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Ideals, Centralizers and Symmetric Bi-derivations on prime rings

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Abstract

The purpose of the present paper is to prove some results concerning symmetric biderivations on an ideal of a ring, which are of independent interest. More precisely, we extend a result of Bresar [7, Theorem 2.1].

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1 Introduction

G. Maksa [3] introduced the concept of a symmetric biderivation. It was shown in [3] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivation in prime and semiprime rings can be found in [1, 2, 3, 6, 9]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f : R \longrightarrow R$ gives rise to a biderivation on R. Namely linearizing $[x, f(x)] = 0$ for all $x, y \in R$ $(x, y) \mapsto [f(x), y]$ is a biderivation (moreover, all derivations appearing are inner). An additive mapping $d : R \longrightarrow R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $d_a(x) = [a, x]$, for all $x \in R$. A mapping $D : R \times R \longrightarrow R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f : R \longrightarrow R$ defined by $f(x) = D(x, x)$, where $D : R \times R \longrightarrow R$ is a symmetric mapping, is called the trace of D. It is obvious that in the case $D : R \times R \longrightarrow R$ is a symmetric mapping which is also biadditive (i.e. additive in both arguments), the trace f of D satisfies the relation $f(x+y) = f(x)+f(y)+2D(x, y)$, for all $x, y \in R$. A biadditive mapping $D : R \times R \longrightarrow R$ is called a biderivation if for every $x \in R$, the map $y \mapsto D(x, y)$ as well as for every $y \in R$, the map $x \mapsto D(x, y)$ is a derivation of R, i.e., $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$ and $D(x, yz) = D(x, y)z + yD(x, z)$ for all $x, y, z \in R$.

For any semiprime (prime) ring R one can construct the Martindale ring of quotients Q of R (see [10]). As R can be embedded isomorphically in Q, we consider R as a subring of Q. If the element $q \in Q$ commutes with every element in R, then $q \in C$, the centre of Q. C contains the centroid of R and it is called the extended centroid of R. For more details on Martindale ring of quotients, one can see [5].

In [4] Herstein determined the structure of a prime ring R admitting a nonzero derivation d such that the values of d commute, that is for which $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$. Perhaps even more natural might be the question of what can be said on a derivaion when elements in a prime ring commute with all the values of a nonzero derivation. Herstein [4] addressed this question by proving the following result: If d is a nonzero derivation of a prime ring R and $a \notin Z(R)$ is such that $[a, d(x)] = 0$ for all $x \in R$, then R has a characteristic 2, $a^2 \in Z(R)$ and $d(x) = [\lambda a, x]$, for all $x \in R$ and $\lambda \in C$, the extended centroid of R.

In [7], Bresar generalized above result of Herstein and gave a description of derivations d, g and h of a prime ring satisfying $d(x) = ag(x) + h(x)b$, $x \in R$, where a, b are some fixed elements in R. Then d, g, h has the following forms $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in R$.

Inspired by all these observations, our aim is to generalize above results for the case of biderivations on two sided ideals of prime rings.

Throughout the paper R denotes a prime ring of characteristic different from 2 and Q stands for the Martindale ring of quotients. Also we make extensive use of the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Moreover, we shall require the following lemma.

Lemma 1.1 [9] Let R be a prime ring and M be any set. Suppose that the map $f, g : M \longrightarrow Q$ satisfy $f(x)tg(z) = g(x)tf(z)$ for all $x, z \in M$ and for all $t \in I$, a nonzero ideal of R. If $f \neq 0$, then there exists a $\lambda \in C$ such that $g(x) = \lambda f(x)$ for all $x \in M$.

2 Main Results

Theorem 2.1. *Let I be a nonzero ideal of R and $D, G, H : I \times I \longrightarrow Q$ be nonzero biderivations of R with trace d, g, h respectively. Suppose there exists $a, b \in R$ such that $D(x, x) = aG(x, x) + H(x, x)b$ for all $x \in I$. If $a, b \notin Z(R)$, then there exists $\lambda \in C$ such that $d(x) = [\lambda ab, x]$, $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$ for all $x \in I$.*

To prove our main result we need the following lemmas:

Lemma 2.1. *Let I be a nonzero ideal of R and $D, G : I \times I \longrightarrow Q$ be biderivations of R with trace d, g respectively. Suppose that $D(x, x)G(y, y) = G(x, x)D(y, y)$ for all $x, y \in I$. If $D \neq 0$, then there exists $\lambda \in C$ such that $G(x, x) = \lambda D(x, x)$ for all $x \in I$.*

Proof. Suppose that $D \neq 0$ and $D(x, x)G(y, y) = G(x, x)D(y, y)$ for all $x, y \in I$, i.e.

$$d(x)g(y) = g(x)d(y) \quad \text{for all } x, y \in I. \tag{2.1}$$

Linearization of (2.1) in x yields that

$$2D(x,z)g(y) = 2G(x,z)d(y) \text{ for all } x,y,z \in I. \tag{2.2}$$

Replacing x by zx in (2.2) and using char$R \neq 2$, we obtain

$$zD(x,z)g(y)+D(z,z)xg(y) = zG(x,z)d(y)+G(z,z)xd(y) \text{ for all } x,y,z \in I. \tag{2.3}$$

Comparing (2.2) and (2.3), we get

$$D(z,z)xg(y) = G(z,z)xd(y) \text{ for all } x,y,z \in I. \tag{2.4}$$

This implies that

$$D(z,z)xG(y,y) = G(z,z)xD(y,y) \text{ for all } x,y,z \in I. \tag{2.5}$$

Application of Lemma 1.1 yields that there exists $\lambda \in C$ such that $G(x,x) = \lambda D(x,x)$ for all $x \in I$.

□

Lemma 2.2. *Let I be a nonzero ideal of R and $D,G,H,F : I \times I \longrightarrow Q$ be biderivations of R with trace d,g,h,f respectively. Suppose that $D(x,x)G(y,y) = H(x,x)F(y,y)$ for all $x,y \in I$. If $F,D \neq 0$, then there exists $\lambda \in C$ such that $H(x,x) = \lambda D(x,x)$ and $G(x,x) = \lambda F(x,x)$for all $x \in I$.*

Proof. Suppose that $F \neq 0$ and $D(x,x)G(y,y) = H(x,x)F(y,y)$ for all $x,y \in I$, i.e.

$$d(x)g(y) = h(x)f(y) \text{ for all } x,y \in I. \tag{2.6}$$

Linearization of (2.6) in x yields that

$$2D(x,z)g(y) = 2H(x,z)f(y) \text{ for all } x,y,z \in I. \tag{2.7}$$

Replacing y by $y+w$ in (2.7) and using char$R \neq 2$, we obtain

$$D(x,z)G(y,w) = H(x,z)F(y,w) \text{ for all } w,x,y,z \in I. \tag{2.8}$$

Substitute xz for x in (2.8) and use (2.8) to get

$$D(x,x)zG(y,w) = H(x,x)zF(y,w) \text{ for all } w,x,y,z \in I. \tag{2.9}$$

Replacing z by $zD(u,v)$ in (2.9), we have

$$D(x,x)zD(u,v)G(y,w) = H(x,x)zD(u,v)F(y,w) \text{ for all } w,x,y,z,u,v \in I. \tag{2.10}$$

In view of (2.8), (2.10) reduces to

$$D(x,x)zH(u,v)F(y,w) = H(x,x)zD(u,v)F(y,w) \text{ for all } w,x,y,z,u,v \in I. \tag{2.11}$$

This implies that

$$\{D(x,x)zH(u,v) - H(x,x)zD(u,v)\}F(y,w) = 0 \text{ for all } w,x,y,z,u,v \in I. \tag{2.12}$$

Replacing y by ty in (2.12) and use (2.12), we get

$$\{D(x,x)zH(u,v) - H(x,x)zD(u,v)\}tF(y,w) = 0 \text{ for all } t,w,x,y,z,u,v \in I. \tag{2.13}$$

Primeness of R yields that either $F(y,w) = 0$ or

$$D(x,x)zH(u,v) - H(x,x)zD(u,v) = 0,$$

for all $x,y,z,u,v,t,w \in I$. Since $F \neq 0$ we have $D(x,x)zH(u,v) = H(x,x)zD(u,v)$ for all $x,z,u,v \in I$. Application of Lemma 2.1, we find that there exists $\lambda \in C$ such that $H(x,x) = \lambda D(x,x)$ for all $x \in I$.

Using last relation (2.9) becomes

$$D(x,x)zG(y,w) = \lambda D(x,x)zF(y,w) \text{ for all } w,x,y,z \in I. \tag{2.14}$$

This yields that $D(x,x)z\{G(y,w) - \lambda F(y,w) = 0\}$ for all $w,x,y,z \in I$. Since R is prime we have either $D(x,x) = 0$ or $G(y,w) - \lambda F(y,w) = 0$ for all $w,x,y,z \in I$. But $D \neq 0$, hence we obtain $G(y,w) - \lambda F(y,w) = 0$, i.e., $G(y,w) = \lambda F(y,w)$ for all $w,y \in I$.

□

Proof of Theorem 2.1 Suppose that

$$D(x,x) = aG(x,x) + H(x,x)b \text{ for all } x \in I. \tag{2.15}$$

Linearization yields that $2D(x,y) = 2aG(x,y) + 2H(x,y)b$ for all $x, y \in I$. Since Char$R \neq 2$ we have

$$D(x,y) = aG(x,y) + H(x,y)b \text{ for all } x, y \in I. \tag{2.16}$$

Replacing x by xz in (2.17), we find

$$xD(z,y)+D(x,y)z = axG(z,y)+aG(x,y)z+xH(z,y)b+H(x,y)zb \text{ for all } x, y, z \in I. \tag{2.17}$$

Comparing (2.17) and (2.16), we obtain

$$[x,a]G(z,y) = H(x,y)[z,b] \text{ for all } x, y, z \in I. \tag{2.18}$$

Replacing x by xu in (2.18), we have

$$[x,a]uG(z,y) = H(x,y)u[z,b] \text{ for all } u, x, y, z \in I. \tag{2.19}$$

Conclusion follows from Lemma 2.2. Hence we have $G(z,y) = \lambda[z,b]$ and $H(x,y) = \lambda[x,a]$ for all $x, y, z \in I$. In particular, we obtain $g(x) = \lambda[x,b]$ and $h(x) = \lambda[x,a]$ for all $x \in I$.

Now substituting the values of $g(x)$ and $h(x)$ in (2.15), we get $d(x) = a\lambda[x,b] + \lambda[x,a]b$ for all $x \in R$. Hence we get $d(x) = \lambda[x,ab]$ for all $x \in R$.

3 Characterization of centralizers

In this section we investigate the some algebraic identities of bi-derivations and additive centralizers to find the characterization of centralizer. Infact, we obtained the form of centralizer if it satisfies some condition on prime ring. In our investigation following lemma play pivotal role:

Lemma 3.1. *[8] All additive mappings which are centralizing on prime rings R of characteristic not 2; it is shown, that every such mapping f is of the form $f(x) = \eta x + \delta(x)$, where η is an element from the extended centroid of R and δ is an additive mapping from R into the extended centroid of R.*

Theorem 3.1. *Let R be a prime ring of characteristic not two and D_1, D_2 be two symmetric bi-derivation on R. If ζ is an additive left centralizer such that $D_1(x,x)\zeta(x) + D_2(x,x)\zeta(x) = 0$, for all $x \in R$, then one of the following condition hold:*

(1) $D_1 = -D_2$

(2) ζ *is centralizing on* R*. Moreover there exist* η *is an element from the extended centroid of* R *and* δ *is an additive mapping from* R *into the extended centroid of* R *such that* $\zeta(x) = \eta x + \delta(x)$*, for all* $x \in R$*.*

Proof. Lets begin with the given identity in the hypothesis

$$D_1(x,x)\zeta(x) + D_2(x,x)\zeta(x) = 0, \text{ for all } x \in R. \tag{3.1}$$

On linearizing above equation we can get for all $x, y \in R$,

$$(D_1(x,x) + D_1(y,y) + 2D_1(x,y))(\zeta(x) + \zeta(y))$$

$$+(D_2(x,x) + D_2(y,y) + 2D_2(x,y))(\zeta(x) + \zeta(y)) = 0.$$

After simplification we arrive at
$D_1(x,x)\zeta(x) + D_1(y,y)\zeta(x) + 2D_1(x,y)\zeta(x) + D_1(x,x)\zeta(y) + D_1(y,y)\zeta(y) + 2D_1(x,y)\zeta(y) + D_2(x,x)\zeta(x) + D_2(y,y)\zeta(x) + 2D_2(x,y)\zeta(x) + D_2(x,x)\zeta(y) + D_2(y,y)\zeta(y) + 2D_2(x,y)\zeta(y) = 0$ for all $x, y \in R$.
An application of (3.1) in the above relation yields that

$$\begin{aligned} &D_1(y,y)\zeta(x) + 2D_1(x,y)\zeta(x) + D_1(x,x)\zeta(y) \\ &+2D_1(x,y)\zeta(y) + D_2(y,y)\zeta(x) + 2D_2(x,y)\zeta(x) \\ &+D_2(x,x)\zeta(y) + 2D_2(x,y)\zeta(y) = 0 \text{ for all } x, y \in R. \end{aligned} \tag{3.2}$$

Substituting $-y$ for y in (3.2) to obtain

$$\begin{aligned} &D_1(y,y)\zeta(x) - 2D_1(x,y)\zeta(x) - D_1(x,x)\zeta(y) \\ &+2D_1(x,y)\zeta(y) + D_2(y,y)\zeta(x) - 2D_2(x,y)\zeta(x) \\ &-D_2(x,x)\zeta(y) + 2D_2(x,y)\zeta(y) = 0 \text{ for all } x, y \in R. \end{aligned} \tag{3.3}$$

Adding (3.2) and (3.3), using characteristic condition we have

$$\begin{aligned} &D_1(y,y)\zeta(x) + 2D_1(x,y)\zeta(y) + D_2(y,y)\zeta(x) \\ &+2D_2(x,y)\zeta(y) = 0 \text{ for all } x, y \in R. \end{aligned} \tag{3.4}$$

This implies that

$$D_1(y,y)\zeta(x) + D_2(y,y)\zeta(x) = -2D_1(x,y)\zeta(y) - 2D_2(x,y)\zeta(y) \text{ for all } x, y \in R. \tag{3.5}$$

Replace x by xz in (3.4) to find

$$D_1(y,y)\zeta(x)z + 2D_1(x,y)z\zeta(y) + 2xD_1(z,y)\zeta(y) + D_2(y,y)\zeta(x)z \\ +2D_2(x,y)z\zeta(y) + 2xD_2(z,y)\zeta(y) = 0 \text{ for all } x,y,z \in R. \tag{3.6}$$

In view of (3.5), (3.6) takes the form

$$-2D_1(x,y)\zeta(y)z + 2D_1(x,y)z\zeta(y) + 2xD_1(z,y)\zeta(y) - 2D_2(x,y)\zeta(y)z \\ +2D_2(x,y)z\zeta(y) + 2xD_2(z,y)\zeta(y) = 0 \text{ for all } x,y,z \in R. \tag{3.7}$$

After simple manipulation and using characteristic restriction above equation can be reduces to the form

$$D_1(x,y)[z,\zeta(y)] + xD_1(z,y)\zeta(y) + D_2(x,y)[z,\zeta(y)] \\ +xD_2(z,y)\zeta(y) = 0 \text{ for all } x,y,z \in R. \tag{3.8}$$

Substitute rx for x in (3.8) to find

$$rD_1(x,y)[z,\zeta(y)] + D_1(r,y)x[z,\zeta(y)] + rxD_1(z,y)\zeta(y) + rD_2(x,y)[z,\zeta(y)] \\ +D_2(r,y)x[z,\zeta(y)] + rxD_2(z,y)\zeta(y) = 0 \text{ for all } r,x,y,z \in R. \tag{3.9}$$

From (3.8) and (3.9), we can get

$$D_1(r,y)x[z,\zeta(y)] + D_2(r,y)x[z,\zeta(y)] = 0 \text{ for all } r,x,y,z \in R. \tag{3.10}$$

Which yields that

$$\{D_1(r,y) + D_2(r,y)\}x[z,\zeta(y)] = 0 \text{ for all } r,x,y,z \in R. \tag{3.11}$$

Primeness of R implies that either we have $D_1(r,y)+D_2(r,y) = 0$ or $[z,\zeta(y)] = 0$ for all $x,y,z,r \in R$. Consider the first case if $D_1(r,y)+D_2(r,y) = 0$, that is, $D_1(r,y) = -D_2(r,y)$ for all $r,y \in R$. On the hand, in later case $[z,\zeta(y)] = 0$ for all $y,z \in R$. Since $\zeta(y)$ is an additive mapping and now is also centralizing. By using Lemma 3.1 there exist η is an element from the extended centroid of R and δ is an additive mapping from R into the extended centroid of R such that $\zeta(x) = \eta x + \delta(x)$, for all $x \in R$. □

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