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An elliptic analogue of Fukuhara's trigonometric identities

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Abstract

We obtain new elliptic function identities, which are an elliptic analogue of Fukuhara's trigonometric identities. By comparing the coefficients of Laurent expansions at $z = 0$ of our elliptic function identities, we give some reciprocity laws for elliptic Dedekind sums explicitly.

1 Introduction

Let a and b be relatively prime positive integers. Our starting point is to observe the identities from (1.1) to (1.5) in [2], which are quoted below.
For any complex number z,

$$\frac{1}{a}\sum_{\nu=1}^{a-1}\cot\left(\frac{\pi b\nu}{a}\right)\cot\left(\pi\left(z+\frac{\nu}{a}\right)\right)+\frac{1}{b}\sum_{\nu=1}^{b-1}\cot\left(\frac{\pi a\nu}{b}\right)\cot\pi\left(\left(z+\frac{\nu}{b}\right)\right)$$
$$=-\cot(\pi az)\cot(\pi bz)+\frac{1}{ab}\csc(\pi z)^2-1. \tag{1.1}$$

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If a is even, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1}(-1)^{\nu}\cot\left(\frac{\pi b\nu}{a}\right)\cot\left(\pi\left(z+\frac{\nu}{a}\right)\right)+\frac{1}{b}\sum_{\nu=1}^{b-1}\csc\left(\frac{\pi a\nu}{b}\right)\cot\left(\pi\left(z+\frac{\nu}{b}\right)\right)$$
$$=-\csc\left(\pi az\right)\cot\left(\pi bz\right)+\frac{1}{ab}\csc(\pi z)^2. \tag{1.2}$$

If $a+b$ is even, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1}(-1)^{\nu}\csc\left(\frac{\pi b\nu}{a}\right)\cot\left(\pi\left(z+\frac{\nu}{a}\right)\right)+\frac{1}{b}\sum_{\nu=1}^{b-1}(-1)^{\nu}\csc\left(\frac{\pi a\nu}{b}\right)\cot\left(\pi\left(z+\frac{\nu}{b}\right)\right)$$
$$=-\csc\left(\pi az\right)\csc\left(\pi bz\right)+\frac{1}{ab}\csc(\pi z)^2. \tag{1.3}$$

If a is odd, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1}(-1)^{\nu}\cot\left(\frac{\pi b\nu}{a}\right)\csc\left(\pi\left(z+\frac{\nu}{a}\right)\right)+\frac{1}{b}\sum_{\nu=1}^{b-1}\csc\left(\frac{\pi a\nu}{b}\right)\csc\left(\pi\left(z+\frac{\nu}{b}\right)\right)$$
$$=-\csc\left(\pi az\right)\cot\left(\pi bz\right)+\frac{1}{ab}\csc(\pi z)\cot(\pi z). \tag{1.4}$$

If $a+b$ is odd, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1}(-1)^{\nu}\csc\left(\frac{\pi b\nu}{a}\right)\csc\left(\pi\left(z+\frac{\nu}{a}\right)\right)+\frac{1}{b}\sum_{\nu=1}^{b-1}(-1)^{\nu}\csc\left(\frac{\pi a\nu}{b}\right)\csc\left(\pi\left(z+\frac{\nu}{b}\right)\right)$$
$$=-\csc\left(\pi az\right)\csc\left(\pi bz\right)+\frac{1}{ab}\csc(\pi z)\cot(\pi z). \tag{1.5}$$

In [2], Fukuhara pointed out that (1.1) is derived from specialization of Dieter's formula (Theorem 2.4 of [1]) which is a product to sum type formula of trigonometric functions, and proved (1.2) - (1.5). By comparing the coefficients of Laurent expansions at $z=0$ of identities (1.1) - (1.5), he obtained reciprocity laws for some Dedekind-Apostol sums

$$\frac{1}{a}\sum_{\nu=1}^{a-1}(\pm 1)^{\nu}\cot\left(\frac{\pi b\nu}{a}\right)\csc^{(N)}\left(\frac{\pi\nu}{a}\right),\quad \frac{1}{a}\sum_{\nu=1}^{a-1}(\pm 1)^{\nu}\csc\left(\frac{\pi b\nu}{a}\right)\cot^{(N)}\left(\frac{\pi\nu}{a}\right)$$

or

$$\frac{1}{a}\sum_{\nu=1}^{a-1}(\pm 1)^{\nu}\cot\left(\frac{\pi b\nu}{a}\right)\cot^{(N)}\left(\frac{\pi\nu}{a}\right),\quad \frac{1}{a}\sum_{\nu=1}^{a-1}(\pm 1)^{\nu}\csc\left(\frac{\pi b\nu}{a}\right)\csc^{(N)}\left(\frac{\pi\nu}{a}\right)$$

where $\cot^{(N)}(z)$ and $\csc^{(N)}(z)$ are the N-th derivative of the $\cot(z)$ and $\csc(z)$. In [2], Fukuhara used special values of Hurwitz zeta functions

$$\zeta(2N-1, z) := \sum_{n\geq 0} \frac{1}{(n+z)^{2N-1}}$$

instead of higher derivative of the $\cot(z)$ and $\csc(z)$.

The simplest case $N = 0$ of his reciprocity laws are the following identities from (1.11) to (1.15) in [2].

If a and b are relatively prime positive integers, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1} \cot\left(\frac{\pi b\nu}{a}\right)\cot\left(\frac{\pi\nu}{a}\right) + \frac{1}{b}\sum_{\nu=1}^{b-1} \cot\left(\frac{\pi a\nu}{b}\right)\cot\left(\frac{\pi\nu}{b}\right) = \frac{a^2+b^2+1-3ab}{3ab}. \tag{1.6}$$

If a is even, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1} (-1)^\nu \cot\left(\frac{\pi b\nu}{a}\right)\cot\left(\frac{\pi\nu}{a}\right) + \frac{1}{b}\sum_{\nu=1}^{b-1} \csc\left(\frac{\pi a\nu}{b}\right)\cot\left(\frac{\pi\nu}{b}\right) = \frac{-a^2+2b^2+2}{6ab}. \tag{1.7}$$

If $a+b$ is even, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1} (-1)^\nu \csc\left(\frac{\pi b\nu}{a}\right)\cot\left(\frac{\pi\nu}{a}\right) + \frac{1}{b}\sum_{\nu=1}^{b-1} (-1)^\nu \csc\left(\frac{\pi a\nu}{b}\right)\cot\left(\frac{\pi\nu}{b}\right) = \frac{-a^2-b^2+2}{6ab}. \tag{1.8}$$

If a is odd, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1} (-1)^\nu \cot\left(\frac{\pi b\nu}{a}\right)\csc\left(\frac{\pi\nu}{a}\right) + \frac{1}{b}\sum_{\nu=1}^{b-1} \csc\left(\frac{\pi a\nu}{b}\right)\csc\left(\frac{\pi\nu}{b}\right) = \frac{-a^2+2b^2-1}{6ab}. \tag{1.9}$$

If $a+b$ is odd, then

$$\frac{1}{a}\sum_{\nu=1}^{a-1} (-1)^\nu \csc\left(\frac{\pi b\nu}{a}\right)\csc\left(\frac{\pi\nu}{a}\right) + \frac{1}{b}\sum_{\nu=1}^{b-1} (-1)^\nu \csc\left(\frac{\pi a\nu}{b}\right)\csc\left(\frac{\pi\nu}{b}\right) = \frac{-a^2-b^2-1}{6ab}. \tag{1.10}$$

Note that (1.1) is the reciprocity law for the classical Dedekind sum

$$s(a;b) := \frac{1}{4b}\sum_{\nu=1}^{b-1}\cot\left(\frac{\pi a\nu}{b}\right)\cot\left(\frac{\pi\nu}{b}\right).$$

On the other hand, Fukuhara and Yui obtained the following elliptic function identity (Theorem 2.1 in [3]) which is regarded as an elliptic analogue of the trigonometric identity (1.1). If $a+b$ is odd, then

$$\begin{aligned}\frac{1}{a}&\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)\\&+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{cs}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)\\&=-\mathrm{cs}\,(2Kaz,k)\mathrm{cs}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ds}\,(2Kz,k)\mathrm{ns}\,(2Kz,k),\end{aligned}\tag{1.11}$$

where $\mathrm{cs}\,(z,k)$ is the Jacobi elliptic function (see Section 2). From this elliptic function identity, Fukuhara and Yui also gave reciprocity laws (Theorem 2.2 in [3]) for the elliptic Dedekind-Apostol sums

$$\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{cs}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}^{(N)}\left(2K\frac{\mu\tau+\nu}{b},k\right),$$

where $\mathrm{cs}^{(N)}(z)$ is the N-th derivative of the $\mathrm{cs}\,(z)$. The simplest case of Fukuhara-Yui's reciprocity is the following (Lemma 3.1 in [3])

$$\begin{aligned}\frac{1}{a}&\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\left(\frac{\mu\tau+\nu}{a}\right),k\right)\\&+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{cs}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\left(\frac{\mu\tau+\nu}{b}\right),k\right)\\&=\frac{a^2+b^2+1}{6ab}\left(2-\lambda(\tau)\right),\end{aligned}\tag{1.12}$$

which is an elliptic analogue of the reciprocity law (1.6).

In this article, we give an elliptic analogue of (1.2) - (1.5) and (1.7) - (1.10). The content of this paper is as follows. In Section 2, we introduce the elliptic functions $\mathrm{cs}\,(z,k)$, $\mathrm{ds}\,(z,k)$, $\mathrm{ns}\,(z,k)$, and list their fundamental properties. Section 3 is the main part of this article and we prove our main results (Theorem 3, Theorem 4 and Corollary 5). In Section 4, we give all the examples of Theorem 3 and Corollary 5.

2 Preliminaries

Throughout the paper, we denote the ring of rational integers by $\mathbb{Z}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$ and the upper half plane $\mathfrak{H} := \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$. For $\tau \in \mathfrak{H}$, we put

$$e(x) := e^{2\pi\sqrt{-1}x}, \quad q := e(\tau).$$

First, we recall the Jacobi theta functions

$$\begin{aligned}
\theta_1(z,\tau) :=& 2\sum_{n=0}^{\infty}(-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2}\sin\left((2n+1)\pi z\right)\\
=& 2q^{\frac{1}{8}}\sin\pi z\prod_{n=1}^{\infty}(1-q^n)(1-q^n e(z))(1-q^n e(-z)),\\
\theta_2(z,\tau) :=& 2\sum_{n=0}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2}\cos\left((2n+1)\pi z\right)\\
=& 2q^{\frac{1}{8}}\cos\pi z\prod_{n=1}^{\infty}(1-q^n)(1+q^n e(z))(1+q^n e(-z)),\\
\theta_3(z,\tau) :=& 1+2\sum_{n=1}^{\infty} q^{\frac{1}{2}n^2}\cos\left(2n\pi z\right)\\
=& \prod_{n=1}^{\infty}(1-q^n)(1+q^{n-\frac{1}{2}}e(z))(1+q^{n-\frac{1}{2}}e(-z)),\\
\theta_4(z,\tau) :=& 1+2\sum_{n=1}^{\infty}(-1)^n q^{\frac{1}{2}n^2}\cos\left(2n\pi z\right)\\
=& \prod_{n=1}^{\infty}(1-q^n)(1-q^{n-\frac{1}{2}}e(z))(1-q^{n-\frac{1}{2}}e(-z)).
\end{aligned}$$

Further we put

$$k = k(\tau) := \frac{\theta_2(0,\tau)^2}{\theta_3(0,\tau)^2}, \quad \lambda = \lambda(\tau) := k(\tau)^2, \quad K = K(\tau) := \frac{\pi}{2}\theta_3(0,\tau)^2$$

and introduce the Jacobi elliptic functions

$$\begin{aligned}
\mathrm{sn}\,(2Kz,k) &:= \frac{\theta_3(0,\tau)}{\theta_2(0,\tau)}\frac{\theta_1(z,\tau)}{\theta_4(z,\tau)},\\
\mathrm{cn}\,(2Kz,k) &:= \frac{\theta_4(0,\tau)}{\theta_2(0,\tau)}\frac{\theta_2(z,\tau)}{\theta_4(z,\tau)},\\
\mathrm{dn}\,(2Kz,k) &:= \frac{\theta_4(0,\tau)}{\theta_3(0,\tau)}\frac{\theta_3(z,\tau)}{\theta_4(z,\tau)}.
\end{aligned}$$

As is well known, $\mathrm{sn}\,(2Kz,k)$, $\mathrm{cn}\,(2Kz,k)$ and $\mathrm{dn}\,(2Kz,k)$ depend on the elliptic lambda function $\lambda(\tau) = k(\tau)^2$ but on the sign of $k(\tau)$. Since $\lambda(\tau)$ is a modular function of the modular subgroup

$$\Gamma(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \pmod{2},\ b \equiv c \equiv 0 \pmod{2} \right\},$$

under the following we restrict τ to the fundamental domain of $\Gamma(2)$

$$\Gamma(2) \setminus \mathfrak{H} \simeq \left\{ \tau \in \mathfrak{H} \,\middle|\, |\mathrm{Re}\,\tau| \le 1,\ \left|\tau \pm \frac{1}{2}\right| \ge \frac{1}{2} \right\}.$$

The elliptic functions $\mathrm{cs}\,(2Kz,k)$, $\mathrm{ds}\,(2Kz,k)$ and $\mathrm{ns}\,(2Kz,k)$ are defined by

$$\begin{aligned}
\mathrm{cs}\,(2Kz,k) &:= \frac{\mathrm{cn}\,(2Kz,k)}{\mathrm{sn}\,(2Kz,k)},\\
\mathrm{ds}\,(2Kz,k) &:= \frac{\mathrm{dn}\,(2Kz,k)}{\mathrm{sn}\,(2Kz,k)},\\
\mathrm{ns}\,(2Kz,k) &:= \frac{1}{\mathrm{sn}\,(2Kz,k)}.
\end{aligned}$$

The elliptic function $\mathrm{cs}\,(2Kz,k)$ is regarded as an elliptic analogue of $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$. Similarly, $\mathrm{ds}\,(2Kz,k)$ and $\mathrm{ns}\,(2Kz,k)$ are regarded as an elliptic analogue of $\csc(\pi z) = \frac{1}{\sin(\pi z)}$. According to the wolfram functions site [6] and [5], we list fundamental properties of $\mathrm{cs}\,(2Kz,k)$, $\mathrm{ds}\,(2Kz,k)$ and $\mathrm{ns}\,(2Kz,k)$.

Lemma 1. (1) *(parity)*

$$\operatorname{cs}(-2Kz,k) = -\operatorname{cs}(2Kz,k), \tag{2.1}$$
$$\operatorname{ds}(-2Kz,k) = -\operatorname{ds}(2Kz,k), \tag{2.2}$$
$$\operatorname{ns}(-2Kz,k) = -\operatorname{ns}(2Kz,k). \tag{2.3}$$

(2) *(periodicity) For any $\mu, \nu \in \mathbb{Z}$,*

$$\operatorname{cs}(2K(z+\mu\tau+\nu),k) = (-1)^{\mu}\operatorname{cs}(2Kz,k), \tag{2.4}$$
$$\operatorname{ds}(2K(z+\mu\tau+\nu),k) = (-1)^{\mu+\nu}\operatorname{ds}(2Kz,k), \tag{2.5}$$
$$\operatorname{ns}(2K(z+\mu\tau+\nu),k) = (-1)^{\nu}\operatorname{ns}(2Kz,k). \tag{2.6}$$

(3) *(Laurent expansions at $z=0$)*

$$2K\operatorname{cs}(2Kz,k) = \frac{1}{z} + \left(-\frac{1}{3}+\frac{1}{6}\lambda\right)(2K)^2 z + \left(-\frac{1}{45}+\frac{1}{45}\lambda+\frac{7}{360}\lambda^2\right)(2K)^4 z^3 + \cdots, \tag{2.7}$$

$$2K\operatorname{ds}(2Kz,k) = \frac{1}{z} + \left(\frac{1}{6}-\frac{1}{3}\lambda\right)(2K)^2 z + \left(\frac{7}{360}+\frac{1}{45}\lambda-\frac{1}{45}\lambda^2\right)(2K)^4 z^3 + \cdots, \tag{2.8}$$

$$2K\operatorname{ns}(2Kz,k) = \frac{1}{z} + \left(\frac{1}{6}+\frac{1}{6}\lambda\right)(2K)^2 z + \left(\frac{7}{360}-\frac{11}{180}\lambda+\frac{7}{360}\lambda^2\right)(2K)^4 z^3 + \cdots. \tag{2.9}$$

(4) *(Partial fraction expansions)*

$$2K\operatorname{cs}(2Kz,k) = \sum_{m\in\mathbb{Z}}{}^{e}\sum_{n\in\mathbb{Z}}{}^{e}\frac{(-1)^m}{m\tau+n+z}, \tag{2.10}$$

$$2K\operatorname{ds}(2Kz,k) = \sum_{m\in\mathbb{Z}}{}^{e}\sum_{n\in\mathbb{Z}}{}^{e}\frac{(-1)^{m+n}}{m\tau+n+z}, \tag{2.11}$$

$$2K\operatorname{ns}(2Kz,k) = \sum_{m\in\mathbb{Z}}{}^{e}\sum_{n\in\mathbb{Z}}{}^{e}\frac{(-1)^{n}}{m\tau+n+z}, \tag{2.12}$$

where $\sum_{n\in\mathbb{Z}}{}^{e}$ is the Eisenstein convention

$$\sum_{n\in\mathbb{Z}}{}^{e} f(n) := f(0) + \sum_{n=1}^{\infty}\{f(n)+f(-n)\}.$$

In particular, for any non zero constant A, we have

$$\lim_{z\to-\frac{\mu\tau+\nu}{A}}\left(z+\frac{\mu\tau+\nu}{A}\right)2K\mathrm{cs}\,(2KAz,k)=\frac{1}{A}(-1)^{\mu}, \tag{2.13}$$

$$\lim_{z\to-\frac{\mu\tau+\nu}{A}}\left(z+\frac{\mu\tau+\nu}{A}\right)2K\mathrm{ds}\,(2KAz,k)=\frac{1}{A}(-1)^{\mu+\nu}, \tag{2.14}$$

$$\lim_{z\to-\frac{\mu\tau+\nu}{A}}\left(z+\frac{\mu\tau+\nu}{A}\right)2K\mathrm{ns}\,(2KAz,k)=\frac{1}{A}(-1)^{\nu}. \tag{2.15}$$

(6) *(Fourier expansions)*

$$2K\mathrm{cs}\,(2Kz,k)=\pi\cot(\pi z)+\sum_{m=1}^{\infty}\frac{(-1)^m\pi\sin(2\pi z)}{\sin(\pi(z+m\tau))\sin(\pi(z-m\tau))}, \tag{2.16}$$

$$2K\mathrm{ds}\,(2Kz,k)=\sum_{m\in\mathbb{Z}}\frac{\pi}{\sin(\pi(z+m\tau))}, \tag{2.17}$$

$$2K\mathrm{ns}\,(2Kz,k)=\sum_{m\in\mathbb{Z}}\frac{(-1)^m\pi}{\sin(\pi(z+m\tau))}. \tag{2.18}$$

(4) *(Derivations)*

$$\mathrm{cs}'(z,k)=-\mathrm{ds}\,(z,k)\mathrm{ns}\,(z,k), \tag{2.19}$$

$$\mathrm{ds}'(z,k)=-\mathrm{cs}\,(z,k)\mathrm{ns}\,(z,k), \tag{2.20}$$

$$\mathrm{ns}'(z,k)=-\mathrm{cs}\,(z,k)\mathrm{ds}\,(z,k). \tag{2.21}$$

(5) *(Relations between the Weierstrass $\wp$ function)*

$$(2K\mathrm{cs}\,(2Kz,k))^2=\wp(z,\tau)-\wp\left(\frac{1}{2},\tau\right), \tag{2.22}$$

$$(2K\mathrm{ds}\,(2Kz,k))^2=\wp(z,\tau)-\wp\left(\frac{1+\tau}{2},\tau\right), \tag{2.23}$$

$$(2K\mathrm{ns}\,(2Kz,k))^2=\wp(z,\tau)-\wp\left(\frac{\tau}{2},\tau\right), \tag{2.24}$$

$$\wp'(z,\tau)^2=4(2K\mathrm{cs}\,(2Kz,k))(2K\mathrm{ds}\,(2Kz,k))(2K\mathrm{ns}\,(2Kz,k)). \tag{2.25}$$

Here, $\wp(z,\tau)$ is the Weierstrass $\wp$ function defined by

$$\wp(z,\tau):=\frac{1}{z^2}+\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\left\{\frac{1}{(m\tau+n+z)^2}-\frac{1}{(m\tau+n)^2}\right\}.$$

(5) *(trigonometric degenerations)*

$$\lim_{\tau\to\sqrt{-1}\infty} k(\tau) = 0, \tag{2.26}$$

$$\lim_{\tau\to\sqrt{-1}\infty} 2K(\tau) = \pi, \tag{2.27}$$

$$\lim_{\tau\to\sqrt{-1}\infty} \mathrm{cs}\,(2K(z+w\tau),k) = \begin{cases} (-1)^w \cot(\pi z) & (w\in\mathbb{Z}) \\ -\sqrt{-1}(-1)^{\lfloor w\rfloor} & (w\notin\mathbb{Z}) \end{cases}, \tag{2.28}$$

$$\lim_{\tau\to\sqrt{-1}\infty} \mathrm{ds}\,(2K(z+w\tau),k) = \begin{cases} (-1)^w \csc(\pi z) & (w\in\mathbb{Z}) \\ 0 & (w\notin\mathbb{Z}) \end{cases}, \tag{2.29}$$

$$\lim_{\tau\to\sqrt{-1}\infty} \mathrm{ns}\,(2K(z+w\tau),k) = \begin{cases} \csc(\pi z) & (w\in\mathbb{Z}) \\ 0 & (w\notin\mathbb{Z}) \end{cases}. \tag{2.30}$$

Here, $\lfloor w\rfloor$ denotes the greatest integer not exceeding w.

For convenience, we put

$$\begin{aligned} f_{1,0}(z,\tau) &:= 2K\mathrm{cs}\,(2Kz,k), \quad C_{1,0}(\tau) := \left(-\frac{1}{3}+\frac{1}{6}\lambda\right)(2K)^2, \\ f_{1,1}(z,\tau) &:= 2K\mathrm{ds}\,(2Kz,k), \quad C_{1,1}(\tau) := \left(\frac{1}{6}-\frac{1}{3}\lambda\right)(2K)^2, \\ f_{0,1}(z,\tau) &:= 2K\mathrm{ns}\,(2Kz,k), \quad C_{0,1}(\tau) := \left(\frac{1}{6}+\frac{1}{6}\lambda\right)(2K)^2. \end{aligned}$$

According to these notations, we have the following expressions of parity (2.1) - (2.3), periodicity (2.4) - (2.6), Laurent expansions at $z=0$ (2.7) - (2.9), partial fraction expansions (2.10) - (2.12), residues at simple poles (2.13) - (2.15), derivations (2.19) - (2.21) and relations between the Weierstrass $\wp$

function (2.22) - (2.24) respectively.

$$f_{i,j}(-z,\tau) = -f_{i,j}(z,\tau), \tag{2.31}$$

$$f_{i,j}(z+\mu\tau+\nu,\tau) = (-1)^{i\mu+j\nu} f_{i,j}(z,\tau), \tag{2.32}$$

$$f_{i,j}(z,\tau) = \frac{1}{z} + C_{i,j}(\tau)z + O(z^3), \tag{2.33}$$

$$f_{i,j}(z,\tau) = \sum_{m\in\mathbb{Z}}{}^{e} \sum_{n\in\mathbb{Z}}{}^{e} \frac{(-1)^{im+jn}}{m\tau+n+z}, \tag{2.34}$$

$$\lim_{z\to -\frac{\mu\tau+\nu}{A}} \left(z + \frac{\mu\tau+\nu}{A}\right) f_{i,j}(Az,\tau) = \frac{1}{A}(-1)^{i\mu+j\nu}, \tag{2.35}$$

$$f'_{i,j}(z,\tau) = -f_{i+1,1}(z,\tau) f_{1,j+1}(z,\tau), \tag{2.36}$$

$$f_{i,j}(z,\tau)^2 = \wp(z,\tau) - \wp\left(\frac{i+j\tau}{2},\tau\right). \tag{2.37}$$

Here indices of $f_{i,j}(z,\tau)$ are regarded as elements in $\mathbb{Z}/2\mathbb{Z}$.

Remark 2. (1) *Fukuhara-Yui's notation is*

$$\varphi(\tau,z) := \sqrt{\wp(z,\tau) - \wp\left(\frac{1}{2},\tau\right)} = \frac{1}{z} + O(z) \quad (z\to 0)$$

which is equal to $2K\mathrm{cs}\,(2Kz,k)$ *from (2.7) and (2.22). However, Fukuhara-Yui did not mention that* $\varphi(\tau,z)$ *is the Jacobi elliptic function* $2K\mathrm{cs}\,(2Kz,k)$ *exactly.*
(2) *If we use Mumford's notations [4]*

$$\theta_{0,0}(z,\tau) := \theta_3(z,\tau), \quad \theta_{1,0}(z,\tau) := \theta_2(z,\tau), \quad \theta_{0,1}(z,\tau) := \theta_4(z,\tau), \quad \theta_{1,1}(z,\tau) := -\theta_1(z,\tau),$$

then our $f_{i,j}(z,\tau)$ *is written by*

$$f_{i,j}(z,\tau) = -\frac{\pi}{2}\theta_{i,0}(0,\tau)\theta_{0,j}(0,\tau)\frac{\theta_{j+1,i+1}(z,\tau)}{\theta_{1,1}(z,\tau)}.$$

3 Main results

Under the following we assume a and b are relatively prime positive numbers and $i,j,m,n \in \{0,1\}$. We mention and prove the main theorem.

Theorem 3. *If $ia+mb$ or $ja+nb$ is odd, then*

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{i\mu+j\nu}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)f_{ia+mb,ja+nb}\left(z+\frac{\mu\tau+\nu}{a},\tau\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{m\mu+n\nu}f_{i,j}\left(a\frac{\mu\tau+\nu}{b},\tau\right)f_{ia+mb,ja+nb}\left(z+\frac{\mu\tau+\nu}{b},\tau\right)$$
$$=-f_{i,j}(az,\tau)f_{m,n}(bz,\tau)+\frac{1}{ab}f_{ia+mb+1,1}(z,\tau)f_{1,ja+nb+1}(z,\tau). \tag{3.1}$$

Proof. We put

$$\Phi_{(i,j),(m,n)}((a,b),z,\tau):=\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{i\mu+j\nu}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)f_{ia+mb,ja+nb}\left(z+\frac{\mu\tau+\nu}{a},\tau\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{m\mu+n\nu}f_{i,j}\left(a\frac{\mu\tau+\nu}{b},\tau\right)f_{ia+mb,ja+nb}\left(z+\frac{\mu\tau+\nu}{b},\tau\right),$$

$$\Psi_{(i,j),(m,n)}((a,b),z,\tau):=-f_{i,j}(az,\tau)f_{m,n}(bz,\tau)+\frac{1}{ab}f_{ia+mb+1,1}(z,\tau)f_{1,ja+nb+1}(z,\tau)$$
$$=-f_{i,j}(az,\tau)f_{m,n}(bz,\tau)-\frac{1}{ab}f'_{ia+mb,ja+nb}(z,\tau).$$

and

$$U_{(i,j),(m,n)}((a,b),z,\tau):=\Phi_{(i,j),(m,n)}((a,b),z,\tau)-\Psi_{(i,j),(m,n)}((a,b),z,\tau).$$

Under the condition $2\nmid ia+mb$ or $2\nmid ja+nb$, we claim that

$$U_{(i,j),(m,n)}((a,b),z,\tau)\equiv 0.$$

First we show that $\Phi_{(i,j),(m,n)}((a,b),z,\tau)$ and $\Psi_{(i,j),(m,n)}((a,b),z,\tau)$ have same periodicity. From periodicity (2.32), for any integers M and N we have

$$\Psi_{(i,j),(m,n)}((a,b),z+M\tau+N,\tau)$$
$$=-f_{i,j}(a(z+M\tau+N),k)f_{m,n}(b(z+M\tau+N),k)$$
$$+\frac{1}{ab}f_{ia+mb+1,1}(z+M\tau+N,\tau)f_{1,ja+nb+1}(z+M\tau+N,\tau)$$
$$=-(-1)^{iaM+jaN}f_{i,j}(az,k)(-1)^{mbM+nbN}f_{m,n}(bz,k)$$
$$+\frac{1}{ab}(-1)^{(ia+mb+1)M+N}f_{ia+mb+1,1}(z,\tau)(-1)^{M+(ja+nb+1)N}f_{1,ja+nb+1}(z,\tau)$$
$$=(-1)^{(ia+mb)M+(ja+nb)N}\Psi_{(i,j),(m,n)}((a,b),z,\tau).$$

Similarly, for $\Phi_{(i,j),(m,n)}((a,b),z,\tau)$ we have

$$\Phi_{(i,j),(m,n)}((a,b),z+M\tau+N,\tau) = (-1)^{(ia+mb)M+(ja+nb)N}\Phi_{(i,j),(m,n)}((a,b),z,\tau).$$

Thus we obtain double periodicity of $U_{(i,j),(m,n)}((a,b),z,\tau)$

$$U_{(i,j),(m,n)}((a,b),z+M\tau+N,\tau) = (-1)^{(ia+mb)M+(ja+nb)N}U_{(i,j),(m,n)}((a,b),z,\tau). \tag{3.2}$$

Next we consider all the poles of $U_{(i,j),(m,n)}((a,b),z,\tau)$ and their Laurent expansions. Note that $\Phi_{(i,j),(m,n)}((a,b),z,\tau)$, $\Psi_{(i,j),(m,n)}((a,b),z,\tau)$ and $U_{(i,j),(m,n)}((a,b),z,\tau)$ are holomorphic at $z=0$. Actually, since a and b are relatively prime positive integers, the Laurent expansions of $\Phi_{(i,j),(m,n)}((a,b),z,\tau)$ at $z=0$ is the following.

$$\begin{aligned}\Phi_{(i,j),(m,n)}((a,b),z,\tau) &= \frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{i\mu+j\nu}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)f_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{a},\tau\right)\\&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{m\mu+n\nu}f_{i,j}\left(a\frac{\mu\tau+\nu}{b},\tau\right)f_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{b},\tau\right)+O(z).\end{aligned}$$

On the other hand, from (2.36) and (2.33), we have

$$\begin{aligned}\Psi_{(i,j),(m,n)}((a,b),z,\tau) &= -f_{i,j}(az,\tau)f_{m,n}(bz,\tau)-\frac{1}{ab}f'_{ia+mb,ja+nb}(z,\tau)\\&= -\left(\frac{1}{az}+C_{i,j}(\tau)az+O(z^3)\right)\left(\frac{1}{bz}+C_{m,n}(\tau)bz+O(z^3)\right)\\&\quad-\frac{1}{ab}\left(-\frac{1}{z^2}+C_{ia+mb,ja+nb}(\tau)+O(z^2)\right)\\&= -\frac{b}{a}C_{m,n}(\tau)-\frac{a}{b}C_{i,j}(\tau)-\frac{1}{ab}C_{ia+mb,ja+nb}(\tau)+O(z^2).\end{aligned}$$

Then we obtain the Laurent expansion of $U_{(i,j),(m,n)}((a,b),z,\tau)$ at $z=0$

$$\begin{aligned} U_{(i,j),(m,n)}((a,b),z,\tau) &= -\frac{b}{a}C_{m,n}(\tau) - \frac{a}{b}C_{i,j}(\tau) - \frac{1}{ab}C_{ia+mb,ja+nb}(\tau) \\ &\quad - \frac{1}{a}\sum_{\substack{\mu,\nu=0 \\ (\mu,\nu)\neq(0,0)}}^{a-1} (-1)^{i\mu+j\nu} f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right) f_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{a},\tau\right) \\ &\quad - \frac{1}{b}\sum_{\substack{\mu,\nu=0 \\ (\mu,\nu)\neq(0,0)}}^{b-1} (-1)^{m\mu+n\nu} f_{i,j}\left(a\frac{\mu\tau+\nu}{b},\tau\right) f_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{b},\tau\right) + O(z). \end{aligned} \tag{3.3}$$

Hence we investigate other poles. By the definition or partial fractional expansion (2.34) of $f_{i,j}(z,\tau)$, all other poles of $\Phi_{(i,j),(m,n)}((a,b),z,\tau)$ and $\Psi_{(i,j),(m,n)}((a,b),z,\tau)$ are

$$-\frac{\mu\tau+\nu}{a} + M\tau + N, \quad \mu,\nu = 0,1,\ldots,a-1, \quad (\mu,\nu)\neq(0,0), \quad M,N\in\mathbb{Z}$$

or

$$-\frac{\mu\tau+\nu}{b} + M\tau + N, \quad \mu,\nu = 0,1,\ldots,b-1, \quad (\mu,\nu)\neq(0,0), \quad M,N\in\mathbb{Z}.$$

Furthermore, all the poles are simple and the residues at these poles are equal. Actually, from (2.35), we have

$$\begin{aligned} &\lim_{z\to-\frac{\mu\tau+\nu}{a}-M\tau-N} \left(z + \frac{\mu\tau+\nu}{a} + M\tau + N\right)\Phi_{(i,j),(m,n)}((a,b),z,\tau) \\ &\quad = -(-1)^{iaM+jaN} f_{m,n}\left(-b\frac{\mu\tau+\nu}{a} - bM\tau - bN,\tau\right) \\ &\qquad \cdot \lim_{z+M\tau+N\to-\frac{\mu\tau+\nu}{a}} \left(z + M\tau + N + \frac{\mu\tau+\nu}{a}\right) f_{i,j}(a(z+M\tau+N),\tau) \\ &\quad = (-1)^{iaM+jaN}(-1)^{mbM+nbN} f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)\frac{(-1)^{i\mu+j\nu}}{a} \\ &\quad = \frac{1}{a}(-1)^{i\mu+j\nu}(-1)^{(ia+mb)M+(ja+nb)N} f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right) \end{aligned}$$

and

$$\lim_{z\to -\frac{\mu\tau+\nu}{a}-M\tau-N}\left(z+\frac{\mu\tau+\nu}{a}+M\tau+N\right)\Psi_{(i,j),(m,n)}((a,b),z,\tau)$$
$$=\frac{1}{a}(-1)^{i\mu+j\nu}(-1)^{(ia+mb)M+(ja+nb)N}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)$$
$$\cdot\lim_{z+M\tau+N+\frac{\mu\tau+\nu}{a}\to 0}\left(z+M\tau+N+\frac{\mu\tau+\nu}{a}\right)f_{ia+mb,ja+nb}\left(z+M\tau+N+\frac{\mu\tau+\nu}{a},\tau\right)$$
$$=\frac{1}{a}(-1)^{i\mu+j\nu}(-1)^{(ia+mb)M+(ja+nb)N}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right).$$

Thus for $M,N\in\mathbb{Z}$, $\mu,\nu\in\{0,1,\dots,a-1\}$ and $(\mu,\nu)\neq(0,0)$,

$$\lim_{z\to -\frac{\mu\tau+\nu}{a}-M\tau-N}\left(z+M\tau+N+\frac{\mu\tau+\nu}{a}\right)U_{(i,j),(m,n)}((a,b),z,\tau)=0.$$

Similarly, for $M,N\in\mathbb{Z}$, $\mu,\nu\in\{0,1,\dots,b-1\}$ and $(\mu,\nu)\neq(0,0)$ we have

$$\lim_{z\to -\frac{\mu\tau+\nu}{b}-M\tau-N}\left(z+M\tau+N+\frac{\mu\tau+\nu}{b}\right)U_{(i,j),(m,n)}((a,b),z,\tau)=0.$$

Therefore $U_{(i,j),(m,n)}((a,b),z,\tau)$ is an entire function.

Summarizing the above discussion, $U_{(i,j),(m,n)}((a,b),z,\tau)$ is a doubly periodic entire function on $\mathbb{C}$. Then by the well-known Liouville theorem, there exists a constant $c_{(i,j),(m,n)}((a,b),\tau)$ such that

$$U_{(i,j),(m,n)}((a,b),z,\tau)=c_{(i,j),(m,n)}((a,b),\tau).$$

If $ia+mb$ is odd, changing the variable from z to $z+\tau$, we have

$$\begin{aligned}c_{(i,j),(m,n)}((a,b),\tau)&=U_{(i,j),(m,n)}((a,b),z+\tau,\tau)\\&=(-1)^{ia+mb}U_{(i,j),(m,n)}((a,b),z,\tau)\\&=-U_{(i,j),(m,n)}((a,b),z,\tau)\\&=-c_{(i,j),(m,n)}((a,b),\tau).\end{aligned}$$

Here the second equality follows from double periodicity (3.2). If $ia+mb$ is even, from the assumption of theorem, then $ja+nb$ is odd. Hence, changing the variable from z to $z+1$, we have

$$\begin{aligned}c_{(i,j),(m,n)}((a,b),\tau)&=U_{(i,j),(m,n)}((a,b),z+1,\tau)\\&=(-1)^{ja+nb}U_{(i,j),(m,n)}((a,b),z,\tau)\\&=-U_{(i,j),(m,n)}((a,b),z,\tau)\\&=-c_{(i,j),(m,n)}((a,b),\tau).\end{aligned}$$

Therefore $c_{(i,j),(m,n)}((a,b),\tau) \equiv 0$ in all cases and we obtain the conclusion. $\square$

Expanding both side of (3.1) and comparing coefficients of z^{2N}, we obtain reciprocity laws for elliptic Dedekind sums, which is a natural generalization of Fukuhara-Yui's main result (Theorem 2.2 (1) in [3]).

Theorem 4 (Reciprocity laws for elliptic Dedekind sums). *If $ia+mb$ or $ja+nb$ is odd, then*

$$\begin{aligned}
&\frac{1}{(2N)!}\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{i\mu+j\nu}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)f^{(2N)}_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{a},\tau\right)\\
&\quad+\frac{1}{(2N)!}\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{m\mu+n\nu}f_{i,j}\left(a\frac{\mu\tau+\nu}{b},\tau\right)f^{(2N)}_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{b},\tau\right)\\
&\quad=-\sum_{s=0}^{N}C_{i,j}(s,\tau)C_{m,n}(N-s,\tau)a^{2s-1}b^{2N-2s-1}-\frac{1}{ab}(2N+1)C_{ia+mb,ja+nb}(N,\tau),
\end{aligned} \tag{3.4}$$

where $f^{(2N)}_{ia+mb,ja+nb}(z)$ is the $2N$-th derivative of the $f_{ia+mb,ja+nb}(z)$, and $C_{i,j}(s,\tau)$ is the coefficients of the Laurent expansions of $f_{i,j}(z,\tau)$ at $z=0$

$$f_{ij}(z,\tau)=\frac{1}{z}\sum_{s=0}^{\infty}C_{i,j}(s,\tau)z^{2s},\quad C_{i,j}(0,\tau)=1,\quad C_{i,j}(1,\tau):=C_{i,j}(\tau).$$

In particular, considering the case of $N=0$ of (3.4) or taking the limit $z\to 0$ in (3.3), we obtain reciprocity laws for elliptic Dedekind sums.

Corollary 5. *If $ia+mb$ or $ja+nb$ is odd, then*

$$\begin{aligned}
&\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{i\mu+j\nu}f_{m,n}\left(b\frac{\mu\tau+\nu}{a},\tau\right)f_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{a},\tau\right)\\
&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{m\mu+n\nu}f_{i,j}\left(a\frac{\mu\tau+\nu}{b},\tau\right)f_{ia+mb,ja+nb}\left(\frac{\mu\tau+\nu}{b},\tau\right)\\
&\quad=-\frac{b}{a}C_{m,n}(\tau)-\frac{a}{b}C_{i,j}(\tau)-\frac{1}{ab}C_{ia+mb,ja+nb}(\tau).
\end{aligned} \tag{3.5}$$

4 All the examples of (3.1) and (3.5)

In this section, we give examples of (3.1) and (3.5) up to the constant factor $(2K)^2$ explicitly.

4.1 $(i,j)=(1,0)$, $(m,n)=(1,0)$

4.1.1 $2 \not| a+b$

In this case, (3.1) and (3.5) are Fukuhara-Yui's results (1.11) and (1.12) respectively.

4.2 $(i,j)=(1,1)$, $(m,n)=(1,0)$

4.2.1 $2 \not| a+b$, $2 \not| a$

$$
\begin{aligned}
&\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)\\
&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)\\
&\quad=-\mathrm{ds}\,(2Kaz,k)\mathrm{cs}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{cs}\,(2Kz,k),
\end{aligned}
\tag{4.1}
$$

$$
\begin{aligned}
&\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{a},k\right)\\
&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{b},k\right)\\
&\quad=\frac{-a^2+2b^2-1}{6ab}+\frac{2a^2-b^2+2}{6ab}\lambda(\tau).
\end{aligned}
\tag{4.2}
$$

By taking the limit $\tau\to\sqrt{-1}\infty$ and (2.26) - (2.30), (4.1) and (4.2) degenerate to (1.4) and (1.9) respectively.

4.2.2 $2 \nmid a+b,\ 2 \mid a$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ds}\,(2Kaz,k)\mathrm{cs}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{ds}\,(2Kz,k), \tag{4.3}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2+2b^2+2}{6ab}+\frac{2a^2-b^2-1}{6ab}\lambda(\tau). \tag{4.4}$$

By taking the limit $\tau\to\sqrt{-1}\infty$ and (2.26) - (2.30), we have

$$\lim_{\tau\to\sqrt{-1}\infty}\Phi_{(1,1),(1,0)}((a,b),z,\tau)$$
$$=\lim_{\tau\to\sqrt{-1}\infty}\frac{1}{a}\sum_{\nu=1}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\nu}{a},k\right)\mathrm{cs}\left(2K\left(z+\frac{\nu}{a}\right),k\right)$$
$$+\lim_{\tau\to\sqrt{-1}\infty}\frac{1}{a}\sum_{\mu=1}^{a-1}\sum_{\nu=0}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\lim_{\tau\to\sqrt{-1}\infty}\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=\frac{1}{a}\sum_{\nu=1}^{a-1}(-1)^{\nu}\cot\left(\frac{\pi b\nu}{a}\right)\cot\left(\pi\left(z+\frac{\nu}{a}\right)\right)+\frac{1}{b}\sum_{\nu=1}^{b-1}\csc\left(\frac{\pi a\nu}{b}\right)\cot\left(\pi\left(z+\frac{\nu}{b}\right)\right)$$
$$+\frac{1}{a}\sum_{\mu=1}^{a-1}(-1)^{\lfloor\frac{b\mu}{a}\rfloor+\mu-1}\sum_{\nu=0}^{a-1}(-1)^{\nu}, \tag{4.5}$$

$\lim_{\tau\to\sqrt{-1}\infty}\Psi_{(1,1),(1,0)}((a,b),z,\tau)=-\csc(\pi az)\cot(\pi bz)+\frac{1}{ab}\csc(\pi z)^2.$

Since a is even and the third term in (4.5) vanishes, (4.3) and (4.4) degenerate to (1.2) and (1.7) respectively.

4.2.3 $2 \mid a+b$, $2 \nmid a$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ds}\,(2Kaz,k)\mathrm{cs}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ds}\,(2Kz,k)\mathrm{cs}\,(2Kz,k), \tag{4.6}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2+2b^2-1}{6ab}+\frac{2a^2-b^2-1}{6ab}\lambda(\tau). \tag{4.7}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.6) and (4.7) degenerate to (1.4) and (1.9) respectively.

4.3 $(i,j)=(0,1)$, $(m,n)=(1,0)$

4.3.1 $2 \nmid b$, $2 \nmid a$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ns}\,(2Kaz,k)\mathrm{cs}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{cs}\,(z,k), \tag{4.8}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2+2b^2-1}{6ab}+\frac{-a^2-b^2+2}{6ab}\lambda(\tau). \tag{4.9}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.8) and (4.9) degenerate to (1.4) and (1.9) respectively.

4.3.2 $2\nmid b,\ 2\mid a$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ns}\,(2Kaz,k)\mathrm{cs}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{ds}\,(2Kz,k), \tag{4.10}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2+2b^2+2}{6ab}+\frac{-a^2-b^2-1}{6ab}\lambda(\tau). \tag{4.11}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.10) and (4.11) degenerate to (1.2) and (1.7) respectively.

4.3.3 $2 \mid b$**,** $2 \nmid a$

$$\begin{aligned}
&\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)\\
&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)\\
&\quad=-\mathrm{ns}\,(2Kaz,k)cs(2Kbz,k)+\frac{1}{ab}\mathrm{ds}\,(2Kz,k)\mathrm{cs}\,(2Kz,k), \qquad (4.12)\\
&\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{cs}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{a},k\right)\\
&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{b},k\right)\\
&\quad=\frac{-a^2+2b^2-1}{6ab}+\frac{-a^2-b^2-1}{6ab}\lambda(\tau). \qquad (4.13)
\end{aligned}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.12) and (4.13) degenerate to (1.4) and (1.9) respectively.

4.4 $(i,j)=(1,1)$, $(m,n)=(1,1)$

4.4.1 $2 \nmid a+b$

$$\begin{aligned}
&\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)\\
&\quad+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)\\
&\quad=-\mathrm{ds}\,(2Kaz,k)\mathrm{ds}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{cs}\,(2Kz,k), \qquad (4.14)
\end{aligned}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\mu+\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ds}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=-\frac{a^2+b^2+1}{6ab}\left(1-2\lambda(\tau)\right). \tag{4.15}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.14) and (4.15) degenerate to (1.5) and (1.10) respectively.

4.5 $(i,j)=(0,1)$, $(m,n)=(1,1)$

4.5.1 $2\nmid b$, $2\nmid a+b$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ns}\,(2Kaz,k)\mathrm{ds}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{cs}\,(2Kz,k), \tag{4.16}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ds}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2-b^2-1}{6ab}+\frac{-a^2+2b^2+2}{6ab}\lambda(\tau). \tag{4.17}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.16) and (4.17) degenerate to (1.5) and (1.10) respectively.

4.5.2 $2 \nmid b$, $2 \mid a+b$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ns}\,(2Kaz,k)\mathrm{ds}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ns}\,(2Kz,k)\mathrm{ds}\,(2Kz,k), \tag{4.18}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{cs}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{cs}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2-b^2+2}{6ab}+\frac{-a^2+2b^2-1}{6ab}\lambda(\tau). \tag{4.19}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.18) and (4.19) degenerate to (1.3) and (1.8) respectively.

4.5.3 $2 \mid b$, $2 \nmid a+b$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ns}\,(2Kaz,k)\mathrm{ds}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ds}\,(2Kz,k)\mathrm{cs}\,(2Kz,k), \tag{4.20}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ds}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\mu+\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=\frac{-a^2-b^2-1}{6ab}+\frac{-a^2+2b^2-1}{6ab}\lambda(\tau). \tag{4.21}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.20) and (4.21) degenerate to (1.5) and (1.10) respectively.

4.6 $(i,j)=(0,1)$, $(m,n)=(0,1)$

4.6.1 $2\nmid a+b$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ns}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{a}\right),k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\left(z+\frac{\mu\tau+\nu}{b}\right),k\right)$$
$$=-\mathrm{ns}\,(2Kaz,k)\mathrm{ns}\,(2Kbz,k)+\frac{1}{ab}\mathrm{ds}\,(2Kz,k)\mathrm{cs}\,(2Kz,k), \tag{4.22}$$

$$\frac{1}{a}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{a-1}(-1)^{\nu}\mathrm{ns}\left(2Kb\frac{\mu\tau+\nu}{a},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{a},k\right)$$
$$+\frac{1}{b}\sum_{\substack{\mu,\nu=0\\(\mu,\nu)\neq(0,0)}}^{b-1}(-1)^{\nu}\mathrm{ns}\left(2Ka\frac{\mu\tau+\nu}{b},k\right)\mathrm{ns}\left(2K\frac{\mu\tau+\nu}{b},k\right)$$
$$=-\frac{a^2+b^2+1}{6ab}(1+\lambda(\tau)). \tag{4.23}$$

Taking the limit $\tau\to\sqrt{-1}\infty$, (4.22) and (4.23) degenerate to (1.5) and (1.10) respectively.

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