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Minimal Quasi Injective S-Acts

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Abstract

The core of the paper is to investigate the class of generalization of quasi injective S-acts where we generalized the notion of quasi-injective act to several types. Minimal quasi injective S-act is introduced and studied. More precisely, we study properties and characterizations of S-acts in which all subacts are simple. We highlight a relationship of the concept of minimal quasi injective S-acts with min-annihilator acts, min-symmetric acts. We give a characterization of minimal quasi injective acts in terms of duality. In addition, we investigate conditions under which subacts inherit the minimal quasi injective property. We prove that for strongly Kasch S-act $\mathrm{M_s}$, if $\mathrm{M_s}$ is a minimal quasi injective S-act, then there is a bijection between the class of minimal subacts of ${}_TM$ and the class of maximal right ideals of its endomorphism monoid S. We give some characterizations and properties of the structure of endomorphism monoid of minimal quasi injective acts and a minimal quasi injective monoid and then mention the relationship between them. Finally, we study the relationship between the act of all maximal right ideals of S and the act of minimal subacts of ${}_TM$.

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1 Introduction

The nice structure of the right mininjective rings have led Zhanmin Zhu and Zhisong Tan to extend this notion to modules and because the S-act is a generalization of an R-module. This motivated me to generalize these results to S-act. Unless otherwise stated, we assume that every S-act is a unitary right S-act with zero element Θ which we denote by M_s. We refer the reader to [9] for basic definitions and terminology relating to S-acts over a monoid. It is well-known that an S-act has alternative terminologies like: S-systems, S-sets, S-operands, S-polygons, transition systems, S-automata [4]. A subact N of an S-act M_s, is a non-empty subact of M_s such that xs ∈ N for all x ∈ N and s ∈ S. An S-act M_s is called simple if it contains no subact other than M_s itself and it is called Θ-simple if it contains no subact other than M_s and one element subact Θ_s [4, p.50]. Let g be a function from an S-act A_s into an S-act B_s. Then g will be called an S-homomorphism, if for any $a \in A_s$ and $s \in S$, we have $g(as) = g(a)s$ [1]. An S-congruence ρ on a right S-act M_s is an equivalence relation on M_s such that whenever $(a,b) \in \rho$, $(as, bs) \in \rho$ for all $s \in S$ [3]. The identity S-congruence on M_s will be denoted by I_M such that $(a,b) \in I_M$ if and only if $a = b$ [3]. In 1966, P. Berthiaume introduced the concept of injective S-act. An S-act M_s is said to be injective if for any S-monomorphism h from S-act A_s into B_s and S-homomorphism f from A_s into M_s, there is an S-homomorphism g from B_s into M_s such that gh=f [2]. In [9] the author introduced the notion of M-mininjective and minimal injective S-acts as generalizations of injective acts. An S-act N_s is called M-mininjective, if for every S-homomorphism from a simple M-cyclic sub-act of S-act M_s into N_s can be extended to M_s. A monoid S is a right mininjective if and only S_s is mininjective as S-act. An S-act N_s is called minimal injective (or M-simple injective), if every S-homomorphism from simple subact A of M_s to N_s can be extended to S-homomorphism from M_s to N_s. In [5], A. Lopez introduced the concept of quasi injective as a proper generalization of injective S-act. An S-act M_s is called quasi injective if for any subact B of M_s and any S-homomorphism $\alpha : B \longrightarrow M_s$, there exists S-homomorphism $\sigma : M_s \longrightarrow M_s$ such that σ is an extension of α; that is, $\sigma\, i = \alpha$, where i is the inclusion map of B into M_s. Various generalizations of quasi injectivity have been made. In this paper, we adopt another generalization of quasi injective act which is minimal quasi injective S-acts. We introduce and investigate a new kind of generalization of quasi-injective S-acts, namely minimal quasi-injective acts. Certain classes of subacts which inherit the property of minimal quasi-injective are considered. Characterizations of this new class of S-acts are investigated. Some known results on minimal quasi-injective for general modules are generalized to S-acts. A Min-annihilator S-act used as a link between minimal quasi injective and min-symmetric S-act. A duality that yields an important characterization of minimal quasi injective S-act is explored. We try to shed some light on relationship between the set of all maximal right ideals of S and the set of minimal subacts of ${}_TM$.

2 Main Results (Minimal Quasi injectivity S-acts)

Definition (2.1): A right S-acts M_s is called minimal quasi injective if every homomorphism from simple subact of M_s to M_s can be extended to an endomorphism of M_s. A monoid S is right min-injective if and only S_s is minimal quasi injective.
Each principally quasi-injective acts [8] is minimal quasi-injective.
The following theorem gives characterization of minimal quasi injective acts and it can represent a generalization of lemma (1.5.4) and proposition (1.5.9) in [10].
Theorem (2.2): Let M_s be a right S-act with T=End(M_s). The following conditions are equivalent:
1. M_s is minimal quasi injective;
2. If mS is simple, where m $\in M_s$, then $l_M\left(\gamma_s\left(m\right)\right) = Tm$;
3. If mS is simple and $\gamma_s\left(m\right) \subseteq \gamma_s\left(n\right)$, where m,n $\in M_s$ and n $\neq \Theta$, then $Tn = Tm$;
4. If mS is simple and α : mS $\longrightarrow M_s$ is an S-homomorphism, where m $\in M_s$, then α (m) $\in$ Tm;
5. If mS is simple, where m $\in M_s$, then $l_M\left[\left(aS \times aS\right) \bigcap \gamma_s\left(m\right)\right] = l_M\left(aS \times aS\right) \bigcup Tm$ for each a $\in$ S.
Proof: (1 $\longrightarrow$ 2) Let α m $\in$ Tm, where $\alpha \in$ T. For each s,t $\in$ S with ms = mt, we have α (ms)=α (mt), so $\alpha m \in l_M\left(\gamma_s\left(m\right)\right)$. Thus $Tm \subseteq l_M\left(\gamma_s\left(m\right)\right)$.
Conversely, if $n \in l_M\left(\gamma_s\left(m\right)\right)$, then define σ :mS$\rightarrow M_s$ by σ (ms) = ns, for s $\in$ S. If ms = mt, for s,t $\in$ S, then $(s,t) \in \gamma_s\left(m\right) \subseteq \gamma_s\left(n\right)$. Hence ns = nt. This shows that σ is well-defined. It is an easy matter to see that σ is an S-homomorphism. By (1), σ can extended to $\sigma \in T$. So n = σ (m) = σ (m) $\in$ Tm. Thus $l_M\left(\gamma_s\left(m\right)\right) \subseteq Tm$ and hence $l_M\left(\gamma_s\left(m\right)\right) = Tm$.
(2 $\longrightarrow$ 3) If $\gamma_s\left(m\right) \subseteq \gamma_s\left(n\right)$ and mS is simple with n $\neq \Theta$, then $\gamma_s\left(m\right) = \gamma_s\left(n\right)$ and nS is also simple. Thus, by (2), we have $Tm = l_M\left(\gamma_s\left(m\right)\right) = l_M\left(\gamma_s\left(n\right)\right) = Tn$. Consequently, $Tm = Tn$.

$(3 \longrightarrow 4)$ Let $(s,t) \in \gamma_s(m)$ for s,t $\in$ S. Then ms = mt. Since α is S-homomorphism, α (ms) = α (mt). Hence $(s,t) \in \gamma_s(\alpha(m))$. Thus $\gamma_s(m) \subseteq \gamma_s(\alpha(m))$. By (3), α m $\in$ Tm.
$(4 \longrightarrow 1)$ Take $\alpha : \mathrm{mS} \rightarrow \mathrm{M_s}$ to be the inclusion homomorphism in (4).
$(5 \longrightarrow 2)$ This is obvious.
$(3 \longrightarrow 5)$ Let $\beta \in l_M[\gamma_s(m) \bigcap (aS \times aS)]$. We claim that $\gamma_s(ma) \subseteq \gamma_s(\beta a)$, for each s,t $\in$ S. If $(s,t) \in \gamma_s(ma)$, then mas = mat which implies that (as,at) $\in \gamma_s(m) \bigcap$ (aS $\times$ aS). So β as = β at and hence $(s,t) \in \gamma_s$ (β a). If ma $\neq \Theta$, then maS is simple and, by (3), we have $T\beta a = Tma$. In particular, β a $\in$ Tma, say β a = σ ma for some $\sigma \in$ T. Thus $\beta \in Tm \bigcup l_M(aS \times aS)$. This shows that $l_M(\gamma_s(m) \bigcap (aS \times aS)) \subseteq Tm \bigcup l_M(aS \times aS)$.
Conversely, let $\beta \in Tm \bigcup l_M(aS \times aS)$. Then $\beta \in Tm$, so $\beta = \sigma$ m for some $\sigma \in$ T or $\beta \in l_M(aS \times aS)$. So β (mas) = β (mat) for all s,t $\in$ S and a,m $\in \mathrm{M_s}$. Now, for each (as, at) $\in \gamma_s(m) \bigcap (aS \times aS)$, we obtain mas=mat. If $\beta = \sigma$ m, then σ (mas) = σ (mat) which implies that β (mas) = β (mat). Thus $\beta \in l_M[\gamma_s(m) \bigcap (aS \times aS)]$. If β (mas) = β (mat), then $\beta \in l_M(aS \times aS)$ and hence $\beta \in l_M[\gamma_s(m) \bigcap (aS \times aS)]$. Thus $Tm \bigcup l_M(aS \times aS) \subseteq l_M[\gamma_s(m) \bigcap (aS \times aS)]$.
The following proposition illustrates when the simple subact will be a retract of S-act:
Proposition (2.3): Let A be a simple subact of S-act $\mathrm{M_s}$. If A is minimal quasi injective, then A is a retract of $\mathrm{M_s}$.
Proof: Let A be a simple subact of $\mathrm{M_s}$ and $I_A : A \longrightarrow A$ be the identity map. Since A is minimal quasi injective, there exists S-homomorphism f: $\mathrm{M_s} \longrightarrow$ A such that fi = I_A, where i is the inclusion map of A into $\mathrm{M_s}$. This means that i has left inverse and so A is retract of M.
The following theorem explains that minimal quasi injective act satisfies Min-C_2 condition:
Theorem (2.4): Let $\mathrm{M_s}$ be a minimal quasi injective S-act with T=End($\mathrm{M_s}$). An S-act $\mathrm{M_s}$ is said to satisfy (Min-C_2) condition if N is simple and N $\cong$ H, where H is a retract of $\mathrm{M_s}$, then N is retract of M.
Proof: Let N is simple sub-act of an S-act $\mathrm{M_s}$ and H is a retract of $\mathrm{M_s}$ with N $\cong$ H. As H is a retract of $\mathrm{M_s}$ and $\mathrm{M_s}$ is minimal M-injective, H is minimal M-injective. Thus, N is minimal M-injective and N is a simple sub-act of $\mathrm{M_s}$. Therefore, N is a retract of $\mathrm{M_s}$ (by proposition 2.3).
The next theorem extends theorem 1.14 in [6].
Theorem (2.5): Let $\mathrm{M_s}$ be a minimal quasi injective S-act with T=End($\mathrm{M_s}$), and m,n $\in \mathrm{M_s}$:
1. If mS is simple, then Tm is also simple.

2. If nS is simple and nS ≅ mS, then Tn ≅ Tm.
3. If mS is simple, then $\mathrm{Soc_{mS}(M_s)}$ = TmS is a simple sub-act of ${}_TM_S$ contained in $\mathrm{Soc_{Tm}}$.
4. $\mathrm{Soc(M_s)} \subseteq \mathrm{Soc({}_TM)}$.
Proof: Let $\Theta \neq \alpha$ m ∈ Tm . Then α : mS $\longrightarrow$ α (mS) is an S-isomorphism by hypothesis , so let σ : α (mS) $\longrightarrow$ mS be the inverse. If σ ∈ T extends σ , then σ (α (m)) = σ (α (m)) = m ∈ Tα m.
2 Let f : nS $\longrightarrow$ mS be an S-isomorphism. Put f(n) = ma, where a ∈ S. It is clear that $\gamma_s(n) = \gamma_s(f(n))$ (for this, let (s,t) ∈ $\gamma_s(n)$. Then ns = nt. As f is isomorphism, f(ns)=f(nt) which implies that f(n)s = f(n)t and this means (s,t) ∈ $\gamma_s(f(n))$.
Since f(n)S = mS is simple, by theorem (2.2), we have Tn = Tf(n) = Tma = T(ma) = (Tm) a. Now, define g: Tm $\longrightarrow$ Tn by g(tm) = (tm)a. Then T is a left T-isomorphism.
3 Let $N \subseteq Soc_{mS}(M_s)$, and f : mS $\longrightarrow$ N be an S-isomorphism, where N ⊆ $\mathrm{M_s}$. Then $\gamma_s(m) = \gamma_s(f(m))$. As a result, $Tm = Tf(m)$ by theorem (2.2). Thus $f(m) \in Tm \subseteq TmS$. Hence, if α is an extension of f to T, we have $N = f(mS) = \alpha(mS) \subseteq TmS$. Thus $Soc_{mS}(M_s) \subseteq TmS$. The other inclusion always holds (that is, $TmS \subseteq Soc_{mS}(M_s)$, since for α ∈ TmS, we have α : mS $\longrightarrow$ mS be the identity map and since mS ≅ mS and mS is a sub-act of $\mathrm{M_s}$, $\alpha(mS) = mS \subseteq Soc_{mS}(M_s)$. Then $TmS \subseteq Soc_{mS}(M_s)$). Therefore, $\mathrm{Soc_{mS}}$ ($\mathrm{M_s}$) = TmS. Now, let X = $Soc_{mS}(M_s)$ and $\Theta \neq_T A_S \subseteq {}_TX_S$. If B is a simple sub-act of A_S, then B ≅ mS. Thus, if C is any sub-act of $\mathrm{M_s}$ isomorphic to mS, let σ : B $\longrightarrow$ C be an S-isomorphism. Then σ extends to an endomorphism σ of $\mathrm{M_s}$. So C = $\sigma(B) = \sigma(B)$ ⊆ A. This means that X ⊆ A. Therefore, X is a simple sub-act of ${}_TM_S$. For any s ∈ S, define g_s : Tm $\longrightarrow$ ${}_TM$ by g_s (tm) = tms. Then g_s is a left T-homomorphism. So Tms ⊆ $Soc_{Tm}({}_TM)$ and thus TmS ⊆ $Soc_{Tm}({}_TM)$.
Definition (2.6): A monoid S is said to be V-monoid if every right S-act is injective.
Theorem (2.7): A monoid S is a right V-monoid if and only if every S-act is minimal quasi injective.
Proof: For the sufficiency, let $\mathrm{M_s}$ be any simple right S-act. Let E (M) be the injective envelope of $\mathrm{M_s}$. Then $\mathrm{M_s}$ is minimal quasi injective in $\mathrm{M_s} \cup$ E(M). Let $\mathrm{i_1}$: $\mathrm{M_s} \longrightarrow$ E(M) be the inclusion map and $\mathrm{j_1}$: E(M) $\longrightarrow$ $\mathrm{M_s} \cup$ E(M) the injection maps. Since $\mathrm{M_s}$ is minimal- $\mathrm{M_s} \cup$ E(M)-injective, the identity map $\mathrm{I_M}$ of $\mathrm{M_s}$ extends to S-homomorphism f : $\mathrm{M_s} \cup$ E(M) $\longrightarrow$ $\mathrm{M_s}$ such that f $\mathrm{j_1}$ $\mathrm{i_1}$ = $\mathrm{I_M}$. Then, put h (=f $\mathrm{j_1}$): E(M) $\longrightarrow$ $\mathrm{M_s}$. So h $\mathrm{i_1}$ = $\mathrm{I_M}$ and $\mathrm{M_s}$ is a retract of E(M). Therefore, $\mathrm{M_s}$ is injective. The following theorem gives a

characterization of minimal quasi injective acts in terms of duality:

Theorem(2.8): The following conditions are equivalent for an S-act M_s with T=End(M_s):

1. M_s is minimal quasi injective;

2. hom(N_s, ${}_TM_s$) is a simple left T-act for all simple right S-act N ;

3. $l_M(A \times A)$ is simple left T-act for all maximal right ideals A of S.

Proof: ($1 \longrightarrow 2$) Let α, $\beta \in$ hom(N_s, ${}_TM_s$), where N_s is simple, and assume that $\alpha \neq \Theta$. Then $\beta\alpha^{-1} : \alpha(N) \longrightarrow M_s$ is homomorphism. Since $\alpha(N)$ is simple. So $\beta\alpha^{-1}$ can be extended to an endomorphism σ of M_s by (1). Thus $\beta = \sigma\alpha$.

($2 \longrightarrow 3$) Let N_s = nS(n $\in$ N) be cyclic S-act and then take A = $\gamma_s(n \times n)$. Thus $l_M(A \times A) \cong$ hom(N_s, ${}_TM_s$) [9, lemma(2.11)] which implies that $l_M(A \times A)$ is simple by (2).

($3 \longrightarrow 1$) Let α : mS $\longrightarrow M_s$ be an S-homomorphism, where mS is simple and let i:mS $\longrightarrow M_s$ be the inclusion map. Put A = $\gamma_s(m \times m)$. Then A is maximal right ideal of S, so $l_M(A \times A) \cong$ hom(mS, M_s) [9, lemma(2.11)]. Hence hom(mS, M_s) is simple. Hence $\alpha = \beta i$ for some $\beta \in$ T. In [7], Nicholson defined a Kasch module M as every simple subquotient of M embeds in M. However, we just need here a weaker form which is called strongly Kasch and we define it as follows:

Definition (2.9): An S-act M_s is called strongly Kasch if $l_M(A \times A) \neq \Theta$ for any maximal right ideal A of S.

Lemma (2.10): Let M_s be a right S-act. Then the following are equivalent:

1. M_s is strongly Kasch;

2. $\gamma_s l_M(A \times A)$ = A × A, for any maximal right ideal A of S.

Proof: ($1 \longrightarrow 2$) Let A be maximal ideal of S. As M_s is strongly Kasch. So $l_M(A \times A) \neq \Theta$. So for $ma_1 = ma_2$ where (a_1,a_2) $\in A \times A$. Then, ma_1s = ma_2s for each s $\in$ S. Thus (a_1,a_2) $\in \gamma_s l_M(A \times A)$ and then A $\times A \subseteq \gamma_s l_M(A \times A)$. Now, since $\gamma_s l_M(A \times A) \neq S \times S$. Thus $\gamma_s l_M(A \times A)$ = A $\times A$ by maximality of A.

($2 \longrightarrow 1$) Let A be maximal right ideal of S and m $\in l_M(A \times A)$. Then, ma_1 = ma_2, where (a_1,a_2) $\neq (\Theta, \Theta) \in A \times A$ Since A $\times A$ is a right ideal of S $\times S$, for (s,t)($\neq (\Theta, \Theta)$) $\in$ S × S, we have ma_1s = ma_2t. Thus (ma_1s , ma_2t) $\in \gamma_s l_M(A \times A)$ and since $\gamma_s l_M(A \times A)$ = A × A, by (2), so (ma_1s ,ma_2t) $\in$ A × A. As A is maximal right ideal, thus (ma_1s ,ma_2t) $\neq$ (Θ, Θ). Hence $l_M(A \times A) \neq \Theta$. The following theorem gives major properties for minimal quasi-injective in terms of strongly Kasch acts. Also, it is a generalization of theorem (2.3) in [11]:

Theorem (2.11): Let M_s be a minimal quasi injective strongly Kasch

S-act with T=End(M_s). Then the maps $\alpha\,(=\beta^{-1}) :$ K $\longrightarrow \gamma_s\,(K)$ and $\beta :$ A $\times$ A $\longrightarrow l_M\,(A \times A)$ are mutually inverse bijections between the set of all minimal subacts K of ${}_TM$ and the set of all maximal right ideals A $\times$ A of S $\times$ S. In particular, $l_M\gamma_s\,(K) = K$ for all minimal subacts K of ${}_TM$.

Proof: To prove β is one-to-one, let X,Y be two maximal right ideals of S and $\beta\,(X \times X) = \beta\,(Y \times Y)$. Then, $l_M\,(X \times X) = l_M\,(Y \times Y)$ (since $\beta\,(X \times X) \subseteq l_M\,(X \times X)$ and $\beta\,(Y \times Y) \subseteq l_M\,(Y \times Y)$) which implies that $\gamma_s l_M\,(X \times X) = \gamma_s l_M\,(Y \times Y)$. Hence, $X \times X = Y \times Y$ by hypothesis and then β is one-to-one. Thus the proof will be complete when we establish the following claims:

Claim (1): $\gamma_s\,(K)$ is a maximal right ideal of S $\times$ S for all minimal subacts K of ${}_TM$.

Proof: Let A be maximal right ideal of S and $\gamma_s\,(K) \subseteq$ A $\times$ A. Then $l_M\,(A \times A) \neq \Theta$ (since M_s is strongly Kasch). Thus, $l_M\,(A \times A) \subseteq l_M\gamma_s\,(K)$ = K by hypothesis and so $l_M\,(A \times A)$ =K by minimality of K. Therefore $\gamma_s l_M\,(A \times A) = \gamma_s\,(K)$ and since $\gamma_s l_M\,(A \times A)$ =A $\times$ A by lemma (2.9), so $\gamma_s\,(K) =$ A $\times$ A.

Claim (2): $l_M\,(A \times A)$ is a minimal subacts of ${}_TM$ for all maximal ideals A of S.

Proof: Let A be maximal ideal of S. As M_s is strongly Kasch S-act, so $l_M\,(A \times A) \neq \Theta$. Thus, there exists m $\in l_M\,(A \times A)$ which implies that A $\times A = \gamma_s\,(m)$ and hence $l_M\,(A \times A) = l_M\gamma_s\,(m) =$ Tm by theorem (2.2) By theorem(2.4) Tm is a minimal subact of ${}_TM$. It follows that $l_M\,(A \times A)$ is minimal.

Definition (2.12): An S-act M_s with T=End(M_s) is called a minannihilator act if, for every minimal subact B of ${}_TM$, there exists a subact A of S such that B = $l_M\,(A \times A)$, equivalently, if $l_M\gamma_s\,(B) =$ B.

Definition (2.13): An S-act M_s with T=End(M_s) is called minsymmetric if mS is simple, where m $\in M_s$ implies that Tm is also simple. The following theorem gives properties of minannihilator S-act in terms of minimal quasi injective:

Theorem (2.14): Let M_s be a minannihilator act. Then the following are equivalent:

1. M_s is minimal quasi injective;
2. M_s is minsymmetric;
3. soc(M_s) $\subseteq$ soc(${}_T$M).

Proof: (1 $\longrightarrow$ 2) follows from theorem (2.4). (2 $\longrightarrow$ 3) Obvious. (3 $\longrightarrow$ 1) Let mS be simple subact. Then by theorem (2.4) Tm is simple. So m $\in$ soc(${}_T$M) by (3). Thus Tm contains a simple subact Tn and hence $\gamma_s\,(m)$

$\subseteq \gamma_s(n)$ and so $\gamma_s(m) = \gamma_s(n)$ because $\gamma_s(m)$ is maximal. Since M_s is a minannihilator act and Tn is simple, Tm $\subseteq l_M\gamma_s(Tm) = l_M\gamma_s(Tn) =$ Tn. This implies that Tm $= l_M\gamma_s(Tm) = l_M\gamma_s(m)$.

Corollary (2.15): If M_s is a minannihilator act such that soc($_T$M) is $\bigcap$-large in $_T$M, where T=End(M_s), then M_s is minimal quasi injective.

The following theorem gives a characterization of minsymmetric acts:

Theorem (2.16): The following are equivalent for a act M_s:

1. M_s is minsymmetric ;

2. If mS is simple, then $l_T((mS \times mS) \bigcap ker\alpha) = l_T(mS \times mS) \bigcup$ T α for all $\alpha \in$ T, where T=End(M_s).

Proof: ($1 \longrightarrow 2$) Assume that mS is simple and let $\alpha \in$ T. Let $\beta \in$ T α $\bigcup l_T$ (mS $\times$ mS). Then $\beta = \sigma\alpha$ for some $\sigma \in$ T or β (ms) = β (mt) for all s, t $\in$ S and m $\in M_s$. For each (ms,mt) $\in$ ker $\alpha \bigcap$ (mS $\times$ mS), if $\beta = \sigma\alpha$, then α (ms) = α (mt) and hence $\sigma\alpha$ (ms) = $\sigma\alpha$ (mt), so β (ms) = β (mt). Thus $\beta \in l_T$ $(mS \times mS) \bigcap ker\alpha$. If β (ms) = β (mt), then $\beta \in l_T$ (mS $\times$ mS) and hence $\beta \in l_T$ $(mS \times mS) \bigcap ker\alpha$. Thus T $\alpha \bigcup l_T$ (mS $\times$ mS) $\subseteq l_T$ (ker $\alpha \bigcap$ (mS $\times$ mS)). If $\alpha = I_M$, then ker $\alpha \bigcap$ (mS $\times$ mS) = (mS $\times$ mS) and so l_T (ker $\alpha \bigcap$ (mS $\times$ mS)) = l_T (mS $\times$ mS) $\subseteq$ T $\alpha \bigcup l_T$ (mS $\times$ mS). If $\alpha \neq I_M$, then ker $\alpha \bigcap$ (mS $\times$ mS) = I_{mS} and so
l_T (ker $\alpha \bigcap$ (mS $\times$ mS)) = l_T (I_{mS})= T = T $\alpha \bigcup l_T$ (mS$\times$ mS) since Tm is simple.

($2 \longrightarrow 1$) Let mS be simple. If $\alpha \in l_T$ (mS $\times$ mS), then α (ms) = α (mt). Thus ker $\alpha \bigcap$ (mS $\times$ mS) = I_{mS} and then T = l_T (I_{mS}) = l_T (ker $\alpha \bigcap$ (mS $\times$mS)) = T $\alpha \bigcup l_T$ (mS $\times$ mS) by (2). This means that T $\alpha \bigcup l_T$ (mS $\times$ mS) = T and l_T (mS $\times$ mS) is maximal.

3 Conclusions

In this paper, our investigation was motivated by [8]. Introducing and studying the topic of this article contributes to the improvement of the vision for finding the correspondence between acts theory and module theory. Besides, the importance of this topic comes from some essential points which we highlight as follow:

1. We found properties and characterizations of S-acts in which all subacts are minimal and conditions on which subact inherit the property of minimal quasi-injectivity.

2. As an of the applications of this topic, we exhibited that if S-act is minimal quasi injective, then there are some interesting results, one of which were properties of minannihilator S-acts which were given in theorem (2.14). The other characterization of minsymmetric S-act was explained in theorem (2.16).

3. In theorem (2.2) we found a characterization of minimal quasi injective S-act and, as such, this represents a generalization of lemma (2.4) and proposition (2.9).

4. Proposition (2.3) gives an answer to the question of when simple subact will be retract

5. Theorem (2.4) illustrates that minimal quasi injective act satisfies the Min-C_2 condition.

6. The relationship between a simple subact of minimal quasi injective act and simple in the monoid T=End(M_s) was demonstrated in theorem (2.5).

7. Among some interesting results is the characterization of V-monoid in terms of minimal quasi injective S-acts in thereom (2.7).

8. Characterization of minimal quasi injective S-act in terms of duality was clarified in theorem (2.8).

9. Another important result is the 1-1 correspondence between the set of minimal subacts of ${}_TM$ and maximal right ideals of S $\times$ S which was illustrated in theorem (2.11).

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