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A new iterative method based on Newton-Cotes formula for system of nonlinear equations

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Abstract

This paper presents a new two-step iterative method based on Newton-Cotes formula. We prove that this method is cubically convergent. Some numerical examples show that our method is comparable with the well-known existing methods.

1 Introduction

Nowadays there are many iterative methods for solving the system of nonlinear equations in the form:

$$F(x) = (f_1(x), f_1(x), \ldots, f_n(x))^t = 0 \tag{1.1}$$

where $F : D \subset \mathbb{R}^n \to \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)^t$ and $f_i, i = 1, 2, \ldots, n$ are maps from a convex subset D of the n-dimensional space $\mathbb{R}^n$ into the n-dimensional space $\mathbb{R}^n$. Solving (1.1) is a process of finding a numerical vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^t$ such that $F(x^*) = 0$. There are recently many iterative

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methods for solving the system of nonlinear equations $F(x) = 0$ which have been developed in order to improve the order of convergence by using different techniques. One of them is using quadrature formulas [1]-[8]. It is known that the quadrature rules play an important part in calculating the numerical value of definite integrals. Motivated and inspired by the on-going activities in this direction, in this paper we analyze a new iterative method for solving the system of nonlinear equations by using Newton-Cotes formula. The new method is a predictor-corrector method which uses Newton's method as a predictor and new method as a corrector. Some numerical examples are given to illustrate the efficiency and the performance of this new method.

2 Description of iterative method

Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a Frechet differentiable function on a convex $D \subset \mathbb{R}^n$. For any $x, x_k \in D$, we use the mean-value theorem of function $F(x)$ as follows:

$$F(x) = F(x_k) + \int_0^1 F'(x_k + t(x - x_k))(x - x_k)dt. \tag{2.2}$$

We approximate the integral on the right-hand side of (2.2) by the rectangular rule. We obtain

$$\int_0^1 F'(x_k + t(x - x_k))(x - x_k)dt \approx F'(x_k)(x - x_k). \tag{2.3}$$

By (2.2), (2.3) and $F(x) = 0$, we have

$$x = x_k - F'(x_k)^{-1}F(x_k). \tag{2.4}$$

This allows us to get a Newton's method for solving the system of nonlinear equations $F(x) = 0$ as follows:

Algorithm 2.1 For a given x_0, compute an approximate solution x_{k+1} by the iterative scheme

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), k \geq 0, \tag{2.5}$$

where $F'(x_k)$ is the Jacobian matrix at the point x_k. Algorithm 2.1 has quadratic convergence.

If we approximate the integral in (2.2) by using the closed Newton-Cote formula, we obtain

$$\int_0^1 F'(x_k + t(x - x_k))(x - x_k)dt$$
$$\approx \tfrac{x-x_k}{90}\left[7F'(x_k) + 32F'(x_k + h) + 12F'(x_k + 2h) + 32F'(x_k + 3h) + 7F'(x_k + 4h)\right] \quad (2.6)$$

where $h = \frac{x-x_k}{4}$.
Replacing (2.6) into (2.2), we have

$$F(x) \approx F(x_k) + \frac{x - x_k}{90}\left[7F'(x_k) + 32F'(x_k + h) + 12F'(x_k + 2h) + 32F'(x_k + 3h) + 7F'(x_k + 4h)\right]. \quad (2.7)$$

Since $F(x) = 0$, we obtain

$$x \approx x_k + \frac{90F(x_k)}{[7F'(x_k) + 32F'(x_k + h) + 12F'(x_k + 2h) + 32F'(x_k + 3h) + 7F'(x_k + 4h)]}. \quad (2.8)$$

To propose an iterative method, we set $x = x_{k+1}$. Then (2.8) can be written as:

$$x_{k+1} \approx x_k + \frac{90F(x_k)}{[7F'(x_k) + 32F'(x_k + h) + 12F'(x_k + 2h) + 32F'(x_k + 3h) + 7F'(x_k + 4h)]}. \quad (2.9)$$

From (2.4), we have $x - x_k = -F'(x_k)^{-1}F(x_k)$. Then (2.9) can be written as:

$$x_{k+1} \approx x_k + \frac{90F(x_k)}{\left[7F'(x_k) + 32F'(x_k + \bar{h}) + 12F'(x_k + 2\bar{h}) + 32F'(x_k + 3\bar{h}) + 7F'(x_k + 4\bar{h})\right]} \quad (2.10)$$

where $\bar{h} = -\frac{1}{4}F'(x_k)^{-1}F(x_k)$.
Therefore we obtain a two-step iterative method for solving the system of nonlinear equations as follows:

Algorithm 2.2 For a given x_0, compute an approximate solution x_{n+1} by the iterative scheme
Predictor Step:

$$y_n = -\frac{1}{4}F'(x_n)^{-1}F(x_n) \quad (2.11)$$

Corrector Step:

$$x_{n+1} = x_n + \frac{90F(x_n)}{[7F'(x_n) + 32F'(x_n + y_n) + 12F'(x_n + 2y_n) + 32F'(x_n + 3y_n) + 7F'(x_n + 4y_n)]} \quad (2.12)$$

where $n = 0, 1, 2, \ldots$.

3 Convergence Analysis

Theorem 3.1. *Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be an $r-$times Frechet differentiable function on a convex set D containing the root α of $F(x) = 0$. Then the iterative method defined by Algorithm 2.2 is cubically convergent and satisfies the error equation*

$$e_{n+1} = c_2^2 e_n^3 + (3c_2c_3 - 3c_2^3)e_n^4 + O(e_n^5). \tag{3.13}$$

Proof. Consider Algorithm 2.2 again,

$$y_n = -\frac{1}{4}F'(x_n)^{-1}F(x_n) \tag{3.14}$$

$$x_{n+1} = x_n + \frac{90F(x_n)}{[7F'(x_n) + 32F'(x_n + y_n) + 12F'(x_n + 2y_n) + 32F'(x_n + 3y_n) + 7F'(x_n + 4y_n)]} \tag{3.15}$$

Let $e_n = x_n - \alpha$. By using Taylor's expansion of $F(x)$ around $x = \alpha$, we obtain

$$F(x) = F'(\alpha)(x-\alpha) + \frac{F"(\alpha)(x-\alpha)^2}{2!} + \frac{F^{(3)}(\alpha)(x-\alpha)^3}{3!} + \frac{F^{(4)}(\alpha)(x-\alpha)^4}{4!} + O((x-\alpha)^5). \tag{3.16}$$

Substituting $e_n = x_n - \alpha$ and rearranging, we get

$$F(x) = F'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)\right] \tag{3.17}$$

where $c_2 = \frac{1}{k!}F'(\alpha)^{-1}F^{(k)}(\alpha)$.
From (3.14), we get

$$y_n = -\frac{1}{4}\left[e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5)\right]. \tag{3.18}$$

Therefore

$$x_n + y_n = \alpha + \frac{3e_n}{4} - \frac{1}{4}\left[-c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5)\right]. \tag{3.19}$$

Using Taylor's expansion for $F'(x_n + y_n)$ at α, we have

$$F'(x_n + y_n) = 1 + \frac{3}{2}c_2e_n + \left(\frac{c_2^2}{2} + \frac{27c_3}{16}\right)e_n^2 + \left(\frac{17c_2c_3}{18} - c_2^3 + \frac{27c_4}{16}\right)e_n^3 + O(e_n^4). \tag{3.20}$$

In a similar way, we obtain

$$\begin{aligned} F'(x_n+2y_n) &= 1+c_2e_n+(\frac{c_2^2}{2}+\frac{3c_3}{4})e_n^2+(\frac{7c_2c_3}{2}-2c_2^3+\frac{c_4}{4})e_n^3+O(e_n^4), \\ F'(x_n+3y_n) &= 1+\frac{1}{2}c_2e_n+\left(\frac{3c_2^2}{2}+\frac{3c_3}{16}\right)e_n^2+(\frac{33c_2c_3}{8}-3c_2^3+\frac{c_4}{16})e_n^3+O(e_n^4), \\ F'(x_n+4y_n) &= 1+2c_2e_n^2-4(c_2c_3-c_2^3)e_n^3+O(e_n^4). \end{aligned}$$

Thus,

$$7F'(x_n)+32F'(x_n+y_n)+12F'(x_n+2y_n)+32F'(x_n+3y_n)+7F'(x_n+4y_n)$$

$$=90+90c_2e_n+90(c_2^2+c_3)e_n^2+(270c_2c_3-180c_2^3+90c_4)e_n^3+O(e_n^4). \quad (3.21)$$

Therefore, from (3.15), (3.17) and (3.21), we get

$$e_{n+1}=c_2^2e_n^3+(3c_2c_3-3c_2^3)e_n^4+O(e_n^5). \quad (3.22)$$

Thus (3.22) shows that the method described by equation (3.14) and (3.15) has third order convergence. □

4 Numerical results.

In this section we will show the performances and efficiency of our new method by comparing the results with the results of Newton's method (NM), the method of Cordero and Torregrosa (CT)[4], the method of Darvishi and Barati (DV)[6], and the method of Noor and Wasseem (NR)[7]. The comparison appears in table 1. All computations have been carried out in MAPLE, using 30 digit floating point arithmetic. The stopping criterion used are $\|x_{n+1}-x_n\|_\infty < 10^{-14}$ and $\|F(x_{n+1})\|_\infty < 10^{-14}$. We analyze the number of iterations and the order of convergence p, approximated by

$$p \approx \frac{\ln\|x_{n+1}-x_n\|_\infty/\|x_n-x_{n-1}\|_\infty}{\ln\|x_{n+1}-x_n\|_\infty/\|x_n-x_{n-1}\|_\infty},$$

we test the new method with the following systems of nonlinear equations with their solutions.

1. $e^{x^2}+8x\sin y=0; x+y=1$
   Solutions are $(-0.14028501081, 0.1402850108)^t$.

2. $x^2-2x-y+0.5=0; x^2+4y^2-4=0$
   Solutions are $(-0.2222145551, 0.9938084186)^t$.

3. $x^2 + y^2 + z^2 = 1; 2x^2 + y^2 - 4z = 0; 3x^2 - 4y^2 + z^2 = 0$
   Solutions are $(0.69828861, 0.62852430, 0.34256419)^t$.

4. $x^2 + y^2 + z^2 = 9; xyz = 1; x + y - z^2 = 0$
   Solutions are $(2.49137570, 0.24274588, 1.65351794)^t$.

5. $yz+w(y+z) = 0; xz+w(x+z) = 0; xy+w(x+y) = 0; xy+xz+yz = 1$.
   Solutions are $(0.57735, 0.57735, 0.57735, -0.28868)^t$.

Table 1. The numerical comparison results.

system	method	initial value	number of iteration (IT)	p
(1)	NM	$(0.2, 0.8)^t$	5	2.0
	CT		4	3.0
	DV		4	3.0
	NR		4	3.0
	new		4	3.0
(2)	NM	$(0.5, 0.5)^t$	7	2.0
	CT		5	3.0
	DV		Fails	-
	NR		5	3.0
	new		5	3.0
(3)	NM	$(0.5, 0.5, 0.5)^t$	6	2.0
	CT		4	3.0
	DV		4	-
	NR		4	3.0
	new		4	3.0
(4)	NM	$(2.5, 0.5, 2.5)^t$	5	2.0
	CT		4	3.0
	DV		4	3.0
	NR		4	3.0
	new		4	3.0
(5)	NM	$(0.6, 0.6, 0.6, -0.2)^t$	5	2.0
	CT		3	3.3
	DV		3	3.3
	NR		3	3.3
	new		3	3.3

Table 1 shows the number of iterations (IT) and the order of convergence of all methods. We see that most of the results of the present method confer with methods CT, DV and NR except for system (2) where method DV fails.

5 Conclusion

In this paper we developed a two-step new iterative method for solving a system of nonlinear equations using the Newton-Cotes formula. The error equation was given theoretically to show that the new method has third order convergence. The new method has been tested on some examples from the literature and show the same results in most cases.

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