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On the stability and admissibility of a singular differential system with constant delay

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Abstract

This paper deals with the problems of asymptotic stability, exponentially stability and admissibility for a mathematical model of singular systems with constant delay. First, the singular system is transformed into a neutral differential system. Secondly, some sufficient conditions are obtained on the stability and the admissibility of solutions of the new neutral differential system using integral inequalities, linear matrix inequality (LMI) technique and meaningful Lyapunov-Krasovskii functionals. At the end, two numerical examples are given to show the effectiveness and applicability of the proposed method and the obtained results. These results generalize the existing ones.

1 Introduction

Singular systems, which are also called implicit or descriptor systems, are dynamic systems. During last years, many books and articles, qualitative properties of solutions of singular and non-singular systems with and without delay have been discussed and many interesting results were obtained

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on the qualitative properties of solutions of these kinds of equations (see, for example, [1-26]). It should be noted that neutral systems are more general classes than those of singular systems. Stability of these systems is a more complex question because these systems include the derivative of the retarded state. More recently, qualitative properties of these systems have been studied by some researchers (see, for example [1, 5, 8, 9, 10, 11, 12, 23, 24, 25, 26] and references therein).

In 2012, Liu [9] considered the following singular system with constant delay:

$$E\dot{x}(t) = Ax(t) + Bx(t-h), t > 0, \tag{1.1}$$

$$x(t_0 + \theta) = \varphi(\theta), -h \leq \theta \leq 0, h > 0,$$

where $x(t) \in \Re^n$ is the state vector, $\varphi(t)$ is a continuous initial function defined on $[-h, 0], h \in \Re, h > 0, h$ is fixed delay, $A \in \Re^{n\times n}$ is a negative definite real constant matrix and $B \in \Re^{n\times n}$ is a real constant matrix, and the matrix $E \in \Re^{n\times n}$ may be singular and we assume $rankE = r \leq n$.

Liu [9] proved the following:

Theorem A(Liu[[9];Theorem 2.1]). For a given constant $h > 0$, the delay singular differential system (1.1) is regular, impulse free and asymptotically stable if there exist positive-definite symmetric matrices P, Q, R and a matrix S of appropriate dimensions and a positive semi-definite matrix,

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$$

such that the following LMIs hold:

$$P^T E = E^T P \geq 0,$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} \end{bmatrix} < 0$$

and

$$E^T(R - X_{33})E \geq 0,$$

where $Z \in \Re^{n\times(n-r)}$ is any matrix satisfying $E^T Z = 0$ and

$$\begin{aligned}
\Omega_{11} &= A^T P + PA + A^T Z S^T + S Z^T A + Q + E^T(hX_{11} + X_{13}^T + X_{13})E,\\
\Omega_{12} &= PB + SZ^T B + E^T(hX_{12} + X_{23}^T - X_{13})E,\\
\Omega_{13} &= hA^T R,\\
\Omega_{22} &= -Q + E^T(hX_{22} - X_{23}^T - X_{23})E,\\
\Omega_{23} &= hB^T R,\\
\Omega_{33} &= -hR.
\end{aligned}$$

Theorem B(Liu[[9];Theorem 3.1]). For given constants $h > 0$ and $\alpha > 0$,the delay singular differential system (1.1) is regular, impulse free and exponential asymptotically stable with decay rate α if there exist positive-definite symmetric matrices P, Q, R, S and a positive semi-definite matrix X

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0$$

such that the following inequalities hold:

$$P^T E = E^T P \geq 0,$$

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} \end{bmatrix} < 0$$

and

$$E^T(R - X_{33})E \geq 0,$$

where $Z \in \Re^{n\times(n-r)}$ is any matrix satisfying $E^T Z = 0$ and

$$
\begin{aligned}
\Psi_{11} =&(A+0.5\alpha E)^T P + P(A+0.5\alpha E) + A^T Z S^T + S Z^T A + Q \\
&+ e^{-\alpha h} E^T (h X_{11} + X_{13}^T + X_{13}) E, \\
\Psi_{12} =&PB + S Z^T B + e^{-\alpha h} E^T (h X_{12} + X_{23}^T - X_{13}) E, \\
\Psi_{13} =&h A^T R, \\
\Psi_{22} =&e^{-\alpha h}[-Q + E^T (h X_{22} - X_{23}^T - X_{23}) E], \\
\Psi_{23} =&h B^T R, \\
\Psi_{33} =&-hR.
\end{aligned}
$$

Theorems A and B in Liu [9] have sufficient conditions. Under these sufficient conditions, system (1.1) is regular, impulse free and asymptotically stable. Both of the theorems in [9] were proved by means of a suitable Lyapunov-Krasovskii functional and linear matrix inequalities. The motivation of this paper has been inspired by the results of Liu [9]; that is, Theorems 1 and 2, and [10],[11]. In this paper, we investigate the same problem discussed by Liu [9], the linear singular system (1.1) with constant delay. The aim of this paper is two-fold. First, we transform the singular delay system (1.1) into a neutral delay system by using two suitable invertible matrices. By this transform, the singular matrix E in system (1.1) will not be included in the new neutral delay system. Secondly, by defining a new Lyapunov-Krasovskii functional, we give two new theorems, which include sufficient conditions, such that the new neutral system is asymptotically and exponentially stable. Hence, the obtained new results are also valid for singular delay system (1.1), (see Liu et al. [10]). Finally, we obtain the results of Liu [9] under weaker and less restrictive conditions.

Before converting the given singular system to a neutral system, we will sate some useful definitions and lemmas.

Definition 1([5]).The pair (E, A) is said to be regular if $det(sE-A) \neq 0$. The pair (E, A) is said to be impulse-free if $deg(det(sE-A)) = rank(E)$.

Definition 2([25]).The singular delay system (1.1) is said to be regular and impulse-free if the pair (E, A) is regular and impulse-free.

Definition 3([25]).The singular delay system (1.1) is said to be admissible if it is regular, impulse-free and stable.

Lemma 1([25]). Suppose the pair (E, A) is regular and impulse-free, then the solution to the singular delay system (1.1) exists and is impulse-free and unique on $[0, \infty)$.

Lemma 2([5]). If the pair (E, A) is regular and impulse-free, then there exist two non-singular matrices $M, N \in \Re^{n\times n}$ such that

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}.$$

By Lemma 2, there exist non-singular matrices $L_1, L_2 \in \Re^{n\times n}$ such that

$$L_1EL_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = \overline{E}, L_1AL_2 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} = \overline{A},$$

$$L_1BL_2 = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \overline{B}, N^{-1}x(t) = \eta(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}.$$

Then, we write the system (1.1) as the following:

$$\overline{E}\dot{\eta}(t) = \overline{A}\eta(t) + \overline{B}\eta(t-h).$$

Hence, this system can be decomposed to the following system:

$$\dot{\eta}_1(t) = A_1\eta_1(t) + B_1\eta_1(t-h) + B_2\eta_2(t-h), \tag{1.2}$$

$$0 = \eta_2(t) + B_3\eta_1(t-h) + B_4\eta_2(t-h). \tag{1.3}$$

If we take the time derivative of the equation (1.3), then we have

$$0 = \dot{\eta}_2(t) + B_3\dot{\eta}_1(t-h) + B_4\dot{\eta}_2(t-h). \tag{1.4}$$

If we combine equations (1.3) and (1.4), then we get

$$\dot{\eta}_2(t) = -\eta_2(t) - B_3\eta_1(t-h) - B_4\eta_2(t-h) - B_3\dot{\eta}_1(t-h) - B_4\dot{\eta}_2(t-h). \tag{1.5}$$

In view of equations (1.2) and equation (1.5), it follows that

$$\begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} A_1\eta_1(t) + B_1\eta_1(t-h) + B_2\eta_2(t-h) \\ -\eta_2(t) - B_3\eta_1(t-h) - B_4\eta_2(t-h) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -B_3 & -B_4 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1(t-h) \\ \dot{\eta}_2(t-h) \end{bmatrix}, \tag{1.6}$$

which is equivalent to the following neutral system with constant delay:

$$\dot{\eta}(t) - \widehat{C}\dot{\eta}(t-h) = \widehat{A}\eta(t) + \widehat{B}\eta(t-h), \tag{1.7}$$

$$\eta(t_0+\theta) = \varphi(\theta), -h \le \theta \le 0, h > 0,$$

where

$$\widehat{A}=\begin{bmatrix} A_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix}, \widehat{B}=\begin{bmatrix} B_1 & B_2 \\ -B_3 & -B_4 \end{bmatrix}, \widehat{C}=\begin{bmatrix} 0 & 0 \\ -B_3 & -B_4 \end{bmatrix}. \quad (1.8)$$

It should be noted that the systems (1.1) and (1.7) are not equivalent, but the stability of the system (1.7) guarantees the stability of system (1.1), and vice versa (see [10]). In light of Definition 3, the system (1.1) is admissible, which is regular, impulse free and stable.

Lemma 3([9]). For any positive semi-definite matrix

$$X=\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0,$$

the following integral inequality holds:

$$-\int_{t-h}^{t} \dot{x}^T(s)X_{33}\dot{x}(s)ds \leq \int_{t-h}^{t} \begin{bmatrix} x^T(t) & x^T(t-h) & \dot{x}^T(s) \end{bmatrix} \times \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} ds.$$

Lemma 4 (Schur Complement)([1]). For a given symmetric matrix

$$S=\begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix},$$

where $S_{11} \in \Re^{r\times r}$, the following inequalities are true:

$$1^0)S<0,$$
$$2^0)S_{11}<0, S_{22}-S_{12}^TS_{11}^{-1}S_{12}<0,$$
$$3^0)S_{22}<0, S_{11}-S_{12}S_{22}^{-1}S_{12}^T<0.$$

Here * denotes the symmetric term in a symmetric matrix.

2 Asymptotic stability and admissibility

A. Assumptions

Throughout this article, we suppose that the following assumptions hold:

(A1) We suppose that the pair (E, A) is regular and impulse-free and all the eigenvalues of $\widehat{C}$ are inside the unit circle and $||\widehat{C}|| < 1$.

(A2) There are symmetric positive definite matrices P, Q, R, W, S with appropriate dimensions and a positive semi-definite matrix

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0,$$

such that the following LMIs hold:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & 0 & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix} < 0, \tag{2.9}$$

and

$$R - X_{33} \geq 0, \tag{2.10}$$

where

$$\begin{aligned}
\Omega_{11} &= \widehat{A}^T P + P\widehat{A} + \widehat{A}^T S + S\widehat{A} + Q + hX_{11} + X_{13}^T + X_{13}, \\
\Omega_{12} &= P\widehat{B} + S\widehat{B} - \widehat{A}^T PC + hX_{12} + X_{23}^T - X_{13}, \\
\Omega_{13} &= S\widehat{C}, \Omega_{14} = \widehat{A}^T [W + hR], \\
\Omega_{22} &= -\widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - Q + hX_{22} - X_{23}^T - X_{23}, \\
\Omega_{24} &= \widehat{B}^T [W + hR], \Omega_{33} = -W, \\
\Omega_{34} &= \widehat{C}^T [W + hR], \Omega_{44} = -[W + hR].
\end{aligned}$$

(A3) There are symmetric positive definite matrices P, Q, R, W, S with appropriate dimensions and a positive semi-definite matrix

$$X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \geq 0,$$

such that the following LMIs hold:

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & 0 & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{bmatrix} < 0, \tag{2.11}$$

and

$$R - X_{33} \geq 0, \tag{2.12}$$

where

$$\begin{aligned} \Theta_{11} =& (\widehat{A} + 0.5\alpha I)^T P + P(\widehat{A} + 0.5\alpha I) + (\widehat{A} + 0.5\alpha I)^T S \\ &+ S(\widehat{A} + 0.5\alpha I) + Q + e^{-\alpha h}(hX_{11} + X_{13}^T + X_{13}), \end{aligned}$$

with $\alpha \in \Re, \alpha > 0, I$ is the identity matrix of appropriate dimension,

$$\begin{aligned} \Theta_{12} &= P\widehat{B} + S\widehat{B} - \widehat{A}^T P\widehat{C} - \alpha P\widehat{C} + e^{-\alpha h}(hX_{12} + X_{23}^T - X_{13}), \\ \Theta_{13} &= S\widehat{C}, \Theta_{14} = \widehat{A}^T[W + hR], \\ \Theta_{22} &= -\widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - e^{-\alpha h}Q + \alpha\widehat{C}^T P\widehat{C} + e^{-\alpha h}(hX_{22} - X_{23}^T - X_{23}), \\ \Theta_{24} &= \widehat{B}^T[W + hR], \\ \Theta_{33} &= -e^{-\alpha h}W, \\ \Theta_{34} &= \widehat{C}^T[W + hR], \Theta_{44} = -[W + hR]. \end{aligned}$$

Theorem 1. If assumptions (A1) and (A2) hold, then the system (1.7) is asymptotically stable and the system (1.1) is asymptotically admissible, that is, the system (1.1) is regular, impulse free and asymptotically stable.

Proof. To prove this theorem, we define the following Lyapunov-Krasovskii functional for the system (1.7):

$$\begin{aligned} V(\eta_t) =& K^T(\eta_t)PK(\eta_t) + \eta^T(t)S\eta(t) + \int_{t-h}^{t} \eta^T(s)Q\eta(s)ds \\ &+ \int_{t-h}^{t} \dot{\eta}^T(s)W\dot{\eta}(s)ds + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{\eta}^T(s)R\dot{\eta}(s)dsd\theta, \end{aligned} \tag{2.13}$$

where

$\eta_t = \eta(t+\theta), -h \leq \theta \leq 0$ and $K(\eta_t) = \eta(t) - \widehat{C}\eta(t-h)$.

In view of the Newton-Leibnitz formula, by the derivative of the functional $V(\eta_t)$ in (2.13) along the system (1.7), we obtain:

$$\begin{aligned}
\dot{V}(\eta_t) =& \eta^T(t)[\widehat{A}^T P + P\widehat{A} + \widehat{A}^T S + S\widehat{A} + Q + \widehat{A}^T(W+hR)\widehat{A}]\eta(t) \\
&+ \eta^T(t)[P\widehat{B} + S\widehat{B} - \widehat{A}^T P\widehat{C} + \widehat{A}^T(W+hR)\widehat{B}]\eta(t-h) \\
&+ \eta^T(t)[S\widehat{C} + \widehat{A}^T(W+hR)\widehat{C}]\dot{\eta}(t-h) \\
&+ \eta^T(t-h)[\widehat{B}^T P + \widehat{B}^T S - \widehat{C}^T P\widehat{A} + \widehat{B}^T(W+hR)\widehat{A}]\eta(t) \\
&+ \eta^T(t-h)[-\widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - Q + \widehat{B}^T(W+hR)\widehat{B}]\eta(t-h) \\
&+ \eta^T(t-h)[\widehat{B}^T W\widehat{C} + h\widehat{B}^T R\widehat{C}]\dot{\eta}(t-h) \\
&+ \dot{\eta}^T(t-h)[\widehat{C}^T S + \widehat{C}^T(W+hR)\widehat{A}]\eta(t) \\
&+ \dot{\eta}^T(t-h)[\widehat{C}^T W\widehat{B} + h\widehat{C}^T R\widehat{B}]\eta(t-h) \\
&+ \dot{\eta}^T(t-h)[\widehat{C}^T W\widehat{C} - W + h\widehat{C}^T R\widehat{C}]\dot{\eta}(t-h) - \int_{t-h}^{t} \dot{\eta}^T(s)R\dot{\eta}(s)ds \\
=& \eta^T(t)[\widehat{A}^T P + P\widehat{A} + \widehat{A}^T S + S\widehat{A} + Q + \widehat{A}^T(W+hR)\widehat{A}]\eta(t) \\
&+ \eta^T(t)[P\widehat{B} + S\widehat{B} - \widehat{A}^T P\widehat{C} + \widehat{A}^T(W+hR)\widehat{B}]\eta(t-h) \\
&+ \eta^T(t)[S\widehat{C} + \widehat{A}^T(W+hR)\widehat{C}]\dot{\eta}(t-h) \\
&+ \eta^T(t-h)[\widehat{B}^T P + \widehat{B}^T S - \widehat{C}^T P\widehat{A} + \widehat{B}^T(W+hR)\widehat{A}]\eta(t) \\
&+ \eta^T(t-h)[-\widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - Q + \widehat{B}^T(W+hR)\widehat{B}]\eta(t-h) \\
&+ \eta^T(t-h)[\widehat{B}^T W\widehat{C} + h\widehat{B}^T R\widehat{C}]\dot{\eta}(t-h) \\
&+ \dot{\eta}^T(t-h)[\widehat{C}^T S + \widehat{C}^T(W+hR)\widehat{A}]\eta(t) \\
&+ \dot{\eta}^T(t-h)[\widehat{C}^T W\widehat{B} + h\widehat{C}^T R\widehat{B}]\eta(t-h) \\
&+ \dot{\eta}^T(t-h)[\widehat{C}^T W\widehat{C} - W + h\widehat{C}^T R\widehat{C}]\dot{\eta}(t-h) \\
&- \int_{t-h}^{t} \dot{\eta}^T(s)(R - X_{33})\dot{\eta}(s)ds - \int_{t-h}^{t} \dot{\eta}^T(s)X_{33}\dot{\eta}(s)ds. \qquad (2.14)
\end{aligned}$$

Now, we consider the last term in the equality (2.14). By means of Lemma

3 and the Newton-Leibnitz formula, we derive the following inequality:

$$
\begin{aligned}
-\int_{t-h}^{t} \dot{\eta}^T(s)X_{33}\dot{\eta}(s)ds \leq & \int_{t-h}^{t} \left[\begin{array}{ccc} \eta^T(t) & \eta^T(t-h) & \dot{\eta}^T(s) \end{array} \right] \\
& \times \left[\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{array} \right] \left[\begin{array}{c} \eta(t) \\ \eta(t-h) \\ \dot{\eta}(s) \end{array} \right] ds \\
\leq & \eta^T(t)hX_{11}\eta(t) + \eta^T(t)hX_{12}\eta(t-h) + \eta^T(t)X_{13}\int_{t-h}^{t} \dot{\eta}(s)ds \\
& + \eta^T(t-h)hX_{12}^T\eta(t) + \eta^T(t-h)hX_{22}\eta(t-h) \\
& + \eta^T(t-h)X_{23}\int_{t-h}^{t} \dot{\eta}(s)ds + \int_{t-h}^{t} \dot{\eta}^T(s)dsX_{13}^T\eta(t) \\
& + \int_{t-h}^{t} \dot{\eta}^T(s)dsX_{23}^T\eta(t-h) \\
\leq & \eta^T(t)[hX_{11} + X_{13}^T + X_{13}]\eta(t) + \eta^T(t)[hX_{12} + X_{23}^T - X_{13}]\eta(t-h) \\
& + \eta^T(t-h)[hX_{12}^T - X_{13}^T + X_{23}]\eta(t) \\
& + \eta^T(t-h)[hX_{22} - X_{23} - X_{23}^T]\eta(t-h). \qquad (2.15)
\end{aligned}
$$

Substituting the right hand side of the inequality (2.15) into the equality (2.14), we obtain:

$$
\dot{V}(\eta_t) < \mu^T(t)\Psi\mu(t) - \int_{t-h}^{t} \dot{\eta}^T(s)(R - X_{33})\dot{\eta}(s)ds, \qquad (2.16)
$$

where

$$
\mu^T(t) = \left[\begin{array}{ccc} \eta^T(t) & \eta^T(t-h) & \dot{\eta}^T(t-h) \end{array} \right]
$$

and

$$
\Psi = \left[\begin{array}{ccc} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{array} \right]
$$

with

$$\begin{aligned}
\Psi_{11} =& \widehat{A}^T P + P\widehat{A} + \widehat{A}^T S + S\widehat{A} + Q + \widehat{A}^T[W + hR]\widehat{A} + hX_{11} + X_{13}^T + X_{13},\\
\Psi_{12} =& P\widehat{B} + S\widehat{B} - \widehat{A}^T P\widehat{C} + \widehat{A}^T[W + hR]\widehat{B} + hX_{12} + X_{23}^T - X_{13},\\
\Psi_{13} =& S\widehat{C} + \widehat{A}^T[W + hR]\widehat{C},\\
\Psi_{22} =& - \widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - Q + \widehat{B}^T[W + hR]\widehat{B} + hX_{22} - X_{23}^T - X_{23},\\
\Psi_{23} =& \widehat{B}^T[W + hR]\widehat{C},\\
\Psi_{33} =& \widehat{C}^T W\widehat{C} - W + h\widehat{C}^T R\widehat{C}.
\end{aligned}$$

In view of the assumptions $\Psi < 0$ and $R - X_{33} \geq 0$, we conclude that $\dot{V}(\eta_t) \leq 0$. Therefore, the system (1.7) is asymptotically stable. Thus, the system (1.1) is also asymptotically stable, regular and impulse free. Namely, the system (1.1) is asymptotically admissible.

Example 1. As a special case of the system (1.1), we consider the following singular system:

$$\begin{aligned}
\frac{d}{dt}\left(\begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) =& \begin{bmatrix} -1.0625 & -0.0625 \\ 0.125 & 0.125 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\\
&+ \begin{bmatrix} -0.3225 & -0.2712 \\ -0.025 & -0.025 \end{bmatrix} \times \begin{bmatrix} x_1(t-0.15) \\ x_2(t-0.15) \end{bmatrix}.
\end{aligned} \tag{2.17}$$

When we compare the system (2.17) with the system (1.1), it follows that

$$E = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1.0625 & -0.0625 \\ 0.125 & 0.125 \end{bmatrix}, B = \begin{bmatrix} -0.3225 & -0.2712 \\ -0.025 & -0.025 \end{bmatrix}$$

and $h = 0.15$.

Clearly, the pair (E, A) is regular and impulse-free. Therefore, there exist two non-singular matrices

$$L_1 = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, L_2 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

such that

$$L_1 E L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, L_1 A L_2 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix},$$

$$L_1BL_2 = \begin{bmatrix} -0.1025 & -1.135 \\ 0 & -0.2 \end{bmatrix}.$$

Hence, in view of Lemma 2, it follows that

$$\widehat{A} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \widehat{B} = \begin{bmatrix} -0.1025 & -1.135 \\ 0 & 0.2 \end{bmatrix}, \widehat{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix}$$

and

$$R - X_{33} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are $\lambda_1 = 0, \lambda_2 = 0.05$. Similarly, we have

$$X = \begin{bmatrix} 9.3050 & -0.1250 & 1.2250 & 0 & 1.2150 & 0.6500 \\ -0.1250 & 9.3650 & 0 & 1.2520 & 0.6500 & 1.4850 \\ 1.2250 & 0 & 4.6250 & 1.2500 & 1.1250 & 1.2250 \\ 0 & 1.2520 & 1.2500 & 5.1050 & 1.2250 & 1.9850 \\ 1.2150 & 0.6500 & 1.1250 & 1.2250 & 1.2350 & 1.2650 \\ 0.6500 & 1.4850 & 1.2250 & 1.9850 & 1.2650 & 1.5050 \end{bmatrix}.$$

The eigenvalues of the matrix X are obtained as the following:

$$m_1 = 0, m_2 = 0.9105, m_3 = 3.3556, m_4 = 6.3567, m_5 = 9.7517, m_2 = 10.7655.$$

It is clear that the matrix X is positive semi-definite. Let

$$P = \begin{bmatrix} 1.65 & 0 \\ 0 & 3.85 \end{bmatrix}, Q = \begin{bmatrix} 1.1 & 0.25 \\ 0.25 & 1.2 \end{bmatrix}, R = \begin{bmatrix} 1.285 & 1.265 \\ 1.265 & 1.505 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}, S = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 2 \end{bmatrix}.$$

In this case, all the eigenvalues of the matrix related to the matrix Ψ are obtained as the following:

$$\lambda_1 = -6.9048, \lambda_2 = -3.0088, \lambda_3 = -2.1437,$$
$$\lambda_4 = -0.8473, \lambda_5 = -0.7564, \lambda_6 = -0.0963.$$

Consequently, it is clear that all the conditions of Theorem 1 are satisfied. Therefore, the singular system (2.17) is asymptotically stable. In addition, the system (2.17) is also asymptotically admissible, since, the system (2.17) is asymptotically stable, regular and impulse free.

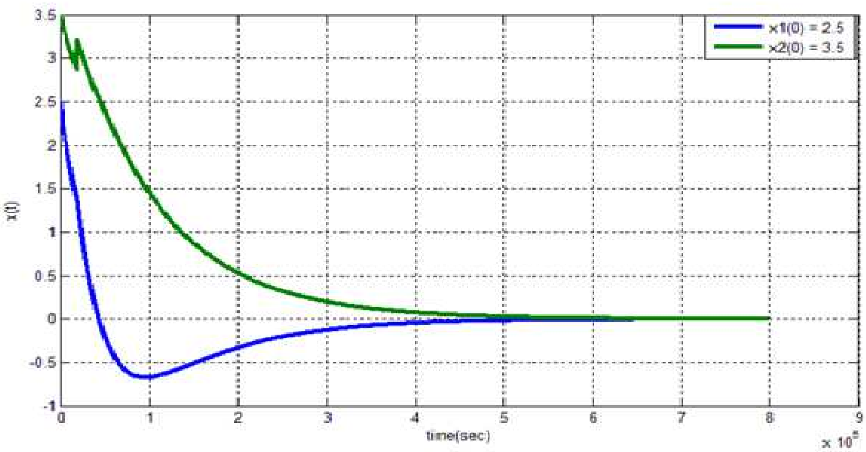

Figure 1: Trajectories of the solution of $x(t)$ of the system (2.17), when $h = 0.15$.

Theorem 2. If assumptions (A1) and (A3) hold, then the system (1.7) is exponentially stable and the system (1.1) is exponentially admissible, hence, the system (1.1) is regular, impulse free and exponentially stable.

Proof. We define the following Lyapunov-Krasovskii functional for system (1.7):

$$V(\eta_t) = e^{\alpha t} K^T(\eta_t) P K(\eta_t) + e^{\alpha t}\eta^T(t) S \eta(t) + \int_{t-h}^{t} e^{\alpha s}\eta^T(s) Q \eta(s) ds + \int_{t-h}^{t} e^{\alpha s}\dot{\eta}^T(s) W \dot{\eta}(s) ds + \int_{-h}^{0}\int_{t+\theta}^{t} e^{\alpha s}\dot{\eta}^T(s) R \dot{\eta}(s) ds d\theta, \quad (2.18)$$

where $\eta_t = \eta(t+\theta), -h \le \theta \le 0$.

By the time derivative of the functional $V(\eta_t)$ in (2.18) along the system

(1.7) and the Newton-Leibniz formula, we obtain:

$$\begin{aligned}\dot{V}(\eta_t) =& e^{\alpha t}\{\eta^T(t)[(\widehat{A}+0.5\alpha I)^T P + P(\widehat{A}+0.5\alpha I) + (\widehat{A}+0.5\alpha I)^T S \\ &+ S(\widehat{A}+0.5\alpha I) + Q + \widehat{A}^T(W+hR)\widehat{A}]\eta(t) \\ &+ \eta^T(t)[P\widehat{B} + S\widehat{B} - \widehat{A}^T P\widehat{C} - \alpha P\widehat{C} + \widehat{A}^T(W+hR)\widehat{B}]\eta(t-h) \\ &+ \eta^T(t)[S\widehat{C} + \widehat{A}^T(W+hR)\widehat{C}]\dot{\eta}(t-h) \\ &+ \eta^T(t-h)[\widehat{B}^T P + \widehat{B}^T S - \widehat{C}^T P\widehat{A} - \alpha\widehat{C}^T P + \widehat{B}^T(W+hR)\widehat{A}]\eta(t) \\ &+ \eta^T(t-h)[-\widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - e^{-\alpha h}Q + \alpha\widehat{C}^T P\widehat{C} \\ &+ \widehat{B}^T(W+hR)\widehat{B}]\eta(t-h) + \eta^T(t-h)[\widehat{B}^T W\widehat{C} + h\widehat{B}^T R\widehat{C}]\dot{\eta}(t-h) \\ &+ \dot{\eta}^T(t-h)[\widehat{C}^T S + \widehat{C}^T(W+hR)\widehat{A}]\eta(t) \\ &+ \dot{\eta}^T(t-h)[\widehat{C}^T W\widehat{B} + h\widehat{C}^T R\widehat{B}]\eta(t-h) \\ &+ \dot{\eta}^T(t-h)[\widehat{C}^T W\widehat{C} - e^{-\alpha h}W + h\widehat{C}^T R\widehat{C}]\dot{\eta}(t-h) \\ &- \int_{t-h}^{t} e^{\alpha(s-t)}\dot{\eta}^T(s)R\dot{\eta}(s)ds\}.\end{aligned}$$

Clearly, for any a scalar $s \in [t-h, t]$, we get $e^{-\alpha h} \le e^{\alpha(s-t)} \le 1$, and

$$-\int_{t-h}^{t} e^{\alpha(s-t)}\dot{\eta}^T(s)R\dot{\eta}(s)ds \le -e^{-\alpha h}\int_{t-h}^{t}\dot{\eta}^T(s)R\dot{\eta}(s)ds.$$

Similar to the proof of Theorem 1, we have

$$\dot{V}(\eta_t) < e^{\alpha t}(\mu^T(t)\overline{\Theta}\mu(t) - \int_{t-h}^{t}\dot{\eta}^T(s)e^{-\alpha h}(R - X_{33})\dot{\eta}(s)ds),$$

where

$$\mu^T(t) = \left[\begin{array}{ccc} \eta^T(t) & \eta^T(t-h) & \dot{\eta}^T(t-h) \end{array}\right]$$

and

$$\overline{\Theta} = \left[\begin{array}{ccc} \overline{\Theta}_{11} & \overline{\Theta}_{12} & \overline{\Theta}_{13} \\ * & \overline{\Theta}_{22} & \overline{\Theta}_{23} \\ * & * & \overline{\Theta}_{33} \end{array}\right]$$

with

$$\begin{aligned}
\overline{\Theta}_{11} =&(\widehat{A}+0.5\alpha I)^T P + P(\widehat{A}+0.5\alpha I) + (\widehat{A}+0.5\alpha I)^T S \\
&+ S(\widehat{A}+0.5\alpha I) + Q + \widehat{A}^T(W+hR)\widehat{A} + e^{-\alpha h}(hX_{11}+X_{13}^T+X_{13}), \\
\overline{\Theta}_{12} =&P\widehat{B} + S\widehat{B} - \widehat{A}^T P\widehat{C} - \alpha P\widehat{C} + \widehat{A}^T(W+hR)\widehat{B} + e^{-\alpha h}(hX_{12}+X_{23}^T - X_{13}), \\
\overline{\Theta}_{13} =&S\widehat{C} + \widehat{A}^T(W+hR)\widehat{C}, \\
\overline{\Theta}_{22} =&- \widehat{B}^T P\widehat{C} - \widehat{C}^T P\widehat{B} - e^{-\alpha h}Q + \alpha\widehat{C}^T P\widehat{C} + \widehat{B}^T(W+hR)\widehat{B} + e^{-\alpha h}(hX_{22} - X_{23}^T - X_{23}), \\
\overline{\Theta}_{23} =&\widehat{B}^T[W+hR]\widehat{C}, \overline{\Theta}_{23} = \widehat{C}^T W\widehat{C} - e^{-\alpha h}W + h\widehat{C}^T R\widehat{C}.
\end{aligned}$$

In order to guarantee $\dot{V}(\eta_t) < 0$, one needs to ensure that $\overline{\Theta} < 0$ and $R - X_{33} \geq 0$. It follows from $\dot{V}(\eta_t) < 0$ and the Schur complement in Lemma 4 that the conditions (2.11) and (2.12) are satisfied. Therefore, the system (1.7) is exponentially stable. Thus, the system (1.1) is exponentially admissible. Hence, the system (1.1) is exponentially stable, regular and impulse free.

Example 2. As a special case of the system (1.1), we consider the following singular system:

$$\frac{d}{dt}\left(\begin{bmatrix} 0.125 & 0 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}\right) = \begin{bmatrix} -0.2813 & -0.1875 \\ -0.125 & 0.25 \end{bmatrix} \times \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -0.0791 & -0.1231 \\ 0.0125 & -0.025 \end{bmatrix} \times \begin{bmatrix} x_1(t-0.20) \\ x_2(t-0.20) \end{bmatrix}. \tag{2.19}$$

When we compare the system (2.19) with the system (1.1), it follows that

$$E = \begin{bmatrix} 0.125 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -0.2813 & -0.1875 \\ -0.125 & 0.25 \end{bmatrix}, B = \begin{bmatrix} -0.0791 & -0.1231 \\ 0.0125 & -0.025 \end{bmatrix},$$

$h = 0.20$ and $\alpha = 0.25$. Clearly, the pair (E, A) is regular and impulse-free. So there exist two regular matrices

$$L_1 = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}, L_2 = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

such that

$$L_1 E L_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, L_1 A L_2 = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix},$$

$$L_1BL_2 = \begin{bmatrix} -0.125 & -1.135 \\ 0 & -0.1 \end{bmatrix}.$$

With respect to Theorem 2, we obtain:

$$\widehat{A} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}, \widehat{B} = \begin{bmatrix} -0.125 & -1.135 \\ 0 & 0.1 \end{bmatrix}, \widehat{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix},$$
$$P = \begin{bmatrix} 3.65 & 0 \\ 0 & 4.85 \end{bmatrix}, Q = \begin{bmatrix} 1.1 & 0.25 \\ 0.25 & 1.2 \end{bmatrix}, R = \begin{bmatrix} 1.285 & 1.265 \\ 1.265 & 1.505 \end{bmatrix},$$
$$W = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix}, S = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 2 \end{bmatrix}.$$

In this case, all the eigenvalues of the matrix related to the matrix $\overline{\Theta}$ are obtained as the following:

$$\lambda_1 = -11.2172, \lambda_2 = -4.2894, \lambda_3 = -4.0434,$$
$$\lambda_4 = -1.5559, \lambda_5 = -1.4056, \lambda_6 = -0.3677.$$

Consequently, it is clear that all the conditions of Theorem 2 can be satisfied. Therefore, the singular system (2.19) is exponentially stable. Thus, the system (2.19) is exponentially admissible, which is exponentially stable, regular and impulse free.

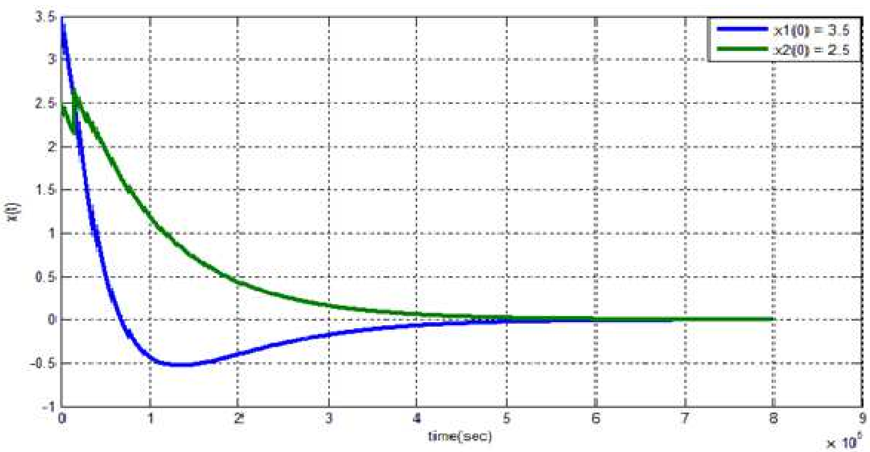

Figure 2: Trajectories of the solution of $x(t)$ of the system (2.19), when $h = 0.20$.

3 Conclusion

Here, it is considered a kind of linear singular systems with a constant delay. By a suitable transform, we reduce the considered system to regular neutral system with constant delay. Then, by defining meaningful Lyapunov-Krasovskii functionals and using LMIs, we prove two new theorems, which include sufficient conditions, on the asymptotic and exponential stability of the considered system. Our results, Theorem 1 and Theorem 2 improve the results of Liu [9, Theorems A, B] under weaker conditions. Finally, two numerical examples are given to show the applicability of Theorem 1 and Theorem 2.

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