a τ_j-open set U_y such that $y \in U_y \subseteq \overline{U_y}^{\tau_i} \subseteq \overline{U_x}^{\tau_j}$, hence $\overline{U_x}^{\tau_j} \cap \overline{U_y}^{\tau_i} = \phi$, in similar way we can find a τ_j-open set V_x contains x and a τ_i-open set V_y contains y such that $\overline{V_x}^{\tau_i} \cap \overline{V_y}^{\tau_j} = \phi$.

□

Theorem 3.4. *A bitopological space $X=(X,\tau_1,\tau_2)$ is pMT_2-space if and only if there exists a function*

$$U(x,y)=\begin{cases} U^i(x,y):X\times X \longrightarrow \tau_i \\ U^j(x,y):X\times X \longrightarrow \tau_j \end{cases},$$

such that for each $x\neq y \in X$, there exists a τ_i- open set $U^i(x,y)$ contains only one point of $\{x,y\}$, such that for $M\subseteq X$, if $x\in \overline{\bigcup\left\{U^i(y,x)|y\in M\right\}}^{\tau_j}$, then $x\in \overline{M}^{\tau_j}$.

Proof. (⇒) Obvious.
(⇐) First, we need to apply Lemma 3.3, X is $p-T_1-$space, assume not, then there exists two distinct points x,y that can not be separated by τ_i-open and τ_j-open sets, with out loss of generally, let a τ_i-open set $U^i(x,y)$ contains x, and $y\in U^i(x,y)\subseteq \overline{U^i(x,y)}^{\tau_j}$, let $M=\{x\}$, since $y\in \overline{U^i(x,y)}^{\tau_j}$, then $y\in \overline{\{x\}}^{\tau_j}$ which is a contradiction , hence X is $p-T_1-$space. Now define

$$V(x,y)=\begin{cases} V^i(x,y):X\times X \longrightarrow \tau_i \\ V^j(x,y):X\times X \longrightarrow \tau_j \end{cases},$$

as

$$V(x,y)=U(x,y) \quad \text{if} \quad x\in U(x,y),$$

and

$$V(x,y)=U(y,x) \quad \text{if} \quad x\notin U(x,y),$$

clearly $V^i(x,y)$ is a τ_i-open set contains x but not y, for $M\subseteq X$, if $x\in \overline{\bigcup\left\{V^i(y,x)|y\in M\right\}}^{\tau_j}$, then $x\in \overline{M}^{\tau_j}$.
Also X is a $p-T_3-$ space, for instance , let A be a τ_j- closed set and

$x \notin A = \overline{A}^{\tau_j}$, then $x \in \overline{\bigcup\left\{V^i(y,x)|y \in A\right\}}^{\tau_j}$, so there exists a τ_j- open set W_x contains x such that $W_x \cap \bigcup\left\{V^i(y,x)|y \in A\right\} = \phi$, hence $A \subseteq \bigcup\left\{V^i(y,x)|y \in A\right\}$ and $x \in W_x$, that means X is $p - T_3-$space, therefore by Lemma 3.3, for $x \neq y$ there exists a τ_i-open set U_x and a τ_j- open set U_y contains x and y ,respectively such that $\overline{U_x}^{\tau_j} \cap \overline{U_y}^{\tau_i} = \phi$, and there exists a τ_j-open set V_x and a τ_i- open set V_y contains x and y ,respectively such that $\overline{U_x}^{\tau_i} \cap \overline{U_y}^{\tau_j} = \phi$. Finally, define

$$W(x,y) = \begin{cases} W^i(x,y) : X \times X \longrightarrow \tau_i \\ W^j(x,y) : X \times X \longrightarrow \tau_j \end{cases},$$

by

$$W^i(x,y) = V^i(x,y) \cap U_x, \quad W^j(y,x) = V^j(y,x) \cap U_y,$$

and

$$W^j(x,y) = V^j(x,y) \cap V_x, \quad W^i(y,x) = V^i(y,x) \cap V_y.$$

Clearly

$$W^i(x,y) \cap W^j(y,x) = \phi,$$

and

$$W^j(x,y) \cap W^i(y,x) = \phi.$$

If $x \in \overline{\bigcup\left\{W^i(y,x)|y \in M\right\}}^{\tau_j}$, then $x \in \overline{\bigcup\left\{V^i(y,x)|y \in M\right\}}^{\tau_j}$, and so $x \in \overline{M}^{\tau_j}$, therefore X is pMT_2-space .

□

We continue in this section by give more characterization of pMT_2-space which is become stronger than Definition 2.1 .

Theorem 3.5. *A bitopological space $X = (X, \tau_1, \tau_2)$ is pMT_2-space, if there is a function:*

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

such that for each pair (x,y) of distinct points in X and for a τ_i- open set $U^i(x,y)$ and a τ_j- open set $U^j(x,y)$ contain x the following conditions are satisfies :

(i) $\overline{U^i(x,y)}^{\tau_j} \cap \overline{U^j(y,x)}^{\tau_i} = \phi$, *and* $\overline{U^j(x,y)}^{\tau_i} \cap \overline{U^i(y,x)}^{\tau_j} = \phi$,

(ii) For each $M \subseteq X$, *if* $x \in \overline{\bigcup\left\{U^i(y,x)|y \in M\right\}}^{\tau_j}$, *then* $x \in \overline{M}^{\tau_j}$.

Proof. (⇐) Trivial.
(⇒) Let X be a pMT_2−space with a function

$$V(x,y) = \begin{cases} V^i(x,y) : X \times X \longrightarrow \tau_i \\ V^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}.$$

Since X is $p-T_3-$ space, then for $x \neq y \in X$, there exists a τ_i- open set $U^i(x,y)$ such that $x \in U^i(x,y) \subseteq \overline{U^i(x,y)}^{\tau_j} \subseteq V^i(x,y)$, similarly for $y \in V^j(y,x)$, there exists a τ_j-open set $U^j(y,x)$ such that $y \in U^j(y,x) \subseteq \overline{U^j(y,x)}^{\tau_i} \subseteq V^j(y,x)$, then

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases},$$

is the needed function, hence the result.

□

The following corollary gives a characterization of strongly−pMT_2−space.

Corollary 3.6. *A bitopological space $X = (X, \tau_1, \tau_2)$ is strongly−pMT_2−space if and only if there exists a function*

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases},$$

such that for each $x \neq y \in X$, there exists a τ_i- open set $U^i(x,y)$ contains only one point of $\{x,y\}$, such that the following conditions satisfies :

(i) For $M \subseteq X$, if $x \in \overline{\bigcup \{U^i(y,x) | y \in M\}}^{\tau_j}$, then $x \in \overline{M}^{\tau_j}$.

(ii) For $z \in U^i(x,y)$, then $U^i(z,y) \subseteq U^i(x,y)$ or if $w \in U^j(x,y)$, then $U^j(w,y) \subseteq U^j(x,y)$.

Proof. From Theorem 3.4. □

Since the concept of pMT_2−space is stronger than $p-T_2$−space . We end this section by a equivalent statement to $p-T_2$−space.

Theorem 3.7. *A bitopological space $X = (X, \tau_1, \tau_2)$ is a $p-T_2$−space, if there is a function:*

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

which assigns to each pair (x,y) of distinct points a τ_i-open set $U^i(x,y)$ contains x but not y such that for $M \subseteq X$, if $x \in \overline{\bigcup \{U^i(y,x) | y \in M\}}^{\tau_j}$, then $x \in \overline{M}^{\tau_j}$.

Proof. ($\Rightarrow$) Let $x \neq y \in X$, since X is $p-T_2-$space, there exists a τ_i-open set U_x contains x and a τ_j- open set U_y contains y such that $U_x \cap U_y = \phi$, now $U_y \subseteq X - U_x$, $\overline{U_y}^{\tau_i} \subseteq \overline{X - U_x}^{\tau_i} = X - U_x$, then $U_x \subseteq X - \overline{U_y}^{\tau_i}$, similarly for $U_y \subseteq X - \overline{U_x}^{\tau_j}$.
Define

$$U^i(x,y) = U_x \cap (X - \overline{U_y}^{\tau_i}),$$

and

$$U^i(y,x) = U_y \cap (X - \overline{U_x}^{\tau_j}),$$

then $U^i(x,y)$ is a τ_i- open set contains x.
Let $x \notin \overline{M}^{\tau_i}$, $\overline{M}^{\tau_i} \subseteq X - \{x\}$, then $\overline{U^j(y,x)}^{\tau_i} = \overline{U_y \cap (X - \overline{U_x}^{\tau_j})}^{\tau_i} \subseteq \overline{U_y}^{\tau_i}$, then $x \notin \overline{\bigcup \{U^j(y,x) | y \in M\}}^{\tau_i}$, as required.

($\Leftarrow$) Let $x \neq y \in X$, by assumption there exists a function

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

with $x \notin \{y\} = \overline{\{y\}}^{\tau_i} \subseteq \overline{U^j(y,x)}^{\tau_i}$. Define

$$U_x = U^i(x,y) \cap (X - \overline{U^j(y,x)}^{\tau_i}),$$

and

$$U_y = U^j(y,x) \cap (X - \overline{U^i(x,y)}^{\tau_j}),$$

then U_x and U_y are the needed τ_i-open, τ_j-open subsets of X , respectively, hence X is $p-T_2-$space . □

4 pMT_2-space and $M_k-(m_k)$-spaces

In this section, we generalize the concept of $p-M_k-$ spaces and $p-m_k-$ spaces $k = 1,2,3$ in bitopological spaces and give some relations among them and pMT_2-space and strongly$-pMT_2-$space.

Definition 4.1. *A pair collection $\mathcal{P} = \{P^i = (P_1^i, P_2^i)\} \cup \{P^j = (P_1^j, P_2^j)\}$ is a pair $p-$ base for $X = (X, \tau_1, \tau_2)$ where $P_1^i \in \tau_i$, $P_1^j \in \tau_j$ and $P_1^i \subseteq P_2^i$, $P_1^j \subseteq P_2^j$, such that for $x \in U$, where U is a τ_i-open (τ_j- open), there exists $P_1^i(P_1^j) \in \mathcal{P}$, such that $x \in P_1^i \subseteq P_2^i \subseteq U$ ($x \in P_1^j \subseteq P_2^j \subseteq U$).*

Definition 4.2. *A pair $p-$base collection $\mathcal{P} = \{P^i = (P^i_1, P^i_2)\} \cup \{P^j = (P^j_1, P^j_2)\}$ in a bitopological space $X = (X, \tau_1, \tau_2)$ is called $p-$ cushioned collection if whenever $\mathcal{P}' \subseteq \mathcal{P}$, then*

$$\overline{\bigcup \{P^i_1 | P^i_1 \in \mathcal{P}'\}}^{\tau_j} \subseteq \bigcup \{P^i_2 | P^i_2 \in \mathcal{P}'\}$$

and

$$\overline{\bigcup \{P^j_1 | P^j_1 \in \mathcal{P}'\}}^{\tau_i} \subseteq \bigcup \{P^j_2 | P^j_2 \in \mathcal{P}'\}.$$

Definition 4.3. *A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p-M_3-$space if X has a $\sigma - p-$ cushioned collection.*

Definition 4.4. *A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p-m_3-$space if each point of X has a $p-$ cushioned local pair $p-$base.*

Theorem 4.5. *If $X = (X, \tau_1, \tau_2)$ is a countable pMT_2- space, then X is $p - M_3-$space.*

Proof. Let X be a countable pMT_2-space, then there exists a function

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

satisfies the conditions of Definition 2.1. Now for $x \in X$, let

$$\mathcal{V}_x = \{V_i(x) | V_i(x) \in \tau_i\} \bigcup \{V_j(x) | Vj(x) \in \tau_j\},$$

be a local $p-$open base for x. Define

$$P_1(V(x)) = \begin{cases} P^i_1(V(x)) = X - \overline{\bigcup \{U^j(y,x) | y \in X - V^i(x)\}}^{\tau_i} \\ P^j_1(V(x)) = X - \overline{\bigcup \{U^i(y,x) | y \in X - V^j(x)\}}^{\tau_j} \end{cases},$$

$$P_2(V(x)) = \begin{cases} P^i_2(V(x)) = V^i(x) \\ P^j_2(V(x)) = V^j(x) \end{cases},$$

and

$$P(V(x)) = \{(P^i_1(V(x)), P^i_2(V(x)))\} \bigcup \{(P^j_1(V(x)), P^j_2(V(x)))\}$$

also define

$$\mathcal{P}_x(V(x)) = \bigcup \{P(V(x)) | V^i(x), V^j(x) \in \mathcal{V}_x\}, \qquad \mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x(V(x)).$$

Let $x \in X$ and $U^i(x)$ is a τ_i-open set, then there exists a $V^i(x) \in \mathcal{V}_x$ with $V^i(x) \subseteq U^i(x)$, similarly for a τ_j-open $U^j(z)$ with $z \in U^j(z)$, so there exists $V^j(z) \in \mathcal{V}_x$ with $V^j(z) \subseteq U^j(z)$, clearly

$$x \in P_1^i(V(x)) \subseteq P_2^i(V(x)) \subseteq U^i(x),$$

and

$$z \in P_1^j(V(x)) \subseteq P_2^j(V(x)) \subseteq U^j(z),$$

hence $\mathcal{P}$ is a pair $p-$ base for X.
For $\mathcal{P'}_x \subseteq \mathcal{P}_x$, assume

$$y \notin \bigcup \{P_2^i(V(x)) | P_2^i(V(x)) \in \mathcal{P}', V^i(x) \in \mathcal{V'}_x \subseteq \mathcal{V}_x\},$$

so

$$y \notin V^i(x) \quad \text{for all} \quad V^i(x) \in \mathcal{V'}_x,$$

and then

$$P_1^i(V(x)) \cap U^j(y,x) = \phi,$$

therefore

$$y \notin \overline{\bigcup \{P_1^i(V(x)) | P_1^i(V(x)) \in \mathcal{P}'\}}^{\tau_j},$$

similarly for

$$w \notin \bigcup \{P_2^j(V(x)) | P_2^j(V(x)) \in \mathcal{P}', V^j(x) \in \mathcal{V'}_x \subseteq \mathcal{V}_x\},$$

hence X is $p - M_3-$space. □

Definition 4.6. *A $p-$base collection $\mathcal{H} = \{H^i | H^i \in \tau_i\} \cup \{H^j | H^j \in \tau_j\}$ of a bitopological space $X = (X, \tau_1, \tau_2)$ is called $p-$ closure preserving if for $\mathcal{H}' \subseteq \mathcal{H}$, we have $\overline{\{\bigcup H^i | H^i \in \tau_i\}}^{\tau_j} = \{\bigcup \overline{H^i}^{\tau_j} | H^i \in \tau_i\}$.*

Definition 4.7. *A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p - M_1-$space if X has $\sigma - p-$closure preserving.*

Definition 4.8. *A bitopological space $X = (X, \tau_1, \tau_2)$ is called $p - m_1-$space if each point of X has a local $p-$closure preserving $p-$ base.*

From the pervious definitions, we give the following obvious theorems.

Theorem 4.9. *Any $p - m_k-$ countable bitopological space $X = (X, \tau_1, \tau_2)$ is $p - M_k-$ spaces for $k = 1, 2, 3$.*

Theorem 4.10. *A first countable bitopological space $X = (X, \tau_1, \tau_2)$ is $p-M_1-$ space.*

Since our concentration is about pMT_2-space , we give the following theorems.

Theorem 4.11. *Every pMT_2-space in a bitopological space $X = (X, \tau_1, \tau_2)$ is $p-m_3-$ space.*

Proof. Since X is a pMT_2-space, there exists a function

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases}$$

satisfies the conditions of Definition 2.1, let $x \in X$ with local $p-$base

$$\mathcal{V} = \{V_i(x)|V_i(x) \in \tau_i\} \cup \{V_j(x)|Vj(x) \in \tau_j\},$$

finally construct $p-$ cushioned local pair $p-$base same as technique in Theorem 4.5. □

The converse of the previous theorem is -in general- not true, but by adding that X is $p-T_2-$ space, we can get the following theorem.

Theorem 4.12. *Let $X = (X, \tau_1, \tau_2)$ be a $p-T_2-$space, $p-m_3-$ space, then X is pMT_2-space.*

Proof. Let $x \in X$ and let $\mathcal{P}_x = \{P_x^i = (P_{1,\alpha}^i, P_{2,\alpha}^i)\} \cup \{P_x^j = (P_{1,\alpha}^j, P_{2,\alpha}^j)\}_{\alpha \in \Delta_x}$ be a $p-$ cushioned pair $p-$base for x. For $x \neq y \in X$, define

$$U(x,y) = \begin{cases} U^i(x,y) = \big(\bigcup\limits_{\alpha \in \Delta_x} \{P_{1,\alpha}^i | y \notin P_{2,\alpha}^i\}\big) - \overline{\big(\bigcup\limits_{\beta \in \Delta_y} \{P_{1,\beta}^j | x \notin P_{2,\beta}^i\}\big)}^{\tau_i} \\ U^j(x,y) = \big(\bigcup\limits_{\alpha \in \Delta_x} \{P_{1,\alpha}^j | y \notin P_{2,\alpha}^j\}\big) - \overline{\big(\bigcup\limits_{\beta \in \Delta_y} \{P_{1,\beta}^i | x \notin P_{2,\beta}^j\}\big)}^{\tau_j} \end{cases},$$

since X is a $p-T_2-$space,then we have

$$x \in \bigcup_{\alpha \in \Delta_x} \{P_{1,\alpha}^i | y \notin P_{2,\alpha}^i\}, \quad \text{and} \quad x \in \bigcup_{\alpha \in \Delta_x} \{P_{1,\alpha}^j | y \notin P_{2,\alpha}^j\},$$

but $\mathcal{P}_y$ is $p-$ cushioned pair $p-$base for y, then

$$x \notin \bigcup_{\beta \in \Delta_y} \{P_{1,\beta}^j | x \notin P_{2,\beta}^i\}, \quad \text{and} \quad x \notin \bigcup_{\beta \in \Delta_y} \{P_{1,\beta}^i | x \notin P_{2,\beta}^j\},$$

hence $U^i(x,y)$, $U^j(x,y)$ is a τ_i-open set ,τ_j-open set of x, respectively.
Also $U^i(x,y)\cap U^j(y,x)=\phi$, assume $z\in U^i(x,y)$ and $z\in U^j(y,x)$, then there exists $\beta\in\Delta_y$ such that

$$z\in\bigcup_{\beta\in\Delta_y}\{P^j_{1,\beta}|x\notin P^i_{2,\beta}\}\subseteq\overline{\bigcup_{\beta\in\Delta_y}\{P^j_{1,\beta}|x\notin P^i_{2,\beta}\}}^{\tau_i},$$

then means $z\notin U^i(x,y)$, therefore $U^i(x,y)\cap U^j(y,x)=\phi$, similarly, we can show that $U^j(x,y)\cap U^i(y,x)=\phi$.
For the second condition, let $x\notin\overline{M}^{\tau_i}$, then for all τ_j-open set U_x contains x we have $U_x\cap M=\phi$, so there exists $\alpha'\in\Delta_x$ such that $\mathcal{P}_x=\{P^i_x=(P^i_{1,\alpha},P^i_{2,\alpha})\}\cup\{P^j_x=(P^j_{1,\alpha},P^j_{2,\alpha})\}$ is a $p-$ cushioned pair $p-$base for x, with $x\in P^j_{1,\alpha'}\subseteq P^j_{2,\alpha'}\subseteq U_x$ and $P^j_{2,\alpha'}\cap M=\phi$. Let $z\in P^j_{1,\alpha'}$ and $y\in M$, so $y\notin P^j_{2,\alpha'}$, and $z\in\overline{\bigcup\{P^j_{1,\alpha'}|y\notin P^j_{2,\alpha'}\}}^{\tau_i}$, and hence $z\notin U^i(y,x)$ for all $y\in M$, moreover

$$P^j_{1,\alpha'}\cap\Big\{\bigcup U^i(y,x)|y\in M\Big\}=\phi,$$

i.e.

$$x\notin\overline{\Big\{\bigcup U^i(y,x)|y\in M\Big\}}^{\tau_j},$$

therefore X is pMT_2-space. □

Corollary 4.13. *Every $p-T_2-$space , $p-m_3-$space in $X=(X,\tau_1,\tau_2)$ is $p-T_3-$space.*

Definition 4.14. *A collection $\mathcal{B}$ is called $p-$quasi base if whenever $x\in U$ and U is a τ_i-open (τ_j- open) there exists $B\in\mathcal{B}$ such that $x\in Int_{\tau_j}(B)\subseteq B\subseteq U$ ($x\in Int_{\tau_i}(B)\subseteq B\subseteq U$).*

Definition 4.15. *A bitopological space $X=(X,\tau_1,\tau_2)$ is called $p-M_2-$space if X has $p-$quasi $p-$closure preserving $p-$base.*

Definition 4.16. *A bitopological space $X=(X,\tau_1,\tau_2)$ is called $p-m_2-$space if each point of X has $p-$quasi $p-$closure preserving local $p-$base.*

Theorem 4.17. *Every $p-T_3-$space ,$p-m_2-$space in $X=(X,\tau_1,\tau_2)$ is $p-m_3-$space.*

Proof. Let $x\in X$, and let $\mathcal{U}_x=\{U_{\alpha,x}\}_{\alpha\in\Delta_x}$ be a $p-$quasi $p-$closure preserving $p-$base for x. Now let

$$\mathcal{P}_x=\{P^i_x=(int_{\tau_i}(U_{\alpha,x}),\overline{U_{\alpha,x}}^{\tau_j})\}\cup\{P^j_x=(int_{\tau_j}U_{\alpha,x},\overline{U_{\alpha,x}}^{\tau_i})\}_{\alpha\in\Delta_x},$$

and let U_x be a τ_i- open contains x , since X is $p-T_3-$ space, there exists a τ_i-open set V_x contains x with $x \in V_x \subseteq \overline{V_x}^{\tau_j} \subseteq U_x$, since $\mathcal{U}_x$ is a $p-$quasi $p-$closure preserving $p-$base for x, there exists $\alpha_1 \in \Delta_x$ such that

$$x \in int_{\tau_j}(U_{\alpha_1,x}) \subseteq U_{\alpha_1,x} \subseteq V_x,$$

therefore

$$x \in int_{\tau_j}(U_{\alpha_1,x}) \subseteq \overline{U_{\alpha_1,x}}^{\tau_j} \subseteq U_x.$$

Finally, $\mathcal{P}_x$ is $p-$ cushioned for x, for instance ,let

$$\mathcal{P'}_x = \{P^{i'}_x = (int_{\tau_i}(U_{\alpha,x}), \overline{U_{\alpha,x}}^{\tau_j})\} \cup \{P^{j'}_x = (int_{\tau_j}(U_{\alpha,x}), \overline{U_{\alpha,x}}^{\tau_i})\} \subseteq \mathcal{P}_x, \quad \text{for } \alpha \in \Delta'_x \subseteq \Delta_x$$

then

$$\overline{\bigcup \{int_{\tau_i}(U_{\alpha,x}) | \alpha \in \Delta'_x \subseteq \Delta_x\}}^{\tau_j} \subseteq \overline{\bigcup \{U_{\alpha,x} | \alpha \in \Delta'_x \subseteq \Delta_x\}}^{\tau_j} = \bigcup \{\overline{U_{\alpha,x}}^{\tau_j} | \alpha \in \Delta'_x \subseteq \Delta_x\},$$

hence the result.

□

We end this paper by the relation between strongly$-pMT_2-$space and $p-m_2-$space .

Theorem 4.18. *If $X = (X, \tau_1, \tau_2)$ is strongly$-pMT_2-$space, then X is $p-m_2-$space.*

Proof. Let $x \in X$ and $\mathcal{V}_x = \{V^i_{\alpha,x} | \alpha \in \Delta, V^i_{\alpha,x} \in \tau_i\} \cup \{V^j_{\alpha,x} | \alpha \in \Delta, V^j_{\alpha,x} \in \tau_j\}$ be a local $p-$base for x, since X is strongly$-pMT_2-$space, there exists a function

$$U(x,y) = \begin{cases} U^i(x,y) : X \times X \longrightarrow \tau_i \\ U^j(x,y) : X \times X \longrightarrow \tau_j \end{cases},$$

satisfies the conditions of Definition 2.1. For $V^j_{\alpha,x} \subseteq \mathcal{V}_x$, define

$$W^* = X - \bigcup \{U^i(y,x) | y \in X - V^j_{\alpha,x}\},$$

clearly W^* is a τ_i-open and $W^* \subseteq V^j_{\alpha,x}$, let $\mathcal{W} = \{W^* | V^j_{\alpha,x} \in \mathcal{V}_x\}$, since X is strongly$-pMT_2-$space and $x \notin X - V^j_{\alpha,x}$, we have

$$x \notin \overline{\bigcup \{U^i(y,x) | y \in X - V^j_{\alpha,x}\}}^{\tau_j},$$

thus

$$X - \overline{\bigcup \{U^i(y,x) | y \in X - V^j_{\alpha,x}\}}^{\tau_j} \subseteq X - \bigcup \{U^i(y,x) | y \in X - V^j_{\alpha,x}\},$$

and hence

$$x \in X - \overline{\bigcup \left\{ U^i(y,x) | y \in X - V^j_{\alpha,x} \right\}}^{\tau_j} \subseteq int_{\tau_j} W^*.$$

Let $x \in U$ and U is a τ_i-open set, there exists $V \in \mathcal{V}_x$ such that $x \in V \subseteq U$, then $x \in int_{\tau_j} W^* \subseteq W^* \subseteq U$, hence $\mathcal{W}$ is $p-$quasi local $p-$base for x, also $\mathcal{W}$ is a $p-$closure preserving, indeed let $\mathcal{W}' \subseteq \mathcal{W}$, clearly

$$\bigcup \left\{ \overline{W^*}^{\tau_i} | V^j_{\alpha,x} \in \mathcal{W}' \right\} \subseteq \overline{\bigcup \left\{ W^* | V^j_{\alpha,x} \in \mathcal{W}' \right\}}^{\tau_i} = \bigcup \left\{ W^* | V^j_{\alpha,x} \in \mathcal{W}' \right\},$$

for the other side, assume $z \notin \bigcup \left\{ \overline{W^*}^{\tau_i} | V^j_{\alpha,x} \in \mathcal{W}' \right\}$, so for each $W^* \in \mathcal{W}'$,

$$z \in \bigcup \left\{ U^i(y,x) | y \in X - V^j_{\alpha,x} \right\},$$

and for each $W^* \subseteq \mathcal{W}'$, there exists $y \in X - V^j_{\alpha,x}$ such that $z \in U^i(y,z)$, but X is strongly$-pMT_2-$space, then

$$U^i(z,x) \subseteq U^i(y,x) \subseteq \{U^i(y,x) | y \in X - V^j_{\alpha,x}\} \quad \text{for all} \quad W^* \in \mathcal{W}',$$

thus $U^i(z,x) \cap W^* = \phi$ for all $W^* \in \mathcal{W}'$, so

$$U^i(z,x) \cup \left(\bigcup \{ W^* | V^j_{\alpha,x} \in \mathcal{W}' \} \right),$$

hence $z \notin \overline{\bigcup \{ W^* | V^j_{\alpha,x} \notin \mathcal{W}' \}}^{\tau_i}$, therefore X is $p - m_2-$space.

□

Acknowledgment

The authors are grateful to the Middle East University, Amman, Jordan for the financial support granted to cover the publication fee of this research article.

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