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New Result on Jeśmanowicz' conjecture

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Abstract

Let (a, b, c) be a primitive Pythagorean triple. The Jeśmanowicz' conjecture, written in 1956, states that the only positive integer solution to the Diophantine equation $(an)^x + (bn)^y = (cn)^z$ is $(x, y, z) = (2, 2, 2)$, where n is an arbitrary positive integer. Let p be an arbitrary prime greater than 3 and let α and β be positive integers and y belongs to the set of even positive integers. In this paper, we show that if either $P(a)|n$ or $P(n) \nmid a$, then the Jeśmanowicz' conjecture is true for Pythagorean triples $(a, b, c) = \big(4k^2 - 1, 4k, 4k^2 + 1\big)$ with $k = 2^\alpha p^\beta$, where $P(r)$ denotes the product of distinct prime factors of r for any positive integer r greater than 1.

1 Introduction

In 1956, Jeśmanowicz [2] showed that the only positive integer solution of the Diophantine equation

$$(an)^x + (bn)^y = (cn)^z, \tag{1.1}$$

is $(x, y, z) = (2, 2, 2)$ for $n = 1$ and $(a, b, c) \in \{(5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\}$. In 1998, Deng and Cohen [1] proved the following two results:
First, if $(a, b, c) = (2k + 1, 2k(k + 1), 2k(k + 1) + 1)$ for some positive integer

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k with a being a prime power and n being a positive integer such that either $P(b) \mid n$ or $P(n) \nmid b$, then Jeśmanowicz' conjecture is true.
Secondly, if $(a, b, c) \in \{(3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\}$, then Jeśmanowicz' conjecture is true as well.
In 1999, Le [3] gave specific conditions for the equation (1.1) to have positive integer solutions (x, y, z) with $(x, y, z) \neq (2, 2, 2)$. In particular, he showed that x, y and z must be distinct. In 2012, Yang and Tang [10] proved that the conjecture is true for $(a, b, c) = (8, 15, 17)$; i.e., the only solution of the Diophantine equation $(8n)^x + (15n)^y = (17n)^z$ is $(x, y, z) = (2, 2, 2)$, for $n \geq 1$. In 2013, Min Tang and Zhi-Juan Yang [9] considered Jeśmanowicz' conjecture for Pythagorean triples (a, b, c), where $a = c - 2$ and c is is a Fermat prime and proved that if $F_k = 2^{2^k} + 1$ is a Fermat prime, then for any positive integer n, the Diophantine equation

$$((F_k - 2)n)^x + \left(2^{2^{k-1}+1} n\right)^y = (F_k n)^z \,, \tag{1.2}$$

has no solution (x, y, z) satisfying $z < \min\{x, y\}$, where k is a positive integer. They also showed that if $F_k = 2^{2^k} + 1$ and $k \in \{1, 2, 3, 4\}$, then, for any positive integer n, (1.2) has no solution other than $(x, y, z) = (2, 2, 2)$. In 2014, Deng [12] proved that Jeśmanowicz' conjecture is true for Pythagorean triples $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ with $k = 2^s$ for some positive integer s and certain divisibility conditions are satisfied. In 2015, Sun and Cheng [7] considered $k = p^m$, where m is some positive integer and p is a prime such that $p \equiv -1 (b \mod 4)$ and they showed that if the positive integer n is such that either $P(4k^2 - 1) | n$ or $P(n) \nmid (4k^2 - 1)$, then the only solution of the Diophantine equation

$$((4k^2 - 1)n)^x + (4kn)^y = ((4k^2 + 1)n)^z \tag{1.3}$$

is $(x, y, z) = (2, 2, 2)$. In the same year 2015, Ma and Wu [16] proved two results:
First, if $P(4k^2 - 1) | n$, then the only solution of the Diophantine equation (1.3) is $(x, y, z) = (2, 2, 2)$.
Secondly, they considered $k = p^m$, p prime and $m \geq 0$ with $p \equiv -1 (b \mod 4)$ and showed that if n is a positive integer such that $P(n) \nmid (4k^2 - 1)$, then the only solution for the equation (1.3) is $(x, y, z) = (2, 2, 2)$. In 2017, Deng and Dong [13] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$, $u \not\equiv v (b \mod 2)$ and $n = 1$. They showed that Jeśmanowicz' conjecture is true if $(u, v) \equiv (2, 3) (b \mod 4)$ and $v < 100$. In

the same year 2017, Mi and Chen [20] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$, $2 \nmid u + v$ and $n = 1$. As a result, they showed that Jeśmanowicz' conjecture is true if $4 \nmid uv$ and $y \geq 2$. In 2013, Miyazaki [22] broadly extended many of classical well-known results on the conjecture for $n = 1$. In 2014, Terai [15] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$, $u \not\equiv v(\text{mod}2)$ and $n = 1$. He showed that if $v = 2$, then the Jeśmanowicz' conjecture is true. In 2014, Miyazaki Yuan and Wu [24] established the conjecture for the case where b is even and either a or c is congruent to ± 1 modulo the product of all prime factors of b. In 2015, Yang and Ruiqin [23] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$ and $2 \mid uv$. They stated that a positive integer solution (x, y, z, n) of the equation (1.1) is called exceptional if $(x, y, z) \neq (2, 2, 2)$ and $n > 1$. They proved the following results: (i) If $x = y$, $y > z$ and $n > 1$, then the equation (1.1) has no positive integer solutions (x, y, z), (ii) If (x, y, z, n) is an exceptional solution of the question (1.1), then either $y > z > x$ or $x > z > y$, (iii) The equation (1.1) has no exceptional solutions (x, y, z, n) with $y > z > x$, if $u = 2^r$ and $v = 2^r - 1$, where r is a positive integer, (iv) The equation (1.1) has no exceptional solutions (x, y, z, n), if $u = 2^r$ and $v = 2^r - 1$ are odd primes, where r is a positive integer; i.e., the Jeśmanowicz' conjecture is true. In 2018, Hu and Le [25] considered a, b, c as fixed coprime positive integers such that $\min\{a, b, c\} > 1$. He proved that if $\max\{a, b, c\} > 5 \times 10^{27}$, then the equation $a^n + b^n = c^z$ has at most three positive integer solutions (x, y, z). In 2015, Miyazaki and Terai [26] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$, $u \not\equiv v(\bmod 2)$ and $n = 1$. As a result, they proved that if v satisfies at least one of three conditions (i) $v/2$ is a power of an odd prime, (ii) $v/2$ has no prime factors congruent to 1 modulo 8, (iii) $v/2$ is a square, then the conjecture is true. In 2017, Deng and Guo [17] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$, $u \not\equiv v(\text{mod}2)$ and $n = 1$. They proved Jeśmanowicz' conjecture in the following cases: (i) $(u, v) \equiv (1, 2)(\text{mod}4)$, (ii) $(u, v) \equiv (3, 2), (7, 6)(\text{mod}8)$ or $(u, v) \equiv (3, 6), (7, 2), (11, 14), (15, 10)(\text{mod}16)$, (iii) $(u, v) \equiv (3, 14), (7, 10), (11, 6), (15, 2)(\text{mod}16)$ and $y > 1$, where $(u, v) \equiv (s, r)(modd)$ denotes $u \equiv s(\text{mod } d)$ and $v \equiv r(\text{mod}d)$. In 2018, Han and Yuan [14] considered Jeśmanowicz' conjecture for Pythagorean triples $(a, b, c) = (u^2 - v^2, 2uv, u^2 + v^2)$, where u and v are positive integers with $u > v$, $gcd(u, v) = 1$, $u \not\equiv v (\text{mod} 2)$ and $n = 1$. They showed that if $2 \parallel uv$ and $u + v$ has a prime factor p with $p \not\equiv 1 (\text{mod} 16)$, then Jeśmanowicz' conjecture is true. Jeśmanowicz' conjecture has been proved for some special cases (See, [5], [8], [11], [6], [18] and [19]). However, in general, the problem is not solved as yet and it is one of the most famous unsolved problems on Pythagorean triples.

2 Main Result

Our main result is the following Theorem:

Theorem 2.1. *Let $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ be a primitive Pythagorean triple with $k = 2^\alpha p^\beta$ where p is arbitrary prime greater than 3 and α, β are positive integers with y belongs to the set of even positive integers. Suppose that n is a positive integer such that either $P(a)|n$ or $P(n) \nmid a$. Then Jeśmanowicz' conjecture is true.*

3 Preliminary Results

In this section, we provide some lemmas which will be used in the proof of Theorem 2.1.

Lemma 3.1. *(see [4], 39-41) The only positive integer solution of the Diophantine equation $(4k^2 - 1)^x + (4k)^y = (4k^2 + 1)^z$ is $(x, y, z) = (2, 2, 2)$.*

Lemma 3.2. *(see [12], Corollary 2.4) Let (a, b, c) be any primitive Pythagorean triple such that the Diophantine equation $a^x + b^y = c^z$ has the only positive integer solution $(x, y, z) = (2, 2, 2)$. If (x, y, z) is a solution of equation (1.1) with $(x, y, z) \neq (2, 2, 2)$, then one of the following conditions is satisfied:*

1. *$x > z > y$ and $P(n)|b$;*
2. *$y > z > x$ and $P(n)|a$.*

Lemma 3.3. *([16], Theorem 1.1) Let $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ be a primitive Pythagorean triple. If n is a positive integer such that $P(a) \mid n$. Then Eq. (1.1) has only a positive integer solution $(x, y, z) = (2, 2, 2)$.*

In the following lemmas, we will assume that $(a, b, c) = (4k^2 - 1, 4k, 4k^2 + 1)$ with $k = 2^{\alpha}p^{\beta}$, where p is arbitrary prime greater than 3 with α, β are positive integers and y belongs to the set of even positive integers.

Lemma 3.4. *If $n = 2^r$ with $r \geq 1$, then Jeśmanowicz' conjecture is true.*

Proof. Suppose that (x, y, z) is a solution of equation (1.1) with $(x, y, z) \neq (2, 2, 2)$. We will show that this leads to a contradiction. Since $P(n) \nmid a$, by **Lemma 3.2**, we have $x > z > y$ and $P(n)|b$. So rewrite equation (1.1) as

$$b^y = n^{z-y}(c^z - a^x n^{x-z}), \tag{3.4}$$

since $b = 4k = 2^{\alpha+2}p^{\beta}$ and $n = 2^r$ with $r \geq 1$. Hence the equation (3.4) becomes

$$\left[2^{\alpha+2}p^{\beta}\right]^y = 2^{r(z-y)}\left[\left(2^{2\alpha+2}p^{2\beta} + 1\right)^z - \left(2^{2\alpha+2}p^{2\beta} - 1\right)^x 2^{r(x-z)}\right].$$

So

$$2^{(\alpha+2)y} = 2^{r(z-y)},$$

and

$$p^{\beta y} = \left(2^{2\alpha+2}p^{2\beta} + 1\right)^z - \left(2^{2\alpha+2}p^{2\beta} - 1\right)^x 2^{r(x-z)}. \tag{3.5}$$

From equation (3.5), we get $1 \equiv 2^z (\mathrm{mod} 3)$, where y is even. Thus $z \equiv 0 (\mathrm{mod} 2)$. So, let $z = 2z_1$ with $z_1 > 1$ and $\beta y = 2y_1$. Then rewrite equation (3.5) as

$$a^x 2^{r(x-z)} = \left(2^{\alpha+1}p^{\beta} - 1\right)^x \left(2^{\alpha+1}p^{\beta} + 1\right)^x 2^{r(x-z)} = \left(c^{z_1} - p^{y_1}\right)\left(c^{z_1} + p^{y_1}\right).$$

Noting that $(c^{z_1} - p^{y_1}, c^{z_1} + p^{y_1}) = 2$, we can write $a = a_1 a_2$, where $\gcd(a_1, a_2) = 1$ with

$$a_1^x |\, c^{z_1} + p^{y_1} \text{ and } a_2^x |\, c^{z_1} - p^{y_1}. \tag{3.6}$$

Now if $a_1 < 2^{\alpha+1}p^{\beta} + 1$ and $a_2 < 2^{\alpha+1}p^{\beta} + 1$, then

$$a_1 \leq 2^{\alpha+1}p^{\beta} - 1 \text{ and } a_2 \leq 2^{\alpha+1}p^{\beta} - 1.$$

Therefore,

$$a = a_1 a_2 \leqslant \left(2^{\alpha+1}p^{\beta} - 1\right)^2 < \left(2^{\alpha+1}p^{\beta} - 1\right)\left(2^{\alpha+1}p^{\beta} + 1\right) = a,$$

which is impossible. So, either

$$a_1 \geqslant 2^{\alpha+1}p^{\beta} + 1 \text{ or } a_2 \geqslant 2^{\alpha+1}p^{\beta} + 1. \tag{3.7}$$

If $a_1 \geqslant 2^{\alpha+1}p^{\beta}+1$, then

$$a_1^2 \geqslant \left(2^{\alpha+1}p^{\beta}+1\right)^2 = c + 2^{\alpha+2}p^{\beta} > c + p^{\beta}.$$

Thus,

$$a_1^x > \left(a_1^2\right)^{z_1} > \left(c+p^{\beta}\right)^{z_1} > c^{z_1} + p^{\beta z_1} > c^{z_1} + p^{y_1},$$

and this contradicts (3.6). Similarly, if $a_2 \geqslant 2^{\alpha+1}p^{\beta}+1$, then

$$a_2^x > \left(a_2^2\right)^{z_1} > \left(c+p^{\beta}\right)^{z_1} > c^{z_1} + p^{\beta z_1} > c^{z_1} + p^{y_1} > c^{z_1} - p^{y_1},$$

and this contradicts (3.6). So the Diophantine equation (1.1) has only positive integer solution $(x, y, z) = (2, 2, 2)$.

□

Lemma 3.5. *If $n = p^s$ with $s \geq 1$, then Jeśmanowicz' conjecture is true.*

Proof. Suppose that (x, y, z) is a solution of equation (1.1) with $(x, y, z) \neq (2, 2, 2)$. We will show that this leads to a contradiction. Since $P(n) \nmid a$, where $a = 2^{2(\alpha+1)}p^{2\beta} - 1$ by Lemma 3.2, we have $x > z > y$ and $P(n)|b$. Since $b = 4k = 2^{\alpha+2}p^{\beta}$ and $n = p^s$ with $s \geq 1$, equation (1.1) becomes

$$2^{(\alpha+2)y}p^{\beta y} = p^{s(z-y)}\left[\left(2^{2\alpha+2}p^{2\beta}+1\right)^z - \left(2^{2\alpha+2}p^{2\beta}-1\right)^x p^{s(x-z)}\right].$$

So

$$\beta y = s(z-y),$$

and

$$2^{(\alpha+2)y} = \left(2^{2\alpha+2}p^{2\beta}+1\right)^z - \left(2^{2\alpha+2}p^{2\beta}-1\right)^x p^{s(x-z)}. \tag{3.8}$$

Then, from equation (3.8), we have $1 \equiv 2^z (\mathrm{mod} 3)$. Thus $z \equiv 0 (\mathrm{mod} 2)$. So, let $z = 2z_1$ with $z_1 > 1$ and $y = 2y_1$. Then rewrite equation (3.8) as

$$a^x p^{s(x-z)} = \left(2^{\alpha+1}p^{\beta}-1\right)^x \left(2^{\alpha+1}p^{\beta}+1\right)^x p^{s(x-z)} = \left(c^{z_1} - 2^{(\alpha+2)y_1}\right)\left(c^{z_1} + 2^{(\alpha+2)y_1}\right).$$

Noting that $\left(c^{z_1} - 2^{(\alpha+2)y_1}, c^{z_1} + 2^{(\alpha+2)y_1}\right) = 1$, we can write $a = a_1 a_2$, where $\gcd(a_1, a_2) = 1$ with

$$a_1^x | \, c^{z_1} + 2^{(\alpha+2)y_1} \ \textit{and} \ \ a_2^x | \, c^{z_1} - 2^{(\alpha+2)y_1}. \tag{3.9}$$

By (3.7) we either have $a_1 \geqslant 2^{\alpha+1}p^{\beta}+1$ *or* $a_2 \geqslant 2^{\alpha+1}p^{\beta}+1$. If $a_1 \geqslant 2^{\alpha+1}p^{\beta}+1$, then

$$a_1^2 \geqslant \left(2^{\alpha+1}p^{\beta}+1\right)^2 = c + 2^{\alpha+2}p^{\beta} > c + 2^{\alpha+2}.$$

Thus,

$$a_1^x > \left(a_1^2\right)^{z_1} > \left(c + 2^{\alpha+2}\right)^{z_1} > c^{z_1} + 2^{(\alpha+2)z_1} > c^{z_1} + 2^{(\alpha+2)y_1},$$

and this contradicts (3.9). Similarly, if $a_2 \geqslant 2^{\alpha+1}p^{\beta} + 1$, then we get

$$a_2^x > \left(a_2^2\right)^{z_1} > \left(c + 2^{\alpha+2}\right)^{z_1} > c^{z_1} + 2^{(\alpha+2)z_1} > c^{z_1} + 2^{(\alpha+2)y_1} > c^{z_1} - 2^{(\alpha+2)y_1},$$

and this contradicts (3.9). So the Diophantine equation (1.1) has only positive integer solution $(x, y, z) = (2, 2, 2)$. □

Lemma 3.6. *If $n = 2^r p^s$ with $r \geq 1$ and $s \geq 1$, then Jeśmanowicz' conjecture is true.*

Proof. Suppose that (x, y, z) is a solution of equation (1.1) with $(x, y, z) \neq (2, 2, 2)$. We will show that this leads to a contradiction. Since $P(n) \nmid a$ where $a = 2^{2(\alpha+1)}p^{2\beta} - 1$, by Lemma 3.2, we have $x > z > y$ and $P(n)|b$. Since $b = 4k = 2^{\alpha+2}p^{\beta}$ and $n = 2^r p^s$ with $r \geq 1$ and $s \geq 1$, equation (1.1) becomes

$$2^{(\alpha+2)y}p^{\beta y} = \left(2^r p^s\right)^{z-y}\left(c^z - a^x\left(2^r p^s\right)^{x-z}\right).$$

So

$$(\alpha + 2)y = r(z - y),$$

and

$$p^{\beta y} = p^{s(z-y)}\left(c^z - a^x\left(2^r p^s\right)^{x-z}\right).$$

Since gcd $\left(p, c^z - a^x\left(2^r p^s\right)^{x-z}\right) = 1$, $\beta y = s(z - y)$ and

$$c^z - a^x\left(2^r p^s\right)^{x-z} = 1, \tag{3.10}$$

Then, by equation (3.10), we have $1 \equiv 2^z (\text{mod} 3)$. Thus $z \equiv 0 (\text{mod} 2)$. So let $z = 2z_1$ and $z_1 > 1$. Then rewrite equation (3.10) as

$$\left(2^{\alpha+1}p^{\beta} - 1\right)^x\left(2^{\alpha+1}p^{\beta} + 1\right)^x\left(2^r p^s\right)^{x-z} = \left(c^{z_1} - 1\right)\left(c^{z_1} + 1\right).$$

Noting that $(c^{z_1} - 1, c^{z_1} + 1) = 2$, we can write $a = a_1 a_2$, where $\gcd(a_1, a_2) = 1$ with

$$a_1^x |\, c^{z_1} + 1 \text{ and } a_2^x |\, c^{z_1} - 1. \tag{3.11}$$

By (3.7) we have either $a_1 \geqslant 2^{\alpha+1}p^{\beta} + 1$ *or* $a_2 \geqslant 2^{\alpha+1}p^{\beta} + 1$.
If $a_1 \geqslant 2^{\alpha+1}p^{\beta} + 1$, then

$$a_1^2 \geqslant \left(2^{\alpha+1}p^{\beta} + 1\right)^2 = c + 2^{\alpha+2}p^{\beta} > c + 1.$$

Thus,

$$a_1^x > \left(a_1^2\right)^{z_1} > (c+1)^{z_1} > c^{z_1} + 1,$$

and this contradicts (3.11). Similarly, if $a_2 \geqslant 2^{\alpha+1}p^{\beta} + 1$, then we get

$$a_2^x > \left(a_2^2\right)^{z_1} > (c+1)^{z_1} > c^{z_1} + 1 > c^{z_1} - 1,$$

and this contradicts (3.11). So the Diophantine equation (1.1) has only positive integer solution $(x, y, z) = (2, 2, 2)$. □

4 Proof of Theorem 2.1

By Lemma 3.1, this is immediate for $n = 1$, it from . We now prove the result for $n \geq 2$. We have two cases to prove:

First Case. If $P(a) \mid n$, then this case is an immediate consequence of Lemma 3.3; i.e., Eq. (1.1) has only a positive integer solution $(x, y, z) = (2, 2, 2)$.

Second Case. If $P(n) \nmid a$. In this case we suppose that (x, y, z) is a solution of equation (1.1) with $(x, y, z) \neq (2, 2, 2)$. By Lemma 3.2, we have $x > z > y$ and $P(n)|b$. Since $b = 4k = 2^{\alpha+2}p^{\beta}$, we can write $n = 2^r p^s$, where $r + s \geqslant 1$. Thus equation (1.1) becomes

$$2^{(\alpha+2)y}p^{\beta y} = (2^r p^s)^{z-y}\left(c^z - a^x (2^r p^s)^{x-z}\right). \tag{4.12}$$

Thus, we consider three subcases as follows:

1. If $r \geq 1$ and $s = 0$, then $n = 2^r$,

2. If $s \geq 1$ and $r = 0$, then $n = p^s$,

3. If $r \geq 1$ and $s \geq 1$, then $n = 2^r p^s$.

By applying Lemmas 3.4, 3.5 and 3.6, we get a contradiction. Therefore, the Diophantine equation (1.1) has only one positive integer solution $(x, y, z) = (2, 2, 2)$. This completes the proof of Theorem 2.1.

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